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NONRECURSIVE TILINGS OF THE PLANE. II

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The result. We show that there is a finite set of tiles which can tile the plane but not in any recursive way. This answers a natural sequel to Hao Wang's problem of the existence of a finite set of tiles which can tile the plane but not in any periodic way. In the proof, an elaboration of Robinson's method of transforming origin-constrained problems into unconstrained problems is applied to Hanf's origin-constrained tiling of Part I. We will assume familiarity with §§2, 3, and 7 of [3].

Tilings. Following Robinson, we will mark the edges of our tiles with symbols and configurations of arrow heads and tails as well as with colors. The matching condition for abutting tiles will be that the symbols on adjacent edges must be identical, every arrow head must match with a tail, every tail with a head, and the colors must be the same. This is, of course, mathematically equivalent to the original condition which involved only colors. A tiling of the plane by a set of tiles is a covering of the plane with translated copies of the tiles such that adjacent edges of abutting tiles satisfy the above matching condition. A set of tiles is *consistent* if the plane can be tiled by the set. A set of tiles with a designated origin tile is *origin-consistent* if there is a tiling of the plane with the origin tile at the center.

A square block of tiles is a tiling if every pair of abutting tiles satisfies the above matching condition. If an origin tile has been designated, a block of tiles is an origin-constrained tiling if it is a tiling with the origin tile at the center. Two blocks have the "*same*" center row if the blocks are of the same size and have identical center rows or if the smaller block's center row is a centered segment of the larger block's center row.

To each origin-consistent finite set of tiles with a designated origin tile we shall associate a consistent finite set of tiles appropriate for tilings with no origin constraint such that every tiling of the plane by the associated set pictures, in a recursive way, arbitrarily large blocks of tiles from the original set all of which are origin-constrained tilings with the "same" center rows. If, as in Part I, the original set has no origin-constrained tiling of the plane with a recursive center row, then the associated set will have no recursive tiling of the plane at all since the pictured blocks effectively determine an infinite center row which is nonrecursive since by König's lemma it must be the center row of some origin-constrained tiling of the plane by the original set of tiles. The sets of tiles that do the picturing are based on those of Robinson; any set of tiles considered in [3], however, can tile the plane recursively if at all. To construct the desired associated sets we extend Robinson's

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sets by adding appropriate markings which "synchronize" the center rows of the pictured blocks.

Extensions. Given two sets of tiles, called the base set and the marked set respectively, such that the tiles of the marked set are marked only with symbols and arrows but not colored edges, the *result of adding the markings of the marked set to the tiles of the base set* is the set of all distinct tiles obtained by placing the markings of a tile from the marked set onto a copy of a tile from the base set. It will always be tacitly assumed that the markings on the tiles of the marked set have been adjusted so that they do not cover up or conflict with any of the markings on the base tiles. A subset of the result of adding markings to the tiles of the base set is an *extension by added markings* if every tile of the base set can be obtained by erasing the added markings on some tile of the subset. The extension is *conservative* (*conservative with respect to a given class of tilings*) if every tiling of the plane (every tiling of the plane in the given class) by the base set can be obtained by erasing the added markings from a tiling of the plane by the extension.

Robinson sets of tiles. In [3, pp. 202, 203] Robinson introduces a set of tiles consisting of reflections and rotations of fifteen tiles derived from his five basic tiles by coloring the side arrows red or green and adding parity markings. In every tiling of the plane by this set, the tiles with red side arrows, call them red border tiles, form hollow squares, called *borders*, of width $m + 1$ for every m which is a nonzero power of four. The tiles with green side arrows, call them green border tiles, form borders of width $m + 1$ for every m which is an odd power of two. The

