tphols-2011

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Contents

```
theory Myhill-1
  imports Main
begin
```

1 Preliminary definitions

types *lang* = *string set*

Sequential composition of two languages

```
definition
  Seq :: lang ⇒ lang ⇒ lang (infixr ;; 100)
where
  A ;; B = {s1 @ s2 | s1 s2. s1 ∈ A ∧ s2 ∈ B}
```

Some properties of operator ;;.

```
lemma seq-add-left:
  assumes a: A = B
  shows C ;; A = C ;; B
using a by simp

lemma seq-union-distrib-right:
  shows (A ∪ B) ;; C = (A ;; C) ∪ (B ;; C)
unfolding Seq-def by auto

lemma seq-union-distrib-left:
  shows C ;; (A ∪ B) = (C ;; A) ∪ (C ;; B)
unfolding Seq-def by  auto

lemma seq-intro:
  assumes a: x ∈ A y ∈ B
  shows x @ y ∈ A ;; B
using a by (auto simp: Seq-def)

lemma seq-assoc:
  shows (A ;; B) ;; C = A ;; (B ;; C)
unfolding Seq-def
apply(auto)
apply(blast)
by (metis append-assoc)

lemma seq-empty [simp]:
  shows A ;; {[]} = A
  and   {[]} ;; A = A
```

```
by (simp-all add: Seq-def)
```

Power and Star of a language

```
fun
  pow :: lang ⇒ nat ⇒ lang (infixl ↑ 100)
where
  A ↑ 0 = {[]}
| A ↑ (Suc n) =  A ;; (A ↑ n)

definition
  Star :: lang ⇒ lang (-⋆ [101] 102)
where
  A⋆ ≡ (⋃n. A ↑ n)


lemma star-start[intro]:
  shows [] ∈ A⋆
proof −
  have [] ∈ A ↑ 0 by auto
  then show [] ∈ A⋆ unfolding Star-def by blast
qed

lemma star-step [intro]:
  assumes a: s1 ∈ A
  and     b: s2 ∈ A⋆
  shows s1 @ s2 ∈ A⋆
proof −
  from b obtain n where s2 ∈ A ↑ n unfolding Star-def by auto
  then have s1 @ s2 ∈ A ↑ (Suc n) using a by (auto simp add: Seq-def)
  then show s1 @ s2 ∈ A⋆ unfolding Star-def by blast
qed

lemma star-induct[consumes 1, case-names start step]:
  assumes a: x ∈ A⋆
  and     b: P []
  and     c: ⋀s1 s2. ⟦s1 ∈ A; s2 ∈ A⋆; P s2⟧ ⟹ P (s1 @ s2)
  shows P x
proof −
  from a obtain n where x ∈ A ↑ n unfolding Star-def by auto
  then show P x
    by (induct n arbitrary: x)
       (auto intro!: b c simp add: Seq-def Star-def)
qed

lemma star-intro1:
  assumes a: x ∈ A⋆
  and     b: y ∈ A⋆
  shows x @ y ∈ A⋆
using a b
```

by (*induct rule*: *star-induct*) (*auto*)

lemma *star-intro2*:
  assumes *a*: $y \in A$
  shows $y \in A\star$
proof −
  from *a* **have** y @ $[] \in A\star$ **by** *blast*
  then show $y \in A\star$ **by** *simp*
qed

lemma *star-intro3*:
  assumes *a*: $x \in A\star$
  and *b*: $y \in A$
  shows x @ $y \in A\star$
using *a b* **by** (*blast intro*: *star-intro1 star-intro2*)

lemma *star-cases*:
  shows $A\star = \{[]\} \cup A$;; $A\star$
proof
  { fix x
    have $x \in A\star \Longrightarrow x \in \{[]\} \cup A$;; $A\star$
      unfolding *Seq-def*
    by (*induct rule*: *star-induct*) (*auto*)
  }
  then show $A\star \subseteq \{[]\} \cup A$;; $A\star$ **by** *auto*
next
  show $\{[]\} \cup A$;; $A\star \subseteq A\star$
    unfolding *Seq-def* **by** *auto*
qed

lemma *star-decom*:
  assumes *a*: $x \in A\star$ $x \neq []$
  shows $\exists\, a\ b.\ x = a$ @ $b \wedge a \neq [] \wedge a \in A \wedge b \in A\star$
using *a*
apply(*induct rule*: *star-induct*)
apply(*simp*)
apply(*blast*)
done

lemma
  shows *seq-Union-left*: B ;; $(\bigcup n.\ A \uparrow n) = (\bigcup n.\ B$;; $(A \uparrow n))$
  and *seq-Union-right*: $(\bigcup n.\ A \uparrow n)$;; $B = (\bigcup n.\ (A \uparrow n)$;; $B)$
unfolding *Seq-def* **by** *auto*

lemma *seq-pow-comm*:
  shows A ;; $(A \uparrow n) = (A \uparrow n)$;; A
by (*induct n*) (*simp-all add*: *seq-assoc*[*symmetric*])

lemma *seq-star-comm*:

shows A ;; $A\star = A\star$;; A
unfolding *Star-def*
unfolding *seq-Union-left*
unfolding *seq-pow-comm*
unfolding *seq-Union-right*
by *simp*

Two lemmas about the length of strings in $A \uparrow n$

lemma *pow-length*:
assumes *a*: $[] \notin A$
and *b*: $s \in A \uparrow Suc\ n$
shows $n < length\ s$
using *b*
proof (*induct n arbitrary*: *s*)
case *0*
have $s \in A \uparrow Suc\ 0$ **by** *fact*
with *a* **have** $s \neq []$ **by** *auto*
then show $0 < length\ s$ **by** *auto*
next
case (*Suc n*)
have *ih*: $\bigwedge s.\ s \in A \uparrow Suc\ n \Longrightarrow n < length\ s$ **by** *fact*
have $s \in A \uparrow Suc\ (Suc\ n)$ **by** *fact*
then obtain *s1 s2* **where** *eq*: $s = s1\ @\ s2$ **and** $*$: $s1 \in A$ **and** $**$: $s2 \in A \uparrow Suc\ n$
by (*auto simp add*: *Seq-def*)
from *ih* $**$ **have** $n < length\ s2$ **by** *simp*
moreover have $0 < length\ s1$ **using** $*$ *a* **by** *auto*
ultimately show $Suc\ n < length\ s$ **unfolding** *eq*
by (*simp only*: *length-append*)
qed

lemma *seq-pow-length*:
assumes *a*: $[] \notin A$
and *b*: $s \in B$;; $(A \uparrow Suc\ n)$
shows $n < length\ s$
proof $-$
from *b* **obtain** *s1 s2* **where** *eq*: $s = s1\ @\ s2$ **and** $*$: $s2 \in A \uparrow Suc\ n$
unfolding *Seq-def* **by** *auto*
from $*$ **have** $n < length\ s2$ **by** (*rule pow-length*[*OF a*])
then show $n < length\ s$ **using** *eq* **by** *simp*
qed

2 A slightly modified version of Arden's lemma

A helper lemma for Arden

lemma *ardens-helper*:
assumes *eq*: $X = X$;; $A \cup B$
shows $X = X$;; $(A \uparrow Suc\ n) \cup (\bigcup m \in \{0..n\}.\ B$;; $(A \uparrow m))$

proof (*induct n*)
  case *0*
  show $X = X$;; $(A \uparrow Suc\ 0) \cup (\bigcup (m{::}nat){\in}\{0..0\}.\ B$;; $(A \uparrow m))$
    using *eq* **by** *simp*
next
  case (*Suc n*)
  have *ih*: $X = X$;; $(A \uparrow Suc\ n) \cup (\bigcup m{\in}\{0..n\}.\ B$;; $(A \uparrow m))$ **by** *fact*
  also have ... $= (X$;; $A \cup B)$;; $(A \uparrow Suc\ n) \cup (\bigcup m{\in}\{0..n\}.\ B$;; $(A \uparrow m))$
using *eq* **by** *simp*
  also have ... $= X$;; $(A \uparrow Suc\ (Suc\ n)) \cup (B$;; $(A \uparrow Suc\ n)) \cup (\bigcup m{\in}\{0..n\}.$
B ;; $(A \uparrow m))$
    by (*simp add*: *seq-union-distrib-right seq-assoc*)
  also have ... $= X$;; $(A \uparrow Suc\ (Suc\ n)) \cup (\bigcup m{\in}\{0..Suc\ n\}.\ B$;; $(A \uparrow m))$
    by (*auto simp add*: *le-Suc-eq*)
  finally show $X = X$;; $(A \uparrow Suc\ (Suc\ n)) \cup (\bigcup m{\in}\{0..Suc\ n\}.\ B$;; $(A \uparrow m))$ **.**
qed

theorem *ardens-revised*:
  assumes *nemp*: $[] \notin A$
  shows $X = X$;; $A \cup B \longleftrightarrow X = B$;; $A\star$
proof
  assume *eq*: $X = B$;; $A\star$
  have $A\star = \{[]\} \cup A\star$;; A
    unfolding *seq-star-comm*[*symmetric*]
    by (*rule star-cases*)
  then have B ;; $A\star = B$;; $(\{[]\} \cup A\star$;; $A)$
    by (*rule seq-add-left*)
  also have ... $= B \cup B$;; $(A\star$;; $A)$
    unfolding *seq-union-distrib-left* **by** *simp*
  also have ... $= B \cup (B$;; $A\star)$;; A
    by (*simp only*: *seq-assoc*)
  finally show $X = X$;; $A \cup B$
    using *eq* **by** *blast*
next
  assume *eq*: $X = X$;; $A \cup B$
  { fix *n::nat*
    have B ;; $(A \uparrow n) \subseteq X$ **using** *ardens-helper*[*OF eq*, *of n*] **by** *auto* **}**
  then have B ;; $A\star \subseteq X$
    unfolding *Seq-def Star-def UNION-def*
    by *auto*
  moreover
  { fix *s::string*
    obtain k **where** $k = length\ s$ **by** *auto*
    then have *not-in*: $s \notin X$;; $(A \uparrow Suc\ k)$
      using *seq-pow-length*[*OF nemp*] **by** *blast*
    assume $s \in X$
    then have $s \in X$;; $(A \uparrow Suc\ k) \cup (\bigcup m{\in}\{0..k\}.\ B$;; $(A \uparrow m))$
      using *ardens-helper*[*OF eq*, *of k*] **by** *auto*
    then have $s \in (\bigcup m{\in}\{0..k\}.\ B$;; $(A \uparrow m))$ **using** *not-in* **by** *auto*

```
  moreover
  have (⋃ m∈{0..k}. B ;; (A ↑ m)) ⊆ (⋃ n. B ;; (A ↑ n)) by auto
  ultimately
  have s ∈ B ;; A⋆
    unfolding seq-Union-left Star-def
    by auto }
then have X ⊆ B ;; A⋆ by auto
ultimately
show X = B ;; A⋆ by simp
qed
```

3 Regular Expressions

```
datatype rexp =
  NULL
| EMPTY
| CHAR char
| SEQ rexp rexp
| ALT rexp rexp
| STAR rexp
```

The following *L* is an overloaded operator, where $L(x)$ evaluates to the language represented by the syntactic object x.

```
consts L:: 'a ⇒ lang
```

