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NONRECURSIVE TILINGS OF THE PLANE. I

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A finite set of tiles (unit squares with colored edges) is said to tile the plane if there exists an arrangement of translated (but not rotated or reflected) copies of the squares which fill the plane in such a way that abutting edges of the squares have the same color. The problem of whether there exists a finite set of tiles which can be used to tile the plane but not in any periodic fashion was proposed by Hao Wang [9] and solved by Robert Berger [1]. Raphael Robinson [7] gives a more detailed history and a very economical solution to this and related problems; we will assume that the reader is familiar with §4 of [7]. In 1971, Dale Myers asked whether there exists a finite set of tiles which can tile the plane but not in any recursive fashion. If we make an additional restriction (called the origin constraint) that a given tile must be used at least once, then the positive answer is given by the main theorem of this paper. Using the Turing machine constructed here and a more complicated version of Berger and Robinson's construction, Myers [5] has recently solved the problem without the origin constraint.

Given a finite set of tiles $T_1, \ldots, T_n$, we can describe a tiling of the plane by a function f of two variables ranging over the integers. $f(i, j) = k$ specifies that the tile T_k is to be placed at the position in the plane with coordinates (i, j). The tiling will be said to be recursive if f is a recursive function.

THEOREM. *There exists a finite set of tiles $T_1, \ldots, T_n$ such that the plane can be tiled using T_1 at the origin but no such tiling is recursive.*

PROOF. We first construct a certain Turing machine which has a two-way infinite tape. The left-hand portion of the tape contains an infinite sequence of 0's and 1's representing the input to the machine. The right-hand portion of the tape is initially blank and is available for working storage. We wish to construct the machine so that it will halt if the input sequence is recursive but will continue forever on certain non-recursive input sequences.

Let A and B be a pair of recursively enumerable sets which are disjoint and recursively inseparable.[1] Let g and h be recursive functions enumerating the elements of A and B respectively. The machine works as follows: It computes $g(0)$ and then checks to see if the $g(0)$th term of the input sequence is a 1. If it is not, the machine halts. Then it computes $h(0)$ and checks to see if the $h(0)$th term of the input sequence is a 0. If not, it halts. It then computes $g(1)$ and checks for a 1, $h(1)$

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[1] The existence of such a pair is well known; see e.g., [8, p. 94]. The classic example is the set of provable and the set of refutable sentences of some axiomatic theory of arithmetic. Taking T to be such a theory, Lemma 2.2 of [2] gives another way of constructing the Turing machine we desire.

and checks for a 0, etc. If the machine never halts, then the input sequence must be the characteristic function of a set C which includes A and is disjoint from B. Since A and B are recursively inseparable, C cannot be recursive. Thus the machine halts on any recursive input sequence but will continue forever if the input sequence is, say, the characteristic function of the set A.

We now construct a set of tiles exactly as in §4 of [7], except that the first tile in Figure 15 is replaced by two tiles, one with the 0 symbol on its top edge and one with the 1 symbol on its top edge. Then it is easy to see that once we have placed the initial tile (the second tile in Figure 15) at the origin, the only way to complete the tiling is to place a nonrecursive sequence of tiles to the left of the origin.

REMARKS. To classify the degree of nonrecursiveness of the tilings, it is convenient to make use of the notions in Putnam [6]. First we note that, given any set of n tiles that can tile the plane at all, there is a tiling that can be described by a trial and error predicate. In fact, for any enumeration of the lattice points of the plane, it is clear that the plane can be tiled by a systematic trial and error procedure in which the tile at the kth lattice point is changed fewer than n^k times. If the machine described in this paper is constructed in the most natural way, there is a tiling which can be described by a 1-trial predicate. On the other hand, if the machine is made to pause an extra step at each 0 (but not at each 1) it passes over in the input sequence, then there does not appear to be a 1-trial tiling.

PROBLEM. *Is it possible to construct a finite set of tiles which has no m-trial tiling for any m?*

Myers observed that this problem is equivalent to constructing a nonempty Π_1^0 class of sets which does not contain any set which is a Boolean combination of r.e. sets. Carl Jockusch [3] has solved this formulation of the problem by constructing a Π_1^0 class containing only bi-immune sets and showing that no bi-immune set is a Boolean combination of r.e. sets. The definition of the Π_1^0 class leads directly to a Turing machine which accepts only input tapes which code bi-immune sets. The Jockusch machine enumerates members of every possible r.e. set. Whenever it has found $e + 3$ members of the r.e. set indexed by e, it checks that this set of $e + 3$ numbers is neither included in nor disjoint from the set represented by the input tape. Thus with the origin constraint, the problem above is solved. Without the origin constraint, however, the problem remains open and, surprisingly enough, Myers method of Part II does not seem to give the solution.

The connection between tiling problems and problems in logic is close. Wang's original problem arose from his study of decision problems for fragments of logic. The existence of a consistent sentence with no recursive models (see Mostowski [4]) is easily derived from the present result. My interest in Berger and Robinson's construction arose from a desire to prove Conjecture I of [2] by eliminating the cycles that made the construction in that paper fall short of the goal. All such attempts have failed, however, and Conjectures I and II of that paper remain open problems.

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