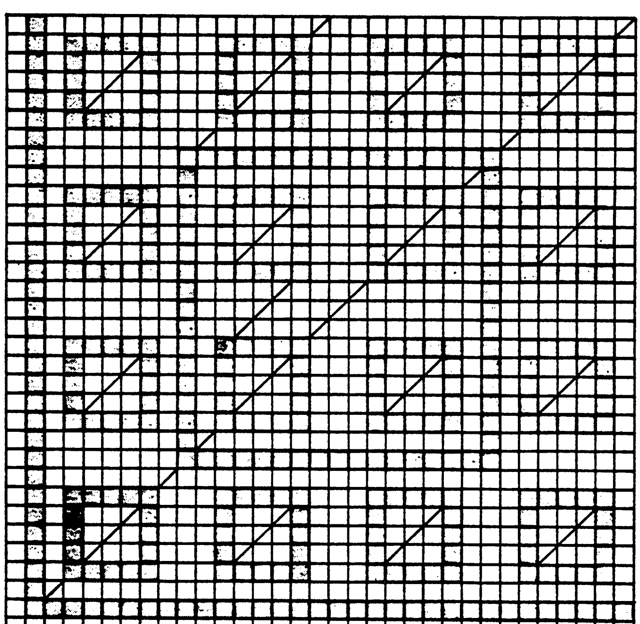

FIGURE 1

green borders of size $m + 1$ are centered on the corners of the red borders of size $2m + 1$.

In the desirable tilings of the plane by this set, the red borders of width $m + 1$ where m is a nonzero power of four are arrayed periodically with period $2m$ on the plane in infinite rows and columns separated by infinite corridors of width $m - 1$ (the width of the blocks enclosed by these borders) and every red border tile is either (a) on a red border whose width is $n + 1$ for some $n \leqq m$ which is a nonzero power of four or (b) in the center row or column of one of the above separating corridors of width $m - 1$. This pattern is illustrated in Figure 1 where the red border tiles are shaded. The diagonal lines are for later use.

The other tilings of the plane by this set consist of two half planes in which the red borders are arrayed periodically as above but for which the borders in one half plane are not always aligned with those in the other. We eliminate such tilings with faults by taking the markings on the tiles obtained by replacing the X on the tile in Figure 2 by a red or green R or L and the Y by a red or green T or B, and adding these markings to the tiles of the given set subject to the condition that tiles which can occur on the left, right, top, or bottom edge (including corners) of a red (green) border must be given red (green) L's, R's, T's, or B's respectively. The result is a consistent set of tiles which has only desirable tilings whose red borders are arrayed periodically as above. Any consistent extension of this set of tiles will be called a *Robinson set of tiles.*

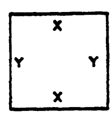

FIGURE 2

Henceforth "red border" and "red border tile" shall be abbreviated to "border" and "border tile" respectively. As seen in Figure 1, every border more than five squares wide includes exactly four borders of the next smaller size. We call them *subborders* of the border.

Basic and auxiliary board and border positions. Given a tiling of the plane by a Robinson set of tiles and a border of the tiling: an *edge* of the border is a row or column consisting of all noncorner tiles on one side of the border; the region within the border but outside all smaller borders is a *board*; the *major diagonal* of the board (see Figure 1) is the string of board tiles between the bottom left and the top right corners of the border; and the *minor diagonal* is the string of board tiles between the top left corner of the bottom left subborder and the bottom right corner of the top left subborder. A *free row* (*column*) is one which runs from one edge of a border to the opposite edge without encountering any smaller border. A *horizontally* (*vertically*) *free tile* is one which occurs in a free row (column). A *free tile* is one which is horizontally and vertically free.

The following will be called the *basic* board and border positions: each of the four edges of a border; the centers of each of these edges; each of the four corners of a border; the square halfway between the bottom left corner and the center of the left edge of a border; and the positions occupied by the free tiles, the horizontally free tiles, and the vertically free tiles.

The following will be called the *auxiliary* board and border positions: the center of a board; the center row of a board; the union of the major and minor diagonals; and the square, called the *minor diagonal's border square*, at the intersection of a border's left edge with the leftward extension of its minor diagonal. The last position occurs only on borders which have subborders. For these borders (borders of size greater than five) the minor diagonal's border square is just the square halfway between the center of the left edge and the bottom left corner. Since this last position is immediately above the bottom left corner only in borders of size five, a minor diagonal's border square may be characterized as a square halfway between the center of the left edge of a border and its bottom left corner but not immediately above the latter.