The *L* (*rexp*) for regular expressions.

```
overloading L-rexp ≡ L::  rexp ⇒ lang
begin
fun
  L-rexp :: rexp ⇒ string set
where
    L-rexp (NULL) = {}
  | L-rexp (EMPTY) = {[]}
  | L-rexp (CHAR c) = {[c]}
  | L-rexp (SEQ r1 r2) = (L-rexp r1) ;; (L-rexp r2)
  | L-rexp (ALT r1 r2) = (L-rexp r1) ∪ (L-rexp r2)
  | L-rexp (STAR r) = (L-rexp r)⋆
end
```

4 Folds for Sets

To obtain equational system out of finite set of equivalence classes, a fold operation on finite sets *folds* is defined. The use of *SOME* makes *folds* more robust than the *fold* in the Isabelle library. The expression *folds f* makes sense when *f* is not *associative* and *commutitive*, while *fold f* does not.

definition

folds :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b$
where
folds f z S $\equiv$ *SOME x. fold-graph f z S x*

abbreviation
Setalt ($\biguplus$- [*1000*] *999*)
where
$\biguplus$ *A* == *folds ALT NULL A*

The following lemma ensures that the arbitrary choice made by the *SOME* in *folds* does not affect the *L*-value of the resultant regular expression.

lemma *folds-alt-simp* [*simp*]:
assumes *a*: *finite rs*
shows *L* ($\biguplus$ *rs*) = $\bigcup$ (*L* ‘ *rs*)
apply(*rule set-eqI*)
apply(*simp add*: *folds-def*)
apply(*rule someI2-ex*)
apply(*rule-tac finite-imp-fold-graph*[*OF a*])
apply(*erule fold-graph.induct*)
apply(*auto*)
done

Just a technical lemma for collections and pairs

lemma *Pair-Collect*[*simp*]:
shows $(x, y) \in \{(x, y).\ P\ x\ y\} \longleftrightarrow P\ x\ y$
by *simp*

$\approx A$ is an equivalence class defined by language A.

definition
str-eq-rel :: *lang* $\Rightarrow$ (*string* $\times$ *string*) *set* ($\approx$- [*100*] *100*)
where
$\approx A \equiv \{(x, y).\ (\forall z.\ x\ @\ z \in A \longleftrightarrow y\ @\ z \in A)\}$

Among the equivalence clases of $\approx A$, the set *finals A* singles out those which contains the strings from A.

definition
finals :: *lang* $\Rightarrow$ *lang set*
where
finals A $\equiv \{\approx A$ ‘‘ $\{x\} \mid x\ .\ x \in A\}$

The following lemma establishes the relationshipt between *finals A* and *A*.

lemma *lang-is-union-of-finals*:
shows $A = \bigcup$ *finals A*
unfolding *finals-def*
unfolding *Image-def*
unfolding *str-eq-rel-def*
apply(*auto*)
apply(*drule-tac x* = [] **in** *spec*)

apply(*auto*)
done

lemma *finals-in-partitions*:
  shows *finals* $A \subseteq (UNIV\ //\ \approx A)$
unfolding *finals-def*
unfolding *quotient-def*
by *auto*

5 Direction *finite partition* ⇒ *regular language*

The relationship between equivalent classes can be described by an equational system. For example, in equational system (1), X_0, X_1 are equivalent classes. The first equation says every string in X_0 is obtained either by appending one b to a string in X_0 or by appending one a to a string in X_1 or just be an empty string (represented by the regular expression λ). Similary, the second equation tells how the strings inside X_1 are composed.

$$\begin{aligned} X_0 &= X_0 b + X_1 a + \lambda \\ X_1 &= X_0 a + X_1 b \end{aligned} \tag{1}$$

The summands on the right hand side is represented by the following data type *rhs-item*, mnemonic for 'right hand side item'. Generally, there are two kinds of right hand side items, one kind corresponds to pure regular expressions, like the λ in (1), the other kind corresponds to transitions from one one equivalent class to another, like the $X_0 b, X_1 a$ etc.

datatype *rhs-item* =
   Lam rexp
 | *Trn lang rexp*

In this formalization, pure regular expressions like λ is repsented by $Lam(EMPTY)$, while transitions like $X_0 a$ is represented by $Trn\ X_0\ (CHAR\ a)$.

The functions *the-r* and *the-Trn* are used to extract subcomponents from right hand side items.

fun
  the-r :: *rhs-item* ⇒ *rexp*
where
  the-r (*Lam r*) = *r*

fun
  the-trn-rexp:: *rhs-item* ⇒ *rexp*
where
  the-trn-rexp (*Trn Y r*) = *r*

Every right-hand side item *itm* defines a language given by $L(itm)$, defined as:

overloading $L\text{-}rhs\text{-}e \equiv L$:: $rhs\text{-}item \Rightarrow lang$
begin
  fun $L\text{-}rhs\text{-}e$:: $rhs\text{-}item \Rightarrow lang$
  where
    $L\text{-}rhs\text{-}e\ (Lam\ r) = L\ r$
  | $L\text{-}rhs\text{-}e\ (Trn\ X\ r) = X\ ;;\ L\ r$
end

The right hand side of every equation is represented by a set of items. The string set defined by such a set *itms* is given by $L(itms)$, defined as:

overloading $L\text{-}rhs \equiv L$:: $rhs\text{-}item\ set \Rightarrow lang$
begin
  fun $L\text{-}rhs$:: $rhs\text{-}item\ set \Rightarrow lang$
  where
    $L\text{-}rhs\ rhs = \bigcup\ (L\ \text{`}\ rhs)$
end

Given a set of equivalence classes CS and one equivalence class X among CS, the term $init\text{-}rhs\ CS\ X$ is used to extract the right hand side of the equation describing the formation of X. The definition of *init-rhs* is:

definition
  $transition :: lang \Rightarrow rexp \Rightarrow lang \Rightarrow bool$ (- $\models$-$\Rightarrow$- [100,100,100] 100)
where
  $Y \models r \Rightarrow X \equiv Y\ ;;\ (L\ r) \subseteq X$

definition
  $init\text{-}rhs\ CS\ X \equiv$
    $if\ ([] \in X)\ then$
      $\{Lam\ EMPTY\} \cup \{Trn\ Y\ (CHAR\ c)\ |\ Y\ c.\ Y \in CS \wedge Y \models (CHAR\ c) \Rightarrow$
$X\}$
    $else$
      $\{Trn\ Y\ (CHAR\ c) |\ Y\ c.\ Y \in CS \wedge Y \models (CHAR\ c) \Rightarrow X\}$

In the definition of *init-rhs*, the term $\{Trn\ Y\ (CHAR\ c) |\ Y\ c.\ Y \in CS \wedge Y\ ;;\ \{[c]\} \subseteq X\}$ appearing on both branches describes the formation of strings in X out of transitions, while the term $\{Lam(EMPTY)\}$ describes the empty string which is intrinsically contained in X rather than by transition. This $\{Lam(EMPTY)\}$ corresponds to the λ in (1).

With the help of *init-rhs*, the equitional system descrbing the formation of every equivalent class inside CS is given by the following $eqs(CS)$.

definition $eqs\ CS \equiv \{(X,\ init\text{-}rhs\ CS\ X)\ |\ X.\ \ X \in CS\}$

The following $trns\text{-}of\ rhs\ X$ returns all X-items in *rhs*.

definition
  $trns\text{-}of\ rhs\ X \equiv \{Trn\ X\ r\ |\ r.\ Trn\ X\ r \in rhs\}$

The following $rexp\text{-}of\ rhs\ X$ combines all regular expressions in X-items using *ALT* to form a single regular expression. It will be used later to implement *arden-variate* and *rhs-subst*.

definition
  $\textit{rexp-of rhs } X \equiv \biguplus \{r.\ \textit{Trn } X\ r \in \textit{rhs}\}$

The following *lam-of rhs* returns all pure regular expression trns in *rhs*.

definition
  $\textit{lam-of rhs} \equiv \{\textit{Lam } r \mid r.\ \textit{Lam } r \in \textit{rhs}\}$

The following *rexp-of-lam rhs* combines pure regular expression items in *rhs* using *ALT* to form a single regular expression. When all variables inside *rhs* are eliminated, *rexp-of-lam rhs* is used to compute compute the regular expression corresponds to *rhs*.

definition
  $\textit{rexp-of-lam rhs} \equiv \biguplus \{r.\ \textit{Lam } r \in \textit{rhs}\}$

The following *attach-rexp rexp′ itm* attach the regular expression *rexp′* to the right of right hand side item *itm*.

fun
  $\textit{attach-rexp} :: \textit{rexp} \Rightarrow \textit{rhs-item} \Rightarrow \textit{rhs-item}$
where
  $\textit{attach-rexp rexp}'\ (\textit{Lam rexp}) \quad = \textit{Lam}\ (\textit{SEQ rexp rexp}')$
$\mid \textit{attach-rexp rexp}'\ (\textit{Trn } X\ \textit{rexp}) = \textit{Trn } X\ (\textit{SEQ rexp rexp}')$

The following *append-rhs-rexp rhs rexp* attaches *rexp* to every item in *rhs*.

definition
  $\textit{append-rhs-rexp rhs rexp} \equiv (\textit{attach-rexp rexp})\ `\ \textit{rhs}$

With the help of the two functions immediately above, Ardens' transformation on right hand side *rhs* is implemented by the following function *arden-variate X rhs*. After this transformation, the recursive occurence of *X* in *rhs* will be eliminated, while the string set defined by *rhs* is kept unchanged.

definition
  $\textit{arden-variate } X\ \textit{rhs} \equiv$
    $\textit{append-rhs-rexp}\ (\textit{rhs} - \textit{trns-of rhs } X)\ (\textit{STAR}\ (\biguplus \{r.\ \textit{Trn } X\ r \in \textit{rhs}\}))$

Suppose the equation defining *X* is $X = xrhs$, the purpose of *rhs-subst* is to substitute all occurences of *X* in *rhs* by *xrhs*. A litte thought may reveal that the final result should be: first append $(a_1|a_2|\ldots|a_n)$ to every item of *xrhs* and then union the result with all non-X-items of *rhs*.

definition
  $\textit{rhs-subst rhs } X\ \textit{xrhs} \equiv$
    $(\textit{rhs} - (\textit{trns-of rhs } X)) \cup (\textit{append-rhs-rexp xrhs}\ (\biguplus \{r.\ \textit{Trn } X\ r \in \textit{rhs}\}))$

Suppose the equation defining *X* is $X = xrhs$, the follwing *eqs-subst ES X xrhs* substitute *xrhs* into every equation of the equational system *ES*.

definition
  eqs-subst ES X xrhs $\equiv$ $\{(Y,\ \textit{rhs-subst yrhs X xrhs}) \mid Y\ \textit{yrhs}.\ (Y,\ \textit{yrhs}) \in ES\}$

The computation of regular expressions for equivalence classes is accomplished using a iteration principle given by the following lemma.

lemma *wf-iter* [*rule-format*]:
  fixes *f*
  assumes *step*: $\bigwedge e.\ [\![P\ e;\ \neg\ Q\ e]\!] \Longrightarrow (\exists\ e'.\ P\ e' \wedge\ (f(e'),\ f(e)) \in \textit{less-than})$
  shows *pe*: $P\ e \longrightarrow (\exists\ e'.\ P\ e' \wedge\ Q\ e')$
proof(*induct e rule*: *wf-induct*
      [*OF wf-inv-image*[*OF wf-less-than*, **where** $f = f$]], *clarify*)
  fix *x*
  assume *h* [*rule-format*]:
    $\forall y.\ (y,\ x) \in \textit{inv-image less-than}\ f \longrightarrow P\ y \longrightarrow (\exists e'.\ P\ e' \wedge Q\ e')$
    and *px*: $P\ x$
  show $\exists e'.\ P\ e' \wedge Q\ e'$
  proof(*cases Q x*)
    assume $Q\ x$ **with** *px* **show** *?thesis* **by** *blast*
  next
    assume *nq*: $\neg\ Q\ x$
    from *step* [*OF px nq*]
    obtain e' **where** *pe'*: $P\ e'$ **and** *ltf*: $(f\ e',\ f\ x) \in \textit{less-than}$ **by** *auto*
    show *?thesis*
    proof(*rule h*)
      from *ltf* **show** $(e',\ x) \in \textit{inv-image less-than}\ f$
        by (*simp add*:*inv-image-def*)
    next
      from *pe'* **show** $P\ e'$ **.**
    qed
  qed
qed

The P in lemma *wf-iter* is an invariant kept throughout the iteration procedure. The particular invariant used to solve our problem is defined by function *Inv*(*ES*), an invariant over equal system *ES*. Every definition starting next till *Inv* stipulates a property to be satisfied by *ES*.

