

# Formalising Computability Theory in Isabelle/HOL

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**Abstract**—We present a formalised theory of computability in the theorem prover Isabelle/HOL. This theorem prover is based on classical logic which precludes *direct* reasoning about computability: every boolean predicate is either true or false because of the law of excluded middle. The only way to reason about computability in a classical theorem prover is to formalise a concrete model for computation. We formalise Turing machines and relate them to abacus machines and recursive functions. Our theory can be used to formalise other computability results: we give one example about the undecidability of Wang tilings, whose proof uses the notion of a universal Turing machine.

**Keywords**—Turing Machines, Computability, Isabelle/HOL, Wang tilings

## I. INTRODUCTION

We formalised in earlier work the correctness proofs for two algorithms in Isabelle/HOL—one about type-checking in LF [5] and another about deciding requests in access control [7]. The formalisations uncovered a gap in the informal correctness proof of the former and made us realise that important details were left out in the informal model for the latter. However, in both cases we were unable to formalise in Isabelle/HOL computability arguments about the algorithms. The reason is that both algorithms are formulated in terms of inductive predicates. Suppose  $P$  stands for one such predicate. Decidability of  $P$  usually amounts to showing whether  $P \vee \neg P$  holds. But this does *not* work in Isabelle/HOL, since it is a theorem prover based on classical logic where the law of excluded middle ensures that  $P \vee \neg P$  is always provable no matter whether  $P$  is constructed by computable means. The same problem would arise if we had formulated the algorithms as recursive functions, because internally in Isabelle/HOL, like in all HOL-based theorem provers, functions are represented as inductively defined predicates too.

The only satisfying way out of this problem in a theorem prover based on classical logic is to formalise a theory of computability. Norrish provided such a formalisation for the HOL4 theorem prover. He chose the  $\lambda$ -calculus as the starting point for his formalisation of computability theory, because of its “simplicity” [3, Page 297]. Part of his formalisation is a clever infrastructure for reducing  $\lambda$ -terms. He also established the computational equivalence between the  $\lambda$ -calculus and recursive functions. Nevertheless he concluded that it would be “appealing” to have formalisations for

more operational models of computations, such as Turing machines or register machines. One reason is that many proofs in the literature use them. He noted however that in the context of theorem provers [3, Page 310]:

*“If register machines are unappealing because of their general fiddliness, Turing machines are an even more daunting prospect.”*

In this paper we took on this daunting prospect and provide a formalisation of Turing machines, as well as abacus machines (a kind of register machines) and recursive functions. To see the difficulties involved with this work, one has to understand that interactive theorem provers, like Isabelle/HOL, are at their best when the data-structures at hand are “structurally” defined, like lists, natural numbers, regular expressions, etc. Such data-structures come with convenient reasoning infrastructures (for example induction principles, recursion combinators and so on). But this is *not* the case with Turing machines (and also not with register machines): underlying their definition is a set of states together with a transition function, both of which are not structurally defined. This means we have to implement our own reasoning infrastructure in order to prove properties about them. This leads to annoyingly fiddly formalisations. We noticed first the difference between both, structural and non-structural, “worlds” when formalising the Myhill-Nerode theorem, where regular expressions fared much better than automata [6]. However, with Turing machines there seems to be no alternative if one wants to formalise the great many proofs from the literature that use them. We will analyse one example—undecidability of Wang tilings—in Section V. The standard proof of this property uses the notion of *universal Turing machines*.

We are not the first who formalised Turing machines in a theorem prover: we are aware of the preliminary work by Asperti and Ricciotti [1]. They describe a complete formalisation of Turing machines in the Matita theorem prover, including a universal Turing machine. They report that the informal proofs from which they started are not “sufficiently accurate to be directly used as a guideline for formalization” [1, Page 2]. For our formalisation we followed the proofs from the textbook [2] and found that the description there is quite detailed. Some details are left out however: for example, it is only shown how the universal Turing machine is constructed for Turing machines comput-

ing unary functions. We had to figure out a way to generalize this result to  $n$ -ary functions. Similarly, when compiling recursive functions to abacus machines, the textbook again only shows how it can be done for 2- and 3-ary functions, but in the formalisation we need arbitrary functions. But the general ideas for how to do this are clear enough in [2]. However, one aspect that is completely left out from the informal description in [2], and similar ones we are aware of, are arguments why certain Turing machines are correct. We will introduce Hoare-style proof rules which help us with such correctness arguments of Turing machines.

The main difference between our formalisation and the one by Asperti and Ricciotti is that their universal Turing machine uses a different alphabet than the machines it simulates. They write [1, Page 23]:

*“In particular, the fact that the universal machine operates with a different alphabet with respect to the machines it simulates is annoying.”*

In this paper we follow the approach by Boolos et al [2], which goes back to Post [4], where all Turing machines operate on tapes that contain only blank or filled cells (represented by  $Bk$  and  $Oc$ , respectively, in our formalisation). Traditionally the content of a cell can be any character from a finite alphabet. Although computationally equivalent, the more restrictive notion of Turing machines make the reasoning more uniform. Unfortunately, it also makes it harder to design programs for Turing machines. Therefore in order to construct a *universal Turing machine* we follow the proof in [2] by relating abacus machines to turing machines and in turn recursive functions to abacus machines.

**Contributions:**

II. TURING MACHINES

Tapes

An action is defined as

$a$	::=	$W0$	write blank ( $Bk$ )
		$W1$	write occupied ( $Oc$ )
		$L$	move left
		$R$	move right
		$Nop$	do nothing

For showing the undecidability of the halting problem, we need to consider two specific Turing machines.

III. ABACUS MACHINES

IV. RECURSIVE FUNCTIONS

V. WANG TILES

Used in texture mappings - graphics

VI. RELATED WORK

The most closely related work is by Norrish [3], and Asperti and Ricciotti [1]. Norrish bases his approach on lambda-terms. For this he introduced a clever rewriting technology based on combinators and de-Brujin indices for rewriting modulo  $\beta$ -equivalence (to keep it manageable)

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