Tiles which can occur on a board, at a bottom left corner, at a minor diagonal's border square, etc., will be called board tiles, bottom left corner tiles, minor diagonal border tiles, etc.

For a given set of tiles, a position on a board or border is *identifiable* iff the tiles which can occur in that position in a tiling of the plane by the given set are distinct from the tiles which can occur elsewhere in a board or border of a tiling. Positions identifiable by a given set of tiles are also identifiable by any extension of the set. The basic board and border positions are identifiable by any Robinson set of tiles. (The tiles which can occur halfway between the bottom left corner and the center of the left edge of a border can be identified as those bottom-half-of-the-left-edge tiles which are also green border tiles.) The auxiliary positions, however, need not be.

Extensions identifying auxiliary positions. Given a Robinson set of tiles we now construct two conservative extensions which make the auxiliary positions identifiable.

The first extension, which makes the center square and center row of a board and the minor diagonal's border square identifiable, is constructed by adding to the given set the markings on the tiles in Figure 3 subject only to the following conditions:

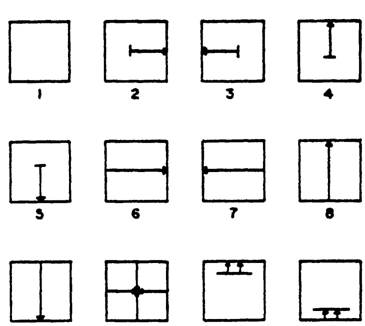

FIGURE 3

The markings on tiles 2, 3, 4, and 5 respectively are to be added only to and no other markings (not even the blank markings on 1) are to be added to the tiles of the given set which can occur at the center of the left, right, bottom, and top edge respectively of a border.

The markings on tiles 6, 7, 8, 9, and 10 are to be added only to board tiles.

The markings on tile 11 are to be added only to and no other markings are to be added to bottom left corner tiles.

The conditions guarantee that a signal, i.e., a sequence of arrows, will be initiated from the center of each edge of a border and will continue across its board to the center and that a double signal will be emitted upward from the bottom left corner and will be absorbed by the tile directly above it as in Figure 4. The tiles which can occur at the center of a board can now be identified as the tiles which bear the markings of tile 10. The center row tiles can be identified as the tiles marked by tiles 6, 7, and 10. The minor diagonal border tiles can be identified as the tiles which can occur midway between the center of the left edge of a border and the bottom left corner (a basis board and border position) but are not marked by tile 12.

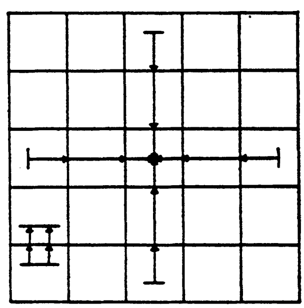

FIGURE 4

Now assume that in the given Robinson set of tiles the minor diagonal's border square is identifiable. The second extension, which makes the diagonals identifiable, is constructed by adding to the given set the markings on the tiles in Figure 5 subject only to the following conditions:

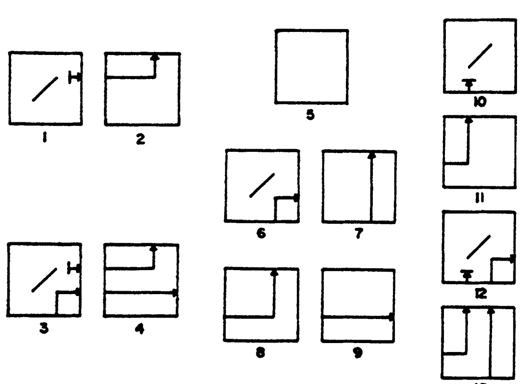

FIGURE 5

Tile 1 is to be added only to bottom left corner tiles and minor diagonal border tiles.