Every variable is defined at most onece in *ES*.

definition
  distinct-equas ES $\equiv$
      $\forall\ X\ \textit{rhs}\ \textit{rhs}'.\ (X,\ \textit{rhs}) \in ES \wedge (X,\ \textit{rhs}') \in ES \longrightarrow \textit{rhs} = \textit{rhs}'$

Every equation in *ES* (represented by $(X,\ \textit{rhs})$) is valid, i.e. $(X = L\ \textit{rhs})$.

definition
  valid-eqns ES $\equiv \forall\ X\ \textit{rhs}.\ (X,\ \textit{rhs}) \in ES \longrightarrow (X = L\ \textit{rhs})$

The following *rhs-nonempty rhs* requires regular expressions occuring in transitional items of *rhs* does not contain empty string. This is necessary for the application of Arden's transformation to *rhs*.

definition
rhs-nonempty rhs ≡ (∀ *Y r*. *Trn Y r* ∈ *rhs* ⟶ [] ∉ *L r*)

The following *ardenable ES* requires that Arden's transformation is applicable to every equation of equational system *ES*.

definition
ardenable ES ≡ ∀ *X rhs*. (*X*, *rhs*) ∈ *ES* ⟶ *rhs-nonempty rhs*

definition
non-empty ES ≡ ∀ *X rhs*. (*X*, *rhs*) ∈ *ES* ⟶ *X* ≠ {}

The following *finite-rhs ES* requires every equation in *rhs* be finite.

definition
finite-rhs ES ≡ ∀ *X rhs*. (*X*, *rhs*) ∈ *ES* ⟶ *finite rhs*

The following *classes-of rhs* returns all variables (or equivalent classes) occuring in *rhs*.

definition
classes-of rhs ≡ {*X*. ∃ *r*. *Trn X r* ∈ *rhs*}

The following *lefts-of ES* returns all variables defined by equational system *ES*.

definition
lefts-of ES ≡ {*Y* | *Y yrhs*. (*Y*, *yrhs*) ∈ *ES*}

The following *self-contained ES* requires that every variable occuring on the right hand side of equations is already defined by some equation in *ES*.

definition
self-contained ES ≡ ∀ (*X*, *xrhs*) ∈ *ES*. *classes-of xrhs* ⊆ *lefts-of ES*

The invariant *Inv*(*ES*) is a conjunction of all the previously defined constaints.

definition
Inv ES ≡ *valid-eqns ES* ∧ *finite ES* ∧ *distinct-equas ES* ∧ *ardenable ES* ∧
non-empty ES ∧ *finite-rhs ES* ∧ *self-contained ES*

5.1 The proof of this direction

5.1.1 Basic properties

The following are some basic properties of the above definitions.

lemma *L-rhs-union-distrib*:
fixes *A B*::*rhs-item set*
shows *L A* ∪ *L B* = *L* (*A* ∪ *B*)
by *simp*

lemma *finite-Trn*:
  assumes *fin*: *finite rhs*
  shows *finite* $\{r.\ Trn\ Y\ r \in rhs\}$
proof −
  have *finite* $\{Trn\ Y\ r \mid Y\ r.\ Trn\ Y\ r \in rhs\}$
    by (*rule rev-finite-subset*[*OF fin*]) (*auto*)
  then have *finite* (*the-trn-rexp* ‘ $\{Trn\ Y\ r \mid Y\ r.\ Trn\ Y\ r \in rhs\}$)
    by *auto*
  then show *finite* $\{r.\ Trn\ Y\ r \in rhs\}$
    apply(*erule-tac rev-finite-subset*)
    apply(*auto simp add*: *image-def*)
    apply(*rule-tac x=Trn Y x* **in** *exI*)
    apply(*auto*)
    done
qed

lemma *finite-Lam*:
  assumes *fin*:*finite rhs*
  shows *finite* $\{r.\ Lam\ r \in rhs\}$
proof −
  have *finite* $\{Lam\ r \mid r.\ Lam\ r \in rhs\}$
    by (*rule rev-finite-subset*[*OF fin*]) (*auto*)
  then have *finite* (*the-r* ‘ $\{Lam\ r \mid r.\ Lam\ r \in rhs\}$)
    by *auto*
  then show *finite* $\{r.\ Lam\ r \in rhs\}$
    apply(*erule-tac rev-finite-subset*)
    apply(*auto simp add*: *image-def*)
    done
qed

lemma *rexp-of-empty*:
  assumes *finite*:*finite rhs*
  and *nonempty*:*rhs-nonempty rhs*
  shows $[] \notin L\ (\biguplus\ \{r.\ Trn\ X\ r \in rhs\})$
using *finite nonempty rhs-nonempty-def*
using *finite-Trn*[*OF finite*]
by (*auto*)

lemma [*intro*!]:
  $P\ (Trn\ X\ r) \Longrightarrow (\exists\, a.\ (\exists\, r.\ a = Trn\ X\ r \wedge P\ a))$ **by** *auto*

lemma *lang-of-rexp-of*:
  assumes *finite*:*finite rhs*
  shows $L\ (\{Trn\ X\ r \mid r.\ Trn\ X\ r \in rhs\}) = X\ ;;\ (L\ (\biguplus\{r.\ Trn\ X\ r \in rhs\}))$
proof −
  have *finite* $\{r.\ Trn\ X\ r \in rhs\}$
    by (*rule finite-Trn*[*OF finite*])
  then show *?thesis*
    apply(*auto simp add*: *Seq-def*)

apply(*rule-tac* $x = s_1$ **in** *exI*, *rule-tac* $x = s_2$ **in** *exI*, *auto*)
apply(*rule-tac x*= *Trn X xa* **in** *exI*)
apply(*auto simp*: *Seq-def*)
done
qed

lemma *rexp-of-lam-eq-lam-set*:
assumes *fin*: *finite rhs*
shows L ($\biguplus$ {*r*. *Lam r* ∈ *rhs*}) = L ({*Lam r* | *r*. *Lam r* ∈ *rhs*})
proof −
have *finite* ({*r*. *Lam r* ∈ *rhs*}) **using** *fin* **by** (*rule finite-Lam*)
then show *?thesis* **by** *auto*
qed

lemma [*simp*]:
L (*attach-rexp r xb*) = L *xb* ;; L *r*
apply (*cases xb*, *auto simp*: *Seq-def*)
apply(*rule-tac* $x = s_1$ @ $s_1{}'$ **in** *exI*, *rule-tac* $x = s_2{}'$ **in** *exI*)
apply(*auto simp*: *Seq-def*)
done

lemma *lang-of-append-rhs*:
L (*append-rhs-rexp rhs r*) = L *rhs* ;; L *r*
apply (*auto simp*:*append-rhs-rexp-def image-def*)
apply (*auto simp*:*Seq-def*)
apply (*rule-tac x* = L *xb* ;; L *r* **in** *exI*, *auto simp add*:*Seq-def*)
by (*rule-tac x* = *attach-rexp r xb* **in** *exI*, *auto simp*:*Seq-def*)

lemma *classes-of-union-distrib*:
classes-of A ∪ *classes-of B* = *classes-of* (*A* ∪ *B*)
by (*auto simp add*:*classes-of-def*)

lemma *lefts-of-union-distrib*:
lefts-of A ∪ *lefts-of B* = *lefts-of* (*A* ∪ *B*)
by (*auto simp*:*lefts-of-def*)

5.1.2 Intialization

The following several lemmas until *init-ES-satisfy-Inv* shows that the initial equational system satisfies invariant *Inv*.

lemma *defined-by-str*:
⟦*s* ∈ *X*; *X* ∈ *UNIV* // (≈*Lang*)⟧ ⟹ *X* = (≈*Lang*) `` {*s*}
by (*auto simp*:*quotient-def Image-def str-eq-rel-def*)

lemma *every-eqclass-has-transition*:
assumes *has-str*: *s* @ [*c*] ∈ *X*
and *in-CS*: *X* ∈ *UNIV* // (≈*Lang*)
obtains *Y* **where** *Y* ∈ *UNIV* // (≈*Lang*) **and** *Y* ;; {[*c*]} ⊆ *X* **and** *s* ∈ *Y*
proof −

```
  def Y ≡ (≈Lang) `` {s}
  have Y ∈ UNIV // (≈Lang)
    unfolding Y-def quotient-def by auto
  moreover
  have X = (≈Lang) `` {s @ [c]}
    using has-str in-CS defined-by-str by blast
  then have Y ;; {[c]} ⊆ X
    unfolding Y-def Image-def Seq-def
    unfolding str-eq-rel-def
    by clarsimp
  moreover
  have s ∈ Y unfolding Y-def
    unfolding Image-def str-eq-rel-def by simp
  ultimately show thesis by (blast intro: that)
qed

lemma l-eq-r-in-eqs:
  assumes X-in-eqs: (X, xrhs) ∈ (eqs (UNIV // (≈Lang)))
  shows X = L xrhs
proof
  show X ⊆ L xrhs
  proof
    fix x
    assume (1): x ∈ X
    show x ∈ L xrhs
    proof (cases x = [])
      assume empty: x = []
      thus ?thesis using X-in-eqs (1)
        by (auto simp:eqs-def init-rhs-def)
    next
      assume not-empty: x ≠ []
      then obtain clist c where decom: x = clist @ [c]
        by (case-tac x rule:rev-cases, auto)
      have X ∈ UNIV // (≈Lang) using X-in-eqs by (auto simp:eqs-def)
      then obtain Y
        where Y ∈ UNIV // (≈Lang)
        and Y ;; {[c]} ⊆ X
        and clist ∈ Y
        using decom (1) every-eqclass-has-transition by blast
      hence
        x ∈ L {Trn Y (CHAR c)| Y c. Y ∈ UNIV // (≈Lang) ∧ Y ⊨(CHAR c)⇒
X}
        unfolding transition-def
        using (1) decom
        by (simp, rule-tac x = Trn Y (CHAR c) in exI, simp add:Seq-def)
      thus ?thesis using X-in-eqs (1)
        by (simp add: eqs-def init-rhs-def)
    qed
  qed
```

next
  show *L xrhs* ⊆ *X* **using** *X-in-eqs*
    by (*auto simp*:*eqs-def init-rhs-def transition-def*)
qed

