Reasoning about Turing Machines and Low-Level Code

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A Trend in Verification

- in the past:

  model a problem mathematically and proof properties about the model

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- needs elegance, is still very hard
- does not help with ensuring the correctness of running programs

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  you would expect the trend would be to for example model C, implement your programs in C and verify the programs written in C (e.g. seL4)

A Trend in Verification

- but actually people start to verify machine code directly (e.g. bignum arithmetic implemented in x86-64 – 700 instructions)
- CPU models exists, but the strategy is to use a small subset which you use in your programs

Why Turing Machines

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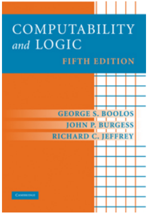

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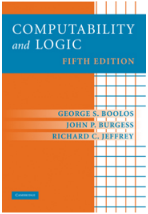
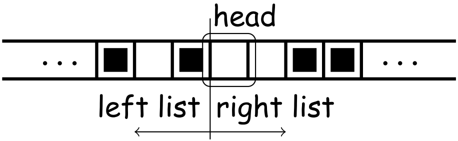


- found an inconsistency in the definition of halting computations (Chap. 3 vs Chap. 8)
- Norrish formalised computability via lambda-calculus (and nominal); Asperti and Riccioti formalised TMs but didn't get proper UTM

Turing Machines

- tapes contain 0 or 1 only



- steps function

  What does the tape look like after the TM has executed n steps?

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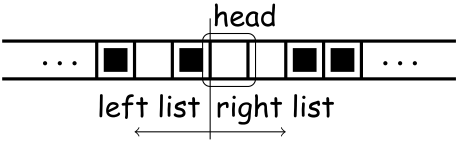


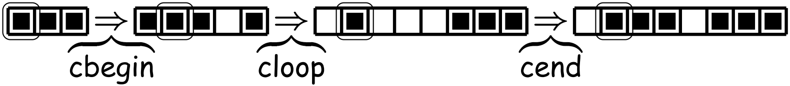
- steps function

  What does the tape look like after the TM has executed n steps?

designate the 0-state as halting state and remain there forever

Copy Turing Machines

- TM that copies a number on the input tape



cbegin $\stackrel{\text{def}}{=}$
$[(W_0, 0), (R, 2), (R, 3),$
$(R, 2), (W_1, 3), (L, 4),$
$(L, 4), (L, 0)]$

cloop $\stackrel{\text{def}}{=}$
$[(R, 0), (R, 2), (R, 3),$
$(W_0, 2), (R, 3), (R, 4),$
$(W_1, 5), (R, 4), (L, 6),$
$(L, 5), (L, 6), (L, 1)]$

cend $\stackrel{\text{def}}{=}$
$[(L, 0), (R, 2), (W_1, 3),$
$(L, 4), (R, 2), (R, 2),$
$(L, 5), (W_0, 4), (R, 0),$
$(L, 5)]$

Hoare Logic for TMs

- Hoare-triples and Hoare-pairs:

$\{P\}\ p\ \{Q\} \stackrel{\text{def}}{=}$
$\quad\forall$ tp.
$\quad\quad$if P tp holds then
$\quad\quad\exists$ n. such that
$\quad\quad$is_final (steps (1, tp) p n) $\wedge$
$\quad\quad$Q holds_for (steps (1, tp) p n)

$\{P\}\ p \uparrow \stackrel{\text{def}}{=}$
$\quad\forall$ tp.
$\quad\quad$if P tp holds then
$\quad\quad\forall$ n. $\neg$ is_final (steps (1, tp) p n)

Hoare Reasoning

- reasoning is still quite difficult—invariants

$$
\begin{array}{lcll}
\hline
I_1\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & (l, r) = ([], 1^n) & \text{(starting state)} \\
I_2\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & \exists\, i\ j.\ 0 < i \wedge i + j = n \wedge (l, r) = (1^i, 1^j) & \\
I_3\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & 0 < n \wedge (l, tl\ r) = (0{::}1^n, []) & \\
I_4\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & 0 < n \wedge (l, r) = (1^n, [0, 1]) \vee (l, r) = (1^{n-1}, [1, 0, 1]) & \\
I_0\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & 1 < n \wedge (l, r) = (1^{n-2}, [1, 1, 0, 1]) \vee & \text{(halting state)} \\
 & & n = 1 \wedge (l, r) = ([], [0, 1, 0, 1]) & \\
\hline
J_1\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & \exists\, i\ j.\ i + j + 1 = n \wedge (l, r) = (1^i, 1{::}1{::}0^j\ @\ 1^j) \wedge 0 < j \vee & \\
 & & 0 < n \wedge (l, r) = ([], 0{::}1{::}0^n\ @\ 1^n) & \text{(starting state)} \\
J_0\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & 0 < n \wedge (l, r) = ([0], 1{::}0^n\ @\ 1^n) & \text{(halting state)} \\
\hline
K_1\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & 0 < n \wedge (l, r) = ([0], 1{::}0^n\ @\ 1^n) & \text{(starting state)} \\
K_0\ n\ (l, r) & \stackrel{\mathrm{def}}{=} & 0 < n \wedge (l, r) = ([0], 1^n\ @\ (0{::}1^n)) & \text{(halting state)} \\
\hline
\end{array}
$$

Register Machines

- instructions

I	::=	Inc R	increment register R by one
	|	Dec R L	if content of R is non-zero, then decrement it by one otherwise jump to instruction L
	|	Goto L	jump to instruction L

Recursive Functions

- addition, multiplication, ...
- logical operations, quantifiers...
- coding of numbers (Cantor encoding)
- UF

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- Recursive Functions $\Rightarrow$ Register Machines
- Register Machines $\Rightarrow$ Turing Machines

Sizes

- UF (size: 140843)
- Register Machine (size: 2 Mio instructions)
- UTM (size: 38 Mio states)

old version: RM (12 Mio) UTM (112 Mio)

Separation Algebra

- introduced a separation algebra framework for register machines and TMs
- we can semi-automate the reasoning for our small TMs
- we can assemble bigger programs out of smaller components

- looks awfully like "real" assembly code

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- we can semi-automate the reasoning for our small TMs
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- looks awfully like “real” assembly code
- Conclusion: we have a playing ground for reasoning about low-level code; we work on automation