No tile other than 1 is to be added to minor diagonal border tiles.

Tile 3 is to be added only to bottom left corner tiles.

No tiles other than 1 and 3 are to be added to bottom left corner tiles.

Tile 6 is to be added only to board tiles and top left corner tiles.

No tiles other than 5, 8, and 10 are to be added to bottom right corner tiles.

Tile 9 is to be added only to bottom edge tiles.

No tiles other than 2, 4, 5, and 9 are to be added to bottom edge tiles.

Tile 7 is to be added only to right edge tiles.

No tiles other than 5, 7, 11, and 13 are to be added to right edge tiles.

Tile 10 is to be added only to top right corner and bottom right corner tiles.

Tile 12 is to be added only to top right corner tiles.

No tiles other than 10 and 12 are to be added to top right corner tiles.

We claim that, in any board of a border of any tiling of the plane by the extension just constructed, the tiles with slash markings, i.e., tiles which receive added markings from 1, 3, 6, 10, or 12, can only occur and must occur on the main diagonal or on the minor diagonal or on the minor diagonal's leftward extension to its border square. To prove this, we show that the added markings must occur essentially as illustrated in Figure 6.

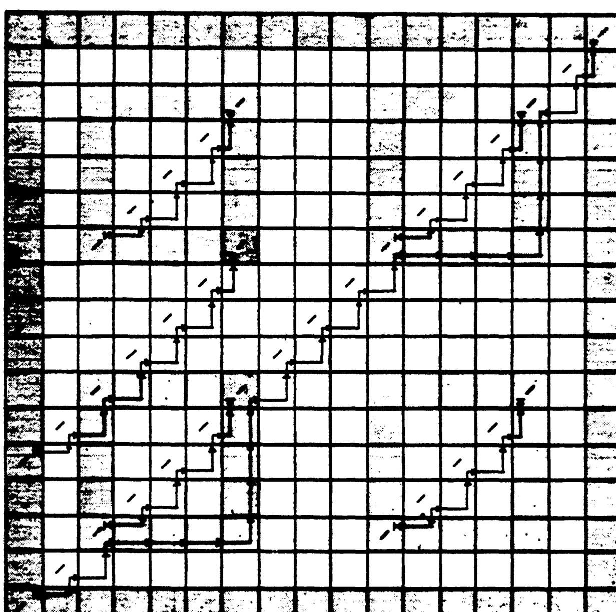

FIGURE 6

The following facts are easily verified:

A signal may only progress upward or to the right.

Tiles marked by 2 and 4 (11 and 13) must always be immediately to the right of (below) tiles marked by 1 and 3 (10 and 12).

While signals may enter and leave a border, no signal ever crosses a border. (Inspect the markings allowed on border tiles.)

The only place a signal may be initiated (terminated) is at a bottom left corner (top right corner) or at a minor diagonal's border square (bottom right corner of a subborder).

Hence at most two signals may travel across a board. The signal emitted at the upper half of the right edge of a bottom left corner tile is sent to a tile marked by 2 or 4 and then upward onto the board. Here it progresses diagonally via an alternating sequence of tiles marked by 6 and 8 (Berger used this type of sequence extensively in [1]) until, possibly, an interior border is encountered. It will enter such a border at the bottom of a bottom left corner tile marked by 3, run along the lower edge, around the corner, up the right edge, exit on the right side of the top right corner and then return to its diagonal path. This continues until the signal is terminated at the top right corner opposite the corner from which it started. The path of a signal initiated at the minor diagonal's border square is similar except that in addition to skirting around smaller interior borders, if any, it has an encounter with the top left corner of the bottom left subborder and an early termination at the bottom right corner of the top left subborder.

The board tiles which can occur on a diagonal can now be identified as the horizontally free tiles with slash markings. The horizontally free requirement serves to eliminate tiles on the leftward extension of the minor diagonal.