lemma *finite-init-rhs*:
  assumes *finite*: *finite CS*
  shows *finite* (*init-rhs CS X*)
proof−
  have *finite* {*Trn Y* (*CHAR c*) |*Y c. Y* ∈ *CS* ∧ *Y* ;; {[*c*]} ⊆ *X*} (**is** *finite ?A*)
  proof −
    def *S* ≡ {(*Y*, *c*)| *Y c. Y* ∈ *CS* ∧ *Y* ;; {[*c*]} ⊆ *X*}
    def *h* ≡ λ (*Y*, *c*). *Trn Y* (*CHAR c*)
    have *finite* (*CS* × (*UNIV*::*char set*)) **using** *finite* **by** *auto*
    hence *finite S* **using** *S-def*
      by (*rule-tac B* = *CS* × *UNIV* **in** *finite-subset*, *auto*)
    moreover have *?A* = *h* ' *S* **by** (*auto simp*: *S-def h-def image-def*)
    ultimately show *?thesis*
      by *auto*
  qed
  thus *?thesis* **by** (*simp add*:*init-rhs-def transition-def*)
qed

lemma *init-ES-satisfy-Inv*:
  assumes *finite-CS*: *finite* (*UNIV* // (≈*Lang*))
  shows *Inv* (*eqs* (*UNIV* // (≈*Lang*)))
proof −
  have *finite* (*eqs* (*UNIV* // (≈*Lang*))) **using** *finite-CS*
    by (*simp add*:*eqs-def*)
  moreover have *distinct-equas* (*eqs* (*UNIV* // (≈*Lang*)))
    by (*simp add*:*distinct-equas-def eqs-def*)
  moreover have *ardenable* (*eqs* (*UNIV* // (≈*Lang*)))
   by (*auto simp add*:*ardenable-def eqs-def init-rhs-def rhs-nonempty-def del*:*L-rhs.simps*)
  moreover have *valid-eqns* (*eqs* (*UNIV* // (≈*Lang*)))
    using *l-eq-r-in-eqs* **by** (*simp add*:*valid-eqns-def*)
  moreover have *non-empty* (*eqs* (*UNIV* // (≈*Lang*)))
    by (*auto simp*:*non-empty-def eqs-def quotient-def Image-def str-eq-rel-def*)
  moreover have *finite-rhs* (*eqs* (*UNIV* // (≈*Lang*)))
    using *finite-init-rhs*[*OF finite-CS*]
    by (*auto simp*:*finite-rhs-def eqs-def*)
  moreover have *self-contained* (*eqs* (*UNIV* // (≈*Lang*)))
    by (*auto simp*:*self-contained-def eqs-def init-rhs-def classes-of-def lefts-of-def*)
  ultimately show *?thesis* **by** (*simp add*:*Inv-def*)
qed

5.1.3 Interation step

From this point until *iteration-step*, it is proved that there exists iteration steps which keep *Inv*(*ES*) while decreasing the size of *ES*.

lemma *arden-variate-keeps-eq*:
  assumes *l-eq-r*: $X = L$ *rhs*
  and *not-empty*: $[] \notin L$ ($\biguplus\{r.$ *Trn* $X\ r \in$ *rhs*$\}$)
  and *finite*: *finite rhs*
  shows $X = L$ (*arden-variate X rhs*)
proof −
  thm *rexp-of-def*
  def $A \equiv L$ ($\biguplus\{r.$ *Trn* $X\ r \in$ *rhs*$\}$)
  def $b \equiv$ *rhs* − *trns-of rhs X*
  def $B \equiv L\ b$
  have $X = B$;; $A\star$
  proof−
    have *L rhs* = *L*(*trns-of rhs X* ∪ *b*) **by** (*auto simp*: *b-def trns-of-def*)
    also have ... = *X* ;; *A* ∪ *B*
      unfolding *trns-of-def*
      unfolding *L-rhs-union-distrib*[*symmetric*]
      by (*simp only*: *lang-of-rexp-of finite B-def A-def*)
    finally show *?thesis*
      using *l-eq-r not-empty*
      apply(*rule-tac ardens-revised*[*THEN iffD1*])
      apply(*simp add*: *A-def*)
      apply(*simp*)
      done
  qed
  moreover have *L* (*arden-variate X rhs*) = (*B* ;; *A*⋆)
    by (*simp only*:*arden-variate-def L-rhs-union-distrib lang-of-append-rhs*
          B-def A-def b-def L-rexp.simps seq-union-distrib-left)
  ultimately show *?thesis* **by** *simp*
qed

lemma *append-keeps-finite*:
  finite rhs ⟹ *finite* (*append-rhs-rexp rhs r*)
by (*auto simp*:*append-rhs-rexp-def*)

lemma *arden-variate-keeps-finite*:
  finite rhs ⟹ *finite* (*arden-variate X rhs*)
by (*auto simp*:*arden-variate-def append-keeps-finite*)

lemma *append-keeps-nonempty*:
  rhs-nonempty rhs ⟹ *rhs-nonempty* (*append-rhs-rexp rhs r*)
apply (*auto simp*:*rhs-nonempty-def append-rhs-rexp-def*)
by (*case-tac x*, *auto simp*:*Seq-def*)

lemma *nonempty-set-sub*:
  rhs-nonempty rhs ⟹ *rhs-nonempty* (*rhs* − *A*)
by (*auto simp*:*rhs-nonempty-def*)

lemma *nonempty-set-union*:
  ⟦*rhs-nonempty rhs*; *rhs-nonempty rhs′*⟧ ⟹ *rhs-nonempty* (*rhs* ∪ *rhs′*)

by (*auto simp*:*rhs-nonempty-def*)

lemma *arden-variate-keeps-nonempty*:
rhs-nonempty rhs $\Longrightarrow$ *rhs-nonempty* (*arden-variate X rhs*)
by (*simp only*:*arden-variate-def append-keeps-nonempty nonempty-set-sub*)

lemma *rhs-subst-keeps-nonempty*:
$[\![$*rhs-nonempty rhs*; *rhs-nonempty xrhs*$]\!]$ $\Longrightarrow$ *rhs-nonempty* (*rhs-subst rhs X xrhs*)
by (*simp only*:*rhs-subst-def append-keeps-nonempty nonempty-set-union nonempty-set-sub*)

lemma *rhs-subst-keeps-eq*:
assumes *substor*: $X = L$ *xrhs*
and *finite*: *finite rhs*
shows L (*rhs-subst rhs X xrhs*) $= L$ *rhs* (**is** *?Left* = *?Right*)
proof−
def $A \equiv L$ (*rhs* − *trns-of rhs X*)
have *?Left* $= A \cup L$ (*append-rhs-rexp xrhs* ($\biguplus$ {*r*. *Trn X r* ∈ *rhs*}))
unfolding *rhs-subst-def*
unfolding *L-rhs-union-distrib*[*symmetric*]
by (*simp add*: *A-def*)
moreover have *?Right* $= A \cup L$ ({*Trn X r* | *r*. *Trn X r* ∈ *rhs*})
proof−
have *rhs* = (*rhs* − *trns-of rhs X*) ∪ (*trns-of rhs X*) **by** (*auto simp add*:
trns-of-def)
thus *?thesis*
unfolding *A-def*
unfolding *L-rhs-union-distrib*
unfolding *trns-of-def*
by *simp*
qed
moreover have L (*append-rhs-rexp xrhs* ($\biguplus$ {*r*. *Trn X r* ∈ *rhs*})) $= L$ ({*Trn X*
r | *r*. *Trn X r* ∈ *rhs*})
using *finite substor* **by** (*simp only*:*lang-of-append-rhs lang-of-rexp-of*)
ultimately show *?thesis* **by** *simp*
qed

lemma *rhs-subst-keeps-finite-rhs*:
$[\![$*finite rhs*; *finite yrhs*$]\!]$ $\Longrightarrow$ *finite* (*rhs-subst rhs Y yrhs*)
by (*auto simp*:*rhs-subst-def append-keeps-finite*)