Synchronizing the center rows. Given a board of a tiling of the plane by a Robinson set of tiles whose free tiles are an extension by added markings of a base set, the result of omitting the nonfree rows and columns, bringing the remaining tiles together, and erasing the added markings is a contiguous block of base tiles called the *simulated* block. Every tiling of the plane by such a set will have simulated blocks of arbitrarily large size. A tiling is *standard* if all the simulated blocks are the "same," i.e., any two such blocks are either identical or the smaller is a centered subblock of the larger.

Assume we are given a Robinson set of tiles whose free tiles are an extension by added markings of a base set and which admits standard tilings. We now construct a consistent extension of the given set such that in every tiling of the plane the simulated blocks of base tiles have the "same" center rows. First assign a symbol to each tile of the base set in a one-to-one way. Let the marked set of tiles be the set of tiles pictured in Figure 2 with X and Y replaced by these symbols. The desired extension is obtained by adding the markings of this marked set to the given Robinson set subject to the conditions that the symbol replacing X on a free center row tile be the symbol assigned to the tile's associated base tile (i.e., the tile obtained by erasing all the added markings), and that the symbols replacing X and Y on a diagonal tile be identical. (Think of the diagonal tiles as reflectors which send matching signals received from above and below out to the left and right.)

In any tiling of the plane by the constructed extension, the condition on the symbols added to the free center row tiles guarantees that the center row of any block of base tiles simulated by any board of the tiling is the same as that simulated by any other board of the same size directly above or below. The condition on diagonal tiles applied to tiles on the main diagonals guarantees that this is also true for blocks simulated by boards of the same size to the left or right of the first board. Finally, the condition on diagonal tiles applied to tiles on the minor diagonal guarantees that the center row of the block simulated by the board of a given border's bottom left subborder is a centered segment of the center row of the block simulated by the given border's board. Together these facts guarantee that in any given tiling of the plane the center rows of the simulated blocks are all the "same."

Proof of main result. Suppose an origin-consistent finite set of tiles with an origin tile has been given. Call it the base set. Robinson has shown how to associate with this set a Robinson set of tiles (or what becomes a Robinson set of tiles after being modified to prevent faults as in the section on Robinson sets of tiles) whose free tiles are the base tiles with added markings and for which the simulated blocks of any tiling of the plane by the set are always tilings and for which there are standard tilings whose simulated blocks are all origin-constrained tilings. By extending this latter set of tiles to a Robinson set for which the center row and center tile are identifiable and adding the restriction that only the origin tile of the base set can bear the markings of the center tile, we obtain a Robinson set for which all simulated blocks of tilings of the plane are origin-constrained tilings. Finally, by extending this set of tiles so that the diagonals are identifiable and then extending the set so that the center rows of the simulated blocks are the "same," we get our desired Robinson set of tiles. Its consistency follows from the fact that the extensions involved are conservative with respect to standard tilings whose simulated blocks are all origin-constrained tilings.

Since the recognition of borders, boards, simulated blocks of base tiles and their center rows are clearly recursive procedures, it is easy to see that for any tiling of the plane by the set just constructed the infinite center row of tiles obtained by piecing together the finite center rows of the simulated blocks is recursive relative to the given tiling of the plane. By König's lemma this infinite row is the center row of an origin-constrained tiling of the plane by the original set of base tiles. Hence if, as in Part I, the center row of an origin-constrained tiling of the plane by the base set is always nonrecursive, then all tilings of the plane (no origin constraint) by the constructed set are nonrecursive.

Conclusion. How much further can one go? We conjecture that there is a consistent finite set of tiles with no tiling of the plane that is a Boolean combination of recursively enumerable sets. (See Part I for the origin-constrained case.) On the other hand, every finite set of tiles has a Δ_2^0 tiling of the plane.

Finally, let us note that the above method of synchronizing center rows provides the basis for a constructively valid proof that the halting problem for Turing machines is equivalent to the tiling problem—the problem of determining the consistency of a finite set of tiles.

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