lemma *eqs-subst-keeps-finite*:
assumes *finite*:*finite* (*ES*:: (*string set* × *rhs-item set*) *set*)
shows *finite* (*eqs-subst ES Y yrhs*)
proof −
have *finite* {(*Ya*, *rhs-subst yrhsa Y yrhs*) |*Ya yrhsa*. (*Ya*, *yrhsa*) ∈ *ES*}
(**is** *finite ?A*)
proof−
def *eqns'* ≡ {((*Ya*::*string set*), *yrhsa*)| *Ya yrhsa*. (*Ya*, *yrhsa*) ∈ *ES*}

```
    def h ≡ λ ((Ya::string set), yrhsa). (Ya, rhs-subst yrhsa Y yrhs)
    have finite (h ‘ eqns′) using finite h-def eqns′-def by auto
    moreover have ?A = h ‘ eqns′ by (auto simp:h-def eqns′-def)
    ultimately show ?thesis by auto
  qed
  thus ?thesis by (simp add:eqs-subst-def)
qed

lemma eqs-subst-keeps-finite-rhs:
  ⟦finite-rhs ES; finite yrhs⟧ ⟹ finite-rhs (eqs-subst ES Y yrhs)
by (auto intro:rhs-subst-keeps-finite-rhs simp add:eqs-subst-def finite-rhs-def)

lemma append-rhs-keeps-cls:
  classes-of (append-rhs-rexp rhs r) = classes-of rhs
apply (auto simp:classes-of-def append-rhs-rexp-def)
apply (case-tac xa, auto simp:image-def)
by (rule-tac x = SEQ ra r in exI, rule-tac x = Trn x ra in bexI, simp+)

lemma arden-variate-removes-cl:
  classes-of (arden-variate Y yrhs) = classes-of yrhs − {Y}
apply (simp add:arden-variate-def append-rhs-keeps-cls trns-of-def)
by (auto simp:classes-of-def)

lemma lefts-of-keeps-cls:
  lefts-of (eqs-subst ES Y yrhs) = lefts-of ES
by (auto simp:lefts-of-def eqs-subst-def)

lemma rhs-subst-updates-cls:
  X ∉ classes-of xrhs ⟹
      classes-of (rhs-subst rhs X xrhs) = classes-of rhs ∪ classes-of xrhs − {X}
apply (simp only:rhs-subst-def append-rhs-keeps-cls
                              classes-of-union-distrib[THEN sym])
by (auto simp:classes-of-def trns-of-def)

lemma eqs-subst-keeps-self-contained:
  fixes Y
  assumes sc: self-contained (ES ∪ {(Y, yrhs)}) (is self-contained ?A)
  shows self-contained (eqs-subst ES Y (arden-variate Y yrhs))
                                                   (is self-contained ?B)
proof−
  { fix X xrhs′
    assume (X, xrhs′) ∈ ?B
    then obtain xrhs
      where xrhs-xrhs′: xrhs′ = rhs-subst xrhs Y (arden-variate Y yrhs)
      and X-in: (X, xrhs) ∈ ES by (simp add:eqs-subst-def, blast)
    have classes-of xrhs′ ⊆ lefts-of ?B
    proof−
      have lefts-of ?B = lefts-of ES by (auto simp add:lefts-of-def eqs-subst-def)
      moreover have classes-of xrhs′ ⊆ lefts-of ES
```

```
      proof−
        have classes-of xrhs' ⊆
                      classes-of xrhs ∪ classes-of (arden-variate Y yrhs) − {Y}
        proof−
          have Y ∉ classes-of (arden-variate Y yrhs)
            using arden-variate-removes-cl by simp
          thus ?thesis using xrhs-xrhs' by (auto simp:rhs-subst-updates-cls)
        qed
        moreover have classes-of xrhs ⊆ lefts-of ES ∪ {Y} using X-in sc
          apply (simp only:self-contained-def lefts-of-union-distrib[THEN sym])
          by (drule-tac x = (X, xrhs) in bspec, auto simp:lefts-of-def)
        moreover have classes-of (arden-variate Y yrhs) ⊆ lefts-of ES ∪ {Y}
          using sc
         by (auto simp add:arden-variate-removes-cl self-contained-def lefts-of-def)
        ultimately show ?thesis by auto
      qed
      ultimately show ?thesis by simp
    qed
  } thus ?thesis by (auto simp only:eqs-subst-def self-contained-def)
qed

lemma eqs-subst-satisfy-Inv:
  assumes Inv-ES: Inv (ES ∪ {(Y, yrhs)})
  shows Inv (eqs-subst ES Y (arden-variate Y yrhs))
proof −
  have finite-yrhs: finite yrhs
    using Inv-ES by (auto simp:Inv-def finite-rhs-def)
  have nonempty-yrhs: rhs-nonempty yrhs
    using Inv-ES by (auto simp:Inv-def ardenable-def)
  have Y-eq-yrhs: Y = L yrhs
    using Inv-ES by (simp only:Inv-def valid-eqns-def, blast)
  have distinct-equas (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES
    by (auto simp:distinct-equas-def eqs-subst-def Inv-def)
  moreover have finite (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES by (simp add:Inv-def eqs-subst-keeps-finite)
  moreover have finite-rhs (eqs-subst ES Y (arden-variate Y yrhs))
  proof−
    have finite-rhs ES using Inv-ES
      by (simp add:Inv-def finite-rhs-def)
    moreover have finite (arden-variate Y yrhs)
    proof −
      have finite yrhs using Inv-ES
        by (auto simp:Inv-def finite-rhs-def)
      thus ?thesis using arden-variate-keeps-finite by simp
    qed
    ultimately show ?thesis
      by (simp add:eqs-subst-keeps-finite-rhs)
  qed
```

```
  moreover have ardenable (eqs-subst ES Y (arden-variate Y yrhs))
  proof -
    { fix X rhs
      assume (X, rhs) ∈ ES
      hence rhs-nonempty rhs  using prems Inv-ES
        by (simp add:Inv-def ardenable-def)
      with nonempty-yrhs
      have rhs-nonempty (rhs-subst rhs Y (arden-variate Y yrhs))
        by (simp add:nonempty-yrhs
               rhs-subst-keeps-nonempty arden-variate-keeps-nonempty)
    } thus ?thesis by (auto simp add:ardenable-def eqs-subst-def)
  qed
  moreover have valid-eqns (eqs-subst ES Y (arden-variate Y yrhs))
  proof-
    have Y = L (arden-variate Y yrhs)
      using Y-eq-yrhs Inv-ES finite-yrhs nonempty-yrhs
      by (rule-tac arden-variate-keeps-eq, (simp add:rexp-of-empty)+)
    thus ?thesis using Inv-ES
      by (clarsimp simp add:valid-eqns-def
              eqs-subst-def rhs-subst-keeps-eq Inv-def finite-rhs-def
                  simp del:L-rhs.simps)
  qed
  moreover have
    non-empty-subst: non-empty (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES by (auto simp:Inv-def non-empty-def eqs-subst-def)
  moreover
  have self-subst: self-contained (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES eqs-subst-keeps-self-contained by (simp add:Inv-def)
  ultimately show ?thesis using Inv-ES by (simp add:Inv-def)
qed

lemma eqs-subst-card-le:
  assumes finite: finite (ES::(string set × rhs-item set) set)
  shows card (eqs-subst ES Y yrhs) <= card ES
proof-
  def f ≡ λ x. ((fst x)::string set, rhs-subst (snd x) Y yrhs)
  have eqs-subst ES Y yrhs = f ` ES
    apply (auto simp:eqs-subst-def f-def image-def)
    by (rule-tac x = (Ya, yrhsa) in bexI, simp+)
  thus ?thesis using finite by (auto intro:card-image-le)
qed

lemma eqs-subst-cls-remains:
  (X, xrhs) ∈ ES ⟹ ∃ xrhs'. (X, xrhs') ∈ (eqs-subst ES Y yrhs)
by (auto simp:eqs-subst-def)

lemma card-noteq-1-has-more:
  assumes card:card S ≠ 1
  and e-in: e ∈ S
```

```
  and finite: finite S
  obtains e′ where e′ ∈ S ∧ e ≠ e′
proof−
  have card (S − {e}) > 0
  proof −
    have card S > 1 using card e-in finite
      by (case-tac card S, auto)
    thus ?thesis using finite e-in by auto
  qed
  hence S − {e} ≠ {} using finite by (rule-tac notI, simp)
  thus (⋀e′. e′ ∈ S ∧ e ≠ e′ ⟹ thesis) ⟹ thesis by auto
qed

lemma iteration-step:
  assumes Inv-ES: Inv ES
  and     X-in-ES: (X, xrhs) ∈ ES
  and     not-T: card ES ≠ 1
  shows ∃ ES′. (Inv ES′ ∧ (∃ xrhs′.(X, xrhs′) ∈ ES′)) ∧
              (card ES′, card ES) ∈ less-than (is ∃ ES′. ?P ES′)
proof −
  have finite-ES: finite ES using Inv-ES by (simp add:Inv-def)
  then obtain Y yrhs
    where Y-in-ES: (Y, yrhs) ∈ ES and not-eq: (X, xrhs) ≠ (Y, yrhs)
    using not-T X-in-ES by (drule-tac card-noteq-1-has-more, auto)
  def ES′ == ES − {(Y, yrhs)}
  let ?ES′′ = eqs-subst ES′ Y (arden-variate Y yrhs)
  have ?P ?ES′′
  proof −
    have Inv ?ES′′ using Y-in-ES Inv-ES
      by (rule-tac eqs-subst-satisfy-Inv, simp add:ES′-def insert-absorb)
    moreover have ∃xrhs′. (X, xrhs′) ∈ ?ES′′  using not-eq X-in-ES
      by (rule-tac ES = ES′ in eqs-subst-cls-remains, auto simp add:ES′-def)
    moreover have (card ?ES′′, card ES) ∈ less-than
    proof −
      have finite ES′ using finite-ES ES′-def by auto
      moreover have card ES′ < card ES using finite-ES Y-in-ES
        by (auto simp:ES′-def card-gt-0-iff intro:diff-Suc-less)
      ultimately show ?thesis
        by (auto dest:eqs-subst-card-le elim:le-less-trans)
    qed
    ultimately show ?thesis by simp
  qed
  thus ?thesis by blast
qed
```

5.1.4 Conclusion of the proof

From this point until *hard-direction*, the hard direction is proved through a simple application of the iteration principle.

```
lemma iteration-conc:
  assumes history: Inv ES
  and     X-in-ES: ∃ xrhs. (X, xrhs) ∈ ES
  shows
  ∃ ES'. (Inv ES' ∧ (∃ xrhs'. (X, xrhs') ∈ ES')) ∧ card ES' = 1
                                                   (is ∃ ES'. ?P ES')
proof (cases card ES = 1)
  case True
  thus ?thesis using history X-in-ES
    by blast
next
  case False
  thus ?thesis using history iteration-step X-in-ES
    by (rule-tac f = card in wf-iter, auto)
qed

lemma last-cl-exists-rexp:
  assumes ES-single: ES = {(X, xrhs)}
  and Inv-ES: Inv ES
  shows ∃ (r::rexp). L r = X (is ∃ r. ?P r)
proof−
  def A ≡ arden-variate X xrhs
  have ?P (rexp-of-lam A)
  proof −
    thm lam-of-def
    thm rexp-of-lam-def
    have L (⨄{r. Lam r ∈ A}) = L ({Lam r | r. Lam r ∈  A})
    proof(rule rexp-of-lam-eq-lam-set)
      show finite A
        unfolding A-def
        using Inv-ES ES-single
        by (rule-tac arden-variate-keeps-finite)
          (auto simp add: Inv-def finite-rhs-def)
    qed
    also have ... = L A
    proof−
      have lam-of A = A
      proof−
        have classes-of A = {} using Inv-ES ES-single
          unfolding A-def
          by (simp add:arden-variate-removes-cl
                  self-contained-def Inv-def lefts-of-def)
        thus ?thesis
          unfolding A-def
          by (auto simp only:lam-of-def classes-of-def, case-tac x, auto)
      qed
      thus ?thesis unfolding lam-of-def by simp
    qed
    also have ... = X
```

```
    unfolding A-def
    proof(rule arden-variate-keeps-eq [THEN sym])
      show X = L xrhs using Inv-ES ES-single
        by (auto simp only:Inv-def valid-eqns-def)
    next
      from Inv-ES ES-single show [] ∉ L (⨄{r. Trn X r ∈ xrhs})
        by(simp add:Inv-def ardenable-def rexp-of-empty finite-rhs-def)
    next
      from Inv-ES ES-single show finite xrhs
        by (simp add:Inv-def finite-rhs-def)
    qed
    finally show ?thesis unfolding rexp-of-lam-def by simp
  qed
  thus ?thesis by auto
qed

lemma every-eqcl-has-reg:
  assumes finite-CS: finite (UNIV // (≈Lang))
  and X-in-CS: X ∈ (UNIV // (≈Lang))
  shows ∃ (reg::rexp). L reg = X (is ∃ r. ?E r)
proof −
  from X-in-CS have ∃ xrhs. (X, xrhs) ∈ (eqs (UNIV // (≈Lang)))
    by (auto simp:eqs-def init-rhs-def)
  then obtain ES xrhs where Inv-ES: Inv ES
    and X-in-ES: (X, xrhs) ∈ ES
    and card-ES: card ES = 1
    using finite-CS X-in-CS init-ES-satisfy-Inv iteration-conc
    by blast
  hence ES-single-equa: ES = {(X, xrhs)}
    by (auto simp:Inv-def dest!:card-Suc-Diff1 simp:card-eq-0-iff)
  thus ?thesis using Inv-ES
    by (rule last-cl-exists-rexp)
qed

theorem hard-direction:
  assumes finite-CS: finite (UNIV // ≈A)
  shows   ∃ r::rexp. A = L r
proof −
  have ∀ X ∈ (UNIV // ≈A). ∃ reg::rexp. X = L reg
    using finite-CS every-eqcl-has-reg by blast
  then obtain f
    where f-prop: ∀ X ∈ (UNIV // ≈A). X = L ((f X)::rexp)
    by (auto dest: bchoice)
  def rs ≡ f ` (finals A)
  have A = ⋃ (finals A) using lang-is-union-of-finals by auto
  also have ... = L (⨄rs)
  proof −
    have finite rs
    proof −
```

have *finite* (*finals A*)
using *finite-CS finals-in-partitions*[*of A*]
by (*erule-tac finite-subset*, *simp*)
thus *?thesis* **using** *rs-def* **by** *auto*
qed
thus *?thesis*
using *f-prop rs-def finals-in-partitions*[*of A*] **by** *auto*
qed
finally show *?thesis* **by** *blast*
qed

end

6 List prefixes and postfixes

theory *List-Prefix*
imports *List Main*
begin

6.1 Prefix order on lists

instantiation *list* :: (*type*) {*order*, *bot*}
begin

definition
prefix-def: $xs \leq ys \longleftrightarrow (\exists zs.\ ys = xs\ @\ zs)$

definition
strict-prefix-def: $xs < ys \longleftrightarrow xs \leq ys \wedge xs \neq (ys{::}'a\ list)$

definition
$bot = []$

instance proof
qed (*auto simp add*: *prefix-def strict-prefix-def bot-list-def*)

end

lemma *prefixI* [*intro?*]: $ys = xs\ @\ zs \Longrightarrow xs \leq ys$
unfolding *prefix-def* **by** *blast*

lemma *prefixE* [*elim?*]:
assumes $xs \leq ys$
obtains *zs* **where** $ys = xs\ @\ zs$
using *assms* **unfolding** *prefix-def* **by** *blast*

lemma *strict-prefixI'* [*intro?*]: $ys = xs\ @\ z\ \#\ zs \Longrightarrow xs < ys$
unfolding *strict-prefix-def prefix-def* **by** *blast*

lemma *strict-prefixE'* [*elim?*]:
  assumes *xs* < *ys*
  obtains *z zs* **where** *ys* = *xs* @ *z* # *zs*
proof −
  from ‹*xs* < *ys*› **obtain** *us* **where** *ys* = *xs* @ *us* **and** *xs* ≠ *ys*
    unfolding *strict-prefix-def prefix-def* **by** *blast*
  with *that* **show** *?thesis* **by** (*auto simp add*: *neq-Nil-conv*)
qed

lemma *strict-prefixI* [*intro?*]: *xs* ≤ *ys* ==> *xs* ≠ *ys* ==> *xs* < (*ys*::'*a list*)
  unfolding *strict-prefix-def* **by** *blast*

lemma *strict-prefixE* [*elim?*]:
  fixes *xs ys* :: '*a list*
  assumes *xs* < *ys*
  obtains *xs* ≤ *ys* **and** *xs* ≠ *ys*
  using *assms* **unfolding** *strict-prefix-def* **by** *blast*

6.2 Basic properties of prefixes

theorem *Nil-prefix* [*iff*]: [] ≤ *xs*
  by (*simp add*: *prefix-def*)

theorem *prefix-Nil* [*simp*]: (*xs* ≤ []) = (*xs* = [])
  by (*induct xs*) (*simp-all add*: *prefix-def*)

lemma *prefix-snoc* [*simp*]: (*xs* ≤ *ys* @ [*y*]) = (*xs* = *ys* @ [*y*] ∨ *xs* ≤ *ys*)
proof
  assume *xs* ≤ *ys* @ [*y*]
  then obtain *zs* **where** *zs*: *ys* @ [*y*] = *xs* @ *zs* **..**
  show *xs* = *ys* @ [*y*] ∨ *xs* ≤ *ys*
    by (*metis append-Nil2 butlast-append butlast-snoc prefixI zs*)
next
  assume *xs* = *ys* @ [*y*] ∨ *xs* ≤ *ys*
  then show *xs* ≤ *ys* @ [*y*]
    by (*metis order-eq-iff strict-prefixE strict-prefixI' xt1*(*7*))
qed

lemma *Cons-prefix-Cons* [*simp*]: (*x* # *xs* ≤ *y* # *ys*) = (*x* = *y* ∧ *xs* ≤ *ys*)
  by (*auto simp add*: *prefix-def*)

lemma *less-eq-list-code* [*code*]:
  ([]::'*a*::{*equal*, *ord*} *list*) ≤ *xs* ⟷ *True*
  (*x*::'*a*::{*equal*, *ord*}) # *xs* ≤ [] ⟷ *False*
  (*x*::'*a*::{*equal*, *ord*}) # *xs* ≤ *y* # *ys* ⟷ *x* = *y* ∧ *xs* ≤ *ys*
  by *simp-all*

lemma *same-prefix-prefix* [*simp*]: (*xs* @ *ys* ≤ *xs* @ *zs*) = (*ys* ≤ *zs*)
  by (*induct xs*) *simp-all*

lemma *same-prefix-nil* [*iff*]: $(xs\ @\ ys \le xs) = (ys = [])$
by (*metis append-Nil2 append-self-conv order-eq-iff prefixI*)

lemma *prefix-prefix* [*simp*]: $xs \le ys ==> xs \le ys\ @\ zs$
by (*metis order-le-less-trans prefixI strict-prefixE strict-prefixI*)

lemma *append-prefixD*: $xs\ @\ ys \le zs \Longrightarrow xs \le zs$
by (*auto simp add*: *prefix-def*)

theorem *prefix-Cons*: $(xs \le y\ \#\ ys) = (xs = [] \lor (\exists zs.\ xs = y\ \#\ zs \land zs \le ys))$
by (*cases xs*) (*auto simp add*: *prefix-def*)

theorem *prefix-append*:
$(xs \le ys\ @\ zs) = (xs \le ys \lor (\exists us.\ xs = ys\ @\ us \land us \le zs))$
apply (*induct zs rule*: *rev-induct*)
 apply *force*
apply (*simp del*: *append-assoc add*: *append-assoc* [*symmetric*])
apply (*metis append-eq-appendI*)
done

lemma *append-one-prefix*:
$xs \le ys ==> length\ xs < length\ ys ==> xs\ @\ [ys\ !\ length\ xs] \le ys$
unfolding *prefix-def*
by (*metis Cons-eq-appendI append-eq-appendI append-eq-conv-conj eq-Nil-appendI nth-drop'*)

theorem *prefix-length-le*: $xs \le ys ==> length\ xs \le length\ ys$
by (*auto simp add*: *prefix-def*)

lemma *prefix-same-cases*:
$(xs_1::'a\ list) \le ys \Longrightarrow xs_2 \le ys \Longrightarrow xs_1 \le xs_2 \lor xs_2 \le xs_1$
unfolding *prefix-def* **by** (*metis append-eq-append-conv2*)

lemma *set-mono-prefix*: $xs \le ys \Longrightarrow set\ xs \subseteq set\ ys$
by (*auto simp add*: *prefix-def*)

lemma *take-is-prefix*: $take\ n\ xs \le xs$
unfolding *prefix-def* **by** (*metis append-take-drop-id*)

lemma *map-prefixI*: $xs \le ys \Longrightarrow map\ f\ xs \le map\ f\ ys$
by (*auto simp*: *prefix-def*)

lemma *prefix-length-less*: $xs < ys \Longrightarrow length\ xs < length\ ys$
by (*auto simp*: *strict-prefix-def prefix-def*)

lemma *strict-prefix-simps* [*simp*, *code*]:
$xs < [] \longleftrightarrow False$
$[] < x\ \#\ xs \longleftrightarrow True$

```
  x # xs < y # ys ⟷ x = y ∧ xs < ys
  by (simp-all add: strict-prefix-def cong: conj-cong)

lemma take-strict-prefix: xs < ys ⟹ take n xs < ys
  apply (induct n arbitrary: xs ys)
   apply (case-tac ys, simp-all)[1]
  apply (metis order-less-trans strict-prefixI take-is-prefix)
  done

lemma not-prefix-cases:
  assumes pfx: ¬ ps ≤ ls
  obtains
    (c1) ps ≠ [] and ls = []
  | (c2) a as x xs where ps = a#as and ls = x#xs and x = a and ¬ as ≤ xs
  | (c3) a as x xs where ps = a#as and ls = x#xs and x ≠ a
proof (cases ps)
  case Nil then show ?thesis using pfx by simp
next
  case (Cons a as)
  note c = ‹ps = a#as›
  show ?thesis
  proof (cases ls)
    case Nil then show ?thesis by (metis append-Nil2 pfx c1 same-prefix-nil)
  next
    case (Cons x xs)
    show ?thesis
    proof (cases x = a)
      case True
      have ¬ as ≤ xs using pfx c Cons True by simp
      with c Cons True show ?thesis by (rule c2)
    next
      case False
      with c Cons show ?thesis by (rule c3)
    qed
  qed
qed

lemma not-prefix-induct [consumes 1, case-names Nil Neq Eq]:
  assumes np: ¬ ps ≤ ls
    and base: ⋀x xs. P (x#xs) []
    and r1: ⋀x xs y ys. x ≠ y ⟹ P (x#xs) (y#ys)
    and r2: ⋀x xs y ys. ⟦ x = y; ¬ xs ≤ ys; P xs ys ⟧ ⟹ P (x#xs) (y#ys)
  shows P ps ls using np
proof (induct ls arbitrary: ps)
  case Nil then show ?case
    by (auto simp: neq-Nil-conv elim!: not-prefix-cases intro!: base)
next
  case (Cons y ys)
  then have npfx: ¬ ps ≤ (y # ys) by simp
```

```
  then obtain x xs where pv: ps = x # xs
    by (rule not-prefix-cases) auto
  show ?case by (metis Cons.hyps Cons-prefix-Cons npfx pv r1 r2)
qed
```

6.3 Parallel lists

```
definition
  parallel :: 'a list => 'a list => bool  (infixl ‖ 50) where
  (xs ‖ ys) = (¬ xs ≤ ys ∧ ¬ ys ≤ xs)

lemma parallelI [intro]: ¬ xs ≤ ys ==> ¬ ys ≤ xs ==> xs ‖ ys
  unfolding parallel-def by blast

lemma parallelE [elim]:
  assumes xs ‖ ys
  obtains ¬ xs ≤ ys ∧ ¬ ys ≤ xs
  using assms unfolding parallel-def by blast

theorem prefix-cases:
  obtains xs ≤ ys | ys < xs | xs ‖ ys
  unfolding parallel-def strict-prefix-def by blast

theorem parallel-decomp:
  xs ‖ ys ==> ∃ as b bs c cs. b ≠ c ∧ xs = as @ b # bs ∧ ys = as @ c # cs
proof (induct xs rule: rev-induct)
  case Nil
  then have False by auto
  then show ?case ..
next
  case (snoc x xs)
  show ?case
  proof (rule prefix-cases)
    assume le: xs ≤ ys
    then obtain ys' where ys: ys = xs @ ys' ..
    show ?thesis
    proof (cases ys')
      assume ys' = []
      then show ?thesis by (metis append-Nil2 parallelE prefixI snoc.prems ys)
    next
      fix c cs assume ys': ys' = c # cs
      then show ?thesis
        by (metis Cons-eq-appendI eq-Nil-appendI parallelE prefixI
          same-prefix-prefix snoc.prems ys)
    qed
  next
    assume ys < xs then have ys ≤ xs @ [x] by (simp add: strict-prefix-def)
    with snoc have False by blast
    then show ?thesis ..
```

next
assume *xs* ∥ *ys*
with *snoc* **obtain** *as b bs c cs* **where** *neq*: $(b{::}'a) \neq c$
and *xs*: *xs* = *as* @ *b* # *bs* **and** *ys*: *ys* = *as* @ *c* # *cs*
by *blast*
from *xs* **have** *xs* @ [*x*] = *as* @ *b* # (*bs* @ [*x*]) **by** *simp*
with *neq ys* **show** *?thesis* **by** *blast*
qed
qed

lemma *parallel-append*: *a* ∥ *b* ⟹ *a* @ *c* ∥ *b* @ *d*
apply (*rule parallelI*)
apply (*erule parallelE*, *erule conjE*,
induct rule: *not-prefix-induct*, *simp*+)+
done

lemma *parallel-appendI*: *xs* ∥ *ys* ⟹ *x* = *xs* @ *xs′* ⟹ *y* = *ys* @ *ys′* ⟹ *x* ∥ *y*
by (*simp add*: *parallel-append*)

lemma *parallel-commute*: *a* ∥ *b* ⟷ *b* ∥ *a*
unfolding *parallel-def* **by** *auto*

6.4 Postfix order on lists

definition
postfix :: *′a list* => *′a list* => *bool* ((-/ >>= -) [*51*, *50*] *50*) **where**
(*xs* >>= *ys*) = (∃ *zs*. *xs* = *zs* @ *ys*)

lemma *postfixI* [*intro?*]: *xs* = *zs* @ *ys* ==> *xs* >>= *ys*
unfolding *postfix-def* **by** *blast*

lemma *postfixE* [*elim?*]:
assumes *xs* >>= *ys*
obtains *zs* **where** *xs* = *zs* @ *ys*
using *assms* **unfolding** *postfix-def* **by** *blast*

lemma *postfix-refl* [*iff*]: *xs* >>= *xs*
by (*auto simp add*: *postfix-def*)
lemma *postfix-trans*: ⟦*xs* >>= *ys*; *ys* >>= *zs*⟧ ⟹ *xs* >>= *zs*
by (*auto simp add*: *postfix-def*)
lemma *postfix-antisym*: ⟦*xs* >>= *ys*; *ys* >>= *xs*⟧ ⟹ *xs* = *ys*
by (*auto simp add*: *postfix-def*)

lemma *Nil-postfix* [*iff*]: *xs* >>= []
by (*simp add*: *postfix-def*)
lemma *postfix-Nil* [*simp*]: ([] >>= *xs*) = (*xs* = [])
by (*auto simp add*: *postfix-def*)

lemma *postfix-ConsI*: *xs* >>= *ys* ⟹ *x*#*xs* >>= *ys*

```
  by (auto simp add: postfix-def)
lemma postfix-ConsD: xs >>= y#ys ⟹ xs >>= ys
  by (auto simp add: postfix-def)

lemma postfix-appendI: xs >>= ys ⟹ zs @ xs >>= ys
  by (auto simp add: postfix-def)
lemma postfix-appendD: xs >>= zs @ ys ⟹ xs >>= ys
  by (auto simp add: postfix-def)

lemma postfix-is-subset: xs >>= ys ==> set ys ⊆ set xs
proof −
  assume xs >>= ys
  then obtain zs where xs = zs @ ys ..
  then show ?thesis by (induct zs) auto
qed

lemma postfix-ConsD2: x#xs >>= y#ys ==> xs >>= ys
proof −
  assume x#xs >>= y#ys
  then obtain zs where x#xs = zs @ y#ys ..
  then show ?thesis
    by (induct zs) (auto intro!: postfix-appendI postfix-ConsI)
qed

lemma postfix-to-prefix [code]: xs >>= ys ⟷ rev ys ≤ rev xs
proof
  assume xs >>= ys
  then obtain zs where xs = zs @ ys ..
  then have rev xs = rev ys @ rev zs by simp
  then show rev ys <= rev xs ..
next
  assume rev ys <= rev xs
  then obtain zs where rev xs = rev ys @ zs ..
  then have rev (rev xs) = rev zs @ rev (rev ys) by simp
  then have xs = rev zs @ ys by simp
  then show xs >>= ys ..
qed

lemma distinct-postfix: distinct xs ⟹ xs >>= ys ⟹ distinct ys
  by (clarsimp elim!: postfixE)

lemma postfix-map: xs >>= ys ⟹ map f xs >>= map f ys
  by (auto elim!: postfixE intro: postfixI)

lemma postfix-drop: as >>= drop n as
  unfolding postfix-def
  apply (rule exI [where x = take n as])
  apply simp
  done
```

```
lemma postfix-take: xs >>= ys ⟹ xs = take (length xs − length ys) xs @ ys
  by (clarsimp elim!: postfixE)

lemma parallelD1: x ∥ y ⟹ ¬ x ≤ y
  by blast

lemma parallelD2: x ∥ y ⟹ ¬ y ≤ x
  by blast

lemma parallel-Nil1 [simp]: ¬ x ∥ []
  unfolding parallel-def by simp

lemma parallel-Nil2 [simp]: ¬ [] ∥ x
  unfolding parallel-def by simp

lemma Cons-parallelI1: a ≠ b ⟹ a # as ∥ b # bs
  by auto

lemma Cons-parallelI2: ⟦ a = b; as ∥ bs ⟧ ⟹ a # as ∥ b # bs
  by (metis Cons-prefix-Cons parallelE parallelI)

lemma not-equal-is-parallel:
  assumes neq: xs ≠ ys
    and len: length xs = length ys
  shows xs ∥ ys
  using len neq
proof (induct rule: list-induct2)
  case Nil
  then show ?case by simp
next
  case (Cons a as b bs)
  have ih: as ≠ bs ⟹ as ∥ bs by fact
  show ?case
  proof (cases a = b)
    case True
    then have as ≠ bs using Cons by simp
    then show ?thesis by (rule Cons-parallelI2 [OF True ih])
  next
    case False
    then show ?thesis by (rule Cons-parallelI1)
  qed
qed

end

theory Prefix-subtract
  imports Main List-Prefix
begin
```

7 A small theory of prefix subtraction

The notion of *prefix-subtract* is need to make proofs more readable.

fun *prefix-subtract* :: *'a list* ⇒ *'a list* ⇒ *'a list* (**infix** − *51*)
where
  prefix-subtract [] *xs* = []
| *prefix-subtract* (*x*#*xs*) [] = *x*#*xs*
| *prefix-subtract* (*x*#*xs*) (*y*#*ys*) = (*if x* = *y then prefix-subtract xs ys else* (*x*#*xs*))

lemma [*simp*]: (*x* @ *y*) − *x* = *y*
apply (*induct x*)
by (*case-tac y*, *simp*+)

lemma [*simp*]: *x* − *x* = []
by (*induct x*, *auto*)

lemma [*simp*]: *x* = *xa* @ *y* ⟹ *x* − *xa* = *y*
by (*induct x*, *auto*)

lemma [*simp*]: *x* − [] = *x*
by (*induct x*, *auto*)

lemma [*simp*]: (*x* − *y* = []) ⟹ (*x* ≤ *y*)
proof−
  have ∃ *xa*. *x* = *xa* @ (*x* − *y*) ∧ *xa* ≤ *y*
    apply (*rule prefix-subtract.induct*[*of - x y*], *simp*+)
    by (*clarsimp*, *rule-tac x* = *y* # *xa* **in** *exI*, *simp*+)
  thus (*x* − *y* = []) ⟹ (*x* ≤ *y*) **by** *simp*
qed

lemma *diff-prefix*:
  ⟦*c* ≤ *a* − *b*; *b* ≤ *a*⟧ ⟹ *b* @ *c* ≤ *a*
by (*auto elim*:*prefixE*)

lemma *diff-diff-appd*:
  ⟦*c* < *a* − *b*; *b* < *a*⟧ ⟹ (*a* − *b*) − *c* = *a* − (*b* @ *c*)
apply (*clarsimp simp*:*strict-prefix-def*)
by (*drule diff-prefix*, *auto elim*:*prefixE*)

lemma *app-eq-cases*[*rule-format*]:
  ∀ *x* . *x* @ *y* = *m* @ *n* ⟶ (*x* ≤ *m* ∨ *m* ≤ *x*)
apply (*induct y*, *simp*)
apply (*clarify*, *drule-tac x* = *x* @ [*a*] **in** *spec*)
by (*clarsimp*, *auto simp*:*prefix-def*)

lemma *app-eq-dest*:
  x @ *y* = *m* @ *n* ⟹
      (*x* ≤ *m* ∧ (*m* − *x*) @ *n* = *y*) ∨ (*m* ≤ *x* ∧ (*x* − *m*) @ *y* = *n*)
by (*frule-tac app-eq-cases*, *auto elim*:*prefixE*)

end

theory *Myhill-2*
  imports *Myhill-1 List-Prefix Prefix-subtract*
begin

8 Direction *regular language* ⇒*finite partition*

8.1 The scheme

The following convenient notation $x \approx A\ y$ means: string x and y are equivalent with respect to language A.

definition
  str-eq :: *string* ⇒ *lang* ⇒ *string* ⇒ *bool* (- ≈- -)
where
  $x \approx A\ y \equiv (x,\ y) \in (\approx A)$

The main lemma (*rexp-imp-finite*) is proved by a structural induction over regular expressions. While base cases (cases for *NULL*, *EMPTY*, *CHAR*) are quite straight forward, the inductive cases are rather involved. What we have when starting to prove these inductive caes is that the partitions induced by the componet language are finite. The basic idea to show the finiteness of the partition induced by the composite language is to attach a tag $tag(x)$ to every string x. The tags are made of equivalent classes from the component partitions. Let *tag* be the tagging function and *Lang* be the composite language, it can be proved that if strings with the same tag are equivalent with respect to *Lang*, expressed as:

$$tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$$

then the partition induced by *Lang* must be finite. There are two arguments for this. The first goes as the following:

1. First, the tagging function *tag* induces an equivalent relation (=*tag*=) (defiintion of *f-eq-rel* and lemma *equiv-f-eq-rel*).

2. It is shown that: if the range of *tag* (denoted $range(tag)$) is finite, the partition given rise by (=*tag*=) is finite (lemma *finite-eq-f-rel*). Since tags are made from equivalent classes from component partitions, and the inductive hypothesis ensures the finiteness of these partitions, it is not difficult to prove the finiteness of $range(tag)$.

3. It is proved that if equivalent relation *R1* is more refined than *R2* (expressed as $R1 \subseteq R2$), and the partition induced by *R1* is finite, then the partition induced by *R2* is finite as well (lemma *refined-partition-finite*).

4. The injectivity assumption $tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$ implies that $(=tag=)$ is more refined than $(\approx Lang)$.

5. Combining the points above, we have: the partition induced by language *Lang* is finite (lemma *tag-finite-imageD*).

```
definition
  f-eq-rel (=-=)
where
  (=f=) = {(x, y) | x y. f x = f y}

lemma equiv-f-eq-rel:equiv UNIV (=f=)
  by (auto simp:equiv-def f-eq-rel-def refl-on-def sym-def trans-def)

lemma finite-range-image: finite (range f) ⟹ finite (f ‘ A)
  by (rule-tac B = {y. ∃x. y = f x} in finite-subset, auto simp:image-def)

lemma finite-eq-f-rel:
  assumes rng-fnt: finite (range tag)
  shows finite (UNIV // (=tag=))
proof -
  let ?f = op ‘ tag and ?A = (UNIV // (=tag=))
  show ?thesis
  proof (rule-tac f = ?f and A = ?A in finite-imageD)
    — The finiteness of f-image is a simple consequence of assumption rng-fnt:
    show finite (?f ‘ ?A)
    proof -
      have ∀ X. ?f X ∈ (Pow (range tag)) by (auto simp:image-def Pow-def)
      moreover from rng-fnt have finite (Pow (range tag)) by simp
      ultimately have finite (range ?f)
        by (auto simp only:image-def intro:finite-subset)
      from finite-range-image [OF this] show ?thesis .
    qed
  next
    — The injectivity of f-image is a consequence of the definition of (=tag=):
    show inj-on ?f ?A
    proof-
      { fix X Y
        assume X-in: X ∈ ?A
          and  Y-in: Y ∈ ?A
          and  tag-eq: ?f X = ?f Y
        have X = Y
        proof -
          from X-in Y-in tag-eq
          obtain x y
            where x-in: x ∈ X and y-in: y ∈ Y and eq-tg: tag x = tag y
            unfolding quotient-def Image-def str-eq-rel-def
                                      str-eq-def image-def f-eq-rel-def
            apply simp by blast
```

```
        with X-in Y-in show ?thesis
          by (auto simp:quotient-def str-eq-rel-def str-eq-def f-eq-rel-def)
      qed
    } thus ?thesis unfolding inj-on-def by auto
  qed
 qed
qed

lemma finite-image-finite: ⟦∀ x ∈ A. f x ∈ B; finite B⟧ ⟹ finite (f ‘ A)
  by (rule finite-subset [of - B], auto)

lemma refined-partition-finite:
  fixes R1 R2 A
  assumes fnt: finite (A // R1)
  and refined: R1 ⊆ R2
  and eq1: equiv A R1 and eq2: equiv A R2
  shows finite (A // R2)
proof −
  let ?f = λ X. {R1 ‘‘ {x} | x. x ∈ X}
    and ?A = (A // R2) and ?B = (A // R1)
  show ?thesis
  proof(rule-tac f = ?f and A = ?A in finite-imageD)
    show finite (?f ‘ ?A)
    proof(rule finite-subset [of - Pow ?B])
      from fnt show finite (Pow (A // R1)) by simp
    next
      from eq2
      show ?f ‘ A // R2 ⊆ Pow ?B
        unfolding image-def Pow-def quotient-def
        apply auto
        by (rule-tac x = xb in bexI, simp,
              unfold equiv-def sym-def refl-on-def, blast)
    qed
  next
    show inj-on ?f ?A
    proof −
      { fix X Y
        assume X-in: X ∈ ?A and Y-in: Y ∈ ?A
          and eq-f: ?f X = ?f Y (is ?L = ?R)
        have X = Y using X-in
        proof(rule quotientE)
          fix x
          assume X = R2 ‘‘ {x} and x ∈ A with eq2
          have x-in: x ∈ X
            unfolding equiv-def quotient-def refl-on-def by auto
          with eq-f have R1 ‘‘ {x} ∈ ?R by auto
          then obtain y where
            y-in: y ∈ Y and eq-r: R1 ‘‘ {x} = R1 ‘‘{y} by auto
          have (x, y) ∈ R1
```

```
          proof -
            from x-in X-in y-in Y-in eq2
            have x ∈ A and y ∈ A
              unfolding equiv-def quotient-def refl-on-def by auto
            from eq-equiv-class-iff [OF eq1 this] and eq-r
            show ?thesis by simp
          qed
          with refined have xy-r2: (x, y) ∈ R2 by auto
          from quotient-eqI [OF eq2 X-in Y-in x-in y-in this]
          show ?thesis .
        qed
      } thus ?thesis by (auto simp:inj-on-def)
    qed
  qed
qed

lemma equiv-lang-eq: equiv UNIV (≈Lang)
  unfolding equiv-def str-eq-rel-def sym-def refl-on-def trans-def
  by blast

lemma tag-finite-imageD:
  fixes tag
  assumes rng-fnt: finite (range tag)
  — Suppose the rang of tagging fucntion tag is finite.
  and same-tag-eqvt: ⋀ m n. tag m = tag (n::string) ⟹ m ≈Lang n
  — And strings with same tag are equivalent
  shows finite (UNIV // (≈Lang))
proof -
  let ?R1 = (=tag=)
  show ?thesis
  proof(rule-tac refined-partition-finite [of - ?R1])
    from finite-eq-f-rel [OF rng-fnt]
     show finite (UNIV // =tag=) .
   next
     from same-tag-eqvt
     show (=tag=) ⊆ (≈Lang)
       by (auto simp:f-eq-rel-def str-eq-def)
   next
     from equiv-f-eq-rel
     show equiv UNIV (=tag=) by blast
   next
     from equiv-lang-eq
     show equiv UNIV (≈Lang) by blast
  qed
qed
```

A more concise, but less intelligible argument for *tag-finite-imageD* is given as the following. The basic idea is still using standard library lemma *finite-imageD*:

$$[\![\mathit{finite}\ (f\ `\ A);\ \mathit{inj\text{-}on}\ f\ A]\!] \Longrightarrow \mathit{finite}\ A$$

which says: if the image of injective function f over set A is finite, then A must be finte, as we did in the lemmas above.

lemma
  fixes *tag*
  assumes *rng-fnt*: *finite* (*range tag*)
  — Suppose the rang of tagging fucntion *tag* is finite.
  and *same-tag-eqvt*: $\bigwedge$ *m n*. *tag m* = *tag* (*n*::*string*) $\Longrightarrow$ *m* ≈*Lang n*
  — And strings with same tag are equivalent
  shows *finite* (*UNIV* // (≈*Lang*))
  — Then the partition generated by (≈*Lang*) is finite.
proof −
  — The particular f and A used in *finite-imageD* are:
  let *?f* = *op* ‘ *tag* **and** *?A* = (*UNIV* // ≈*Lang*)
  show *?thesis*
  proof (*rule-tac f* = *?f* **and** *A* = *?A* **in** *finite-imageD*)
    — The finiteness of f-image is a simple consequence of assumption *rng-fnt*:
    show *finite* (*?f* ‘ *?A*)
    proof −
      have ∀ *X*. *?f X* ∈ (*Pow* (*range tag*)) **by** (*auto simp*:*image-def Pow-def*)
      moreover from *rng-fnt* **have** *finite* (*Pow* (*range tag*)) **by** *simp*
      ultimately have *finite* (*range ?f*)
        by (*auto simp only*:*image-def intro*:*finite-subset*)
      from *finite-range-image* [*OF this*] **show** *?thesis* **.**
    qed
  next
    — The injectivity of f is the consequence of assumption *same-tag-eqvt*:
    show *inj-on ?f ?A*
    proof−
      { fix *X Y*
        assume *X-in*: *X* ∈ *?A*
          and *Y-in*: *Y* ∈ *?A*
          and *tag-eq*: *?f X* = *?f Y*
        have *X* = *Y*
        proof −
          from *X-in Y-in tag-eq*
          obtain *x y* **where** *x-in*: *x* ∈ *X* **and** *y-in*: *y* ∈ *Y* **and** *eq-tg*: *tag x* = *tag y*
            unfolding *quotient-def Image-def str-eq-rel-def str-eq-def image-def*
            apply *simp* **by** *blast*
          from *same-tag-eqvt* [*OF eq-tg*] **have** *x* ≈*Lang y* **.**
          with *X-in Y-in x-in y-in*
          show *?thesis* **by** (*auto simp*:*quotient-def str-eq-rel-def str-eq-def*)
        qed
      } **thus** *?thesis* **unfolding** *inj-on-def* **by** *auto*
    qed
  qed
qed

8.2 The proof

Each case is given in a separate section, as well as the final main lemma. Detailed explainations accompanied by illustrations are given for non-trivial cases.

For ever inductive case, there are two tasks, the easier one is to show the range finiteness of of the tagging function based on the finiteness of component partitions, the difficult one is to show that strings with the same tag are equivalent with respect to the composite language. Suppose the composite language be *Lang*, tagging function be *tag*, it amounts to show:

$$tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$$

expanding the definition of $\approx Lang$, it amounts to show:

$$tag(x) = tag(y) \Longrightarrow (\forall\ z.\ x@z \in Lang \longleftrightarrow y@z \in Lang)$$

Because the assumed tag equlity $tag(x) = tag(y)$ is symmetric, it is suffcient to show just one direction:

$$\bigwedge x\ y\ z.\ [\![tag(x) = tag(y);\ x@z \in Lang]\!] \Longrightarrow y@z \in Lang$$

This is the pattern followed by every inductive case.

8.2.1 The base case for *NULL*

lemma *quot-null-eq*:
  shows $(UNIV\ //\ \approx\{\}) = (\{UNIV\}{::}lang\ set)$
  unfolding *quotient-def Image-def str-eq-rel-def* **by** *auto*

lemma *quot-null-finiteI* [*intro*]:
  shows *finite* $((UNIV\ //\ \approx\{\}){::}lang\ set)$
unfolding *quot-null-eq* **by** *simp*

8.2.2 The base case for *EMPTY*

lemma *quot-empty-subset*:
  $UNIV\ //\ (\approx\{[]\}) \subseteq \{\{[]\},\ UNIV - \{[]\}\}$
proof
  fix x
  assume $x \in UNIV\ //\ \approx\{[]\}$
  then obtain y **where** h: $x = \{z.\ (y, z) \in \approx\{[]\}\}$
    unfolding *quotient-def Image-def* **by** *blast*
  show $x \in \{\{[]\},\ UNIV - \{[]\}\}$
  proof (*cases* $y = []$)
    case *True* **with** h
    have $x = \{[]\}$ **by** (*auto simp*: *str-eq-rel-def*)
    thus *?thesis* **by** *simp*

```
  next
    case False with h
    have x = UNIV − {[]} by (auto simp: str-eq-rel-def)
    thus ?thesis by simp
  qed
qed

lemma quot-empty-finiteI [intro]:
  shows finite (UNIV // (≈{[]}))
by (rule finite-subset[OF quot-empty-subset]) (simp)
```

8.2.3 The base case for *CHAR*

```
lemma quot-char-subset:
  UNIV // (≈{[c]}) ⊆ {{[]},{[c]}, UNIV − {[], [c]}}
proof
  fix x
  assume x ∈ UNIV // ≈{[c]}
  then obtain y where h: x = {z. (y, z) ∈ ≈{[c]}}
    unfolding quotient-def Image-def by blast
  show x ∈ {{[]},{[c]}, UNIV − {[], [c]}}
  proof −
    { assume y = [] hence x = {[]} using h
        by (auto simp:str-eq-rel-def)
    } moreover {
      assume y = [c] hence x = {[c]} using h
        by (auto dest!:spec[where x = []] simp:str-eq-rel-def)
    } moreover {
      assume y ≠ [] and y ≠ [c]
      hence ∀ z. (y @ z) ≠ [c] by (case-tac y, auto)
      moreover have ⋀ p. (p ≠ [] ∧ p ≠ [c]) = (∀ q. p @ q ≠ [c])
        by (case-tac p, auto)
      ultimately have x = UNIV − {[],[c]} using h
        by (auto simp add:str-eq-rel-def)
    } ultimately show ?thesis by blast
  qed
qed

lemma quot-char-finiteI [intro]:
  shows finite (UNIV // (≈{[c]}))
by (rule finite-subset[OF quot-char-subset]) (simp)
```

8.2.4 The inductive case for *ALT*

```
definition
  tag-str-ALT :: lang ⇒ lang ⇒ string ⇒ (lang × lang)
where
  tag-str-ALT L1 L2 = (λx. (≈L1 `` {x}, ≈L2 `` {x}))

lemma quot-union-finiteI [intro]:
```

fixes *L1 L2*::*lang*
assumes *finite1*: *finite* (*UNIV* // ≈*L1*)
and *finite2*: *finite* (*UNIV* // ≈*L2*)
shows *finite* (*UNIV* // ≈(*L1* ∪ *L2*))
proof (*rule-tac tag* = *tag-str-ALT L1 L2* **in** *tag-finite-imageD*)
show $\bigwedge x\ y.$ *tag-str-ALT L1 L2 x* = *tag-str-ALT L1 L2 y* ⟹ *x* ≈(*L1* ∪ *L2*) *y*
unfolding *tag-str-ALT-def*
unfolding *str-eq-def*
unfolding *Image-def*
unfolding *str-eq-rel-def*
by *auto*
next
have ∗: *finite* ((*UNIV* // ≈*L1*) × (*UNIV* // ≈*L2*))
using *finite1 finite2* **by** *auto*
show *finite* (*range* (*tag-str-ALT L1 L2*))
unfolding *tag-str-ALT-def*
apply(*rule finite-subset*[*OF* - ∗])
unfolding *quotient-def*
by *auto*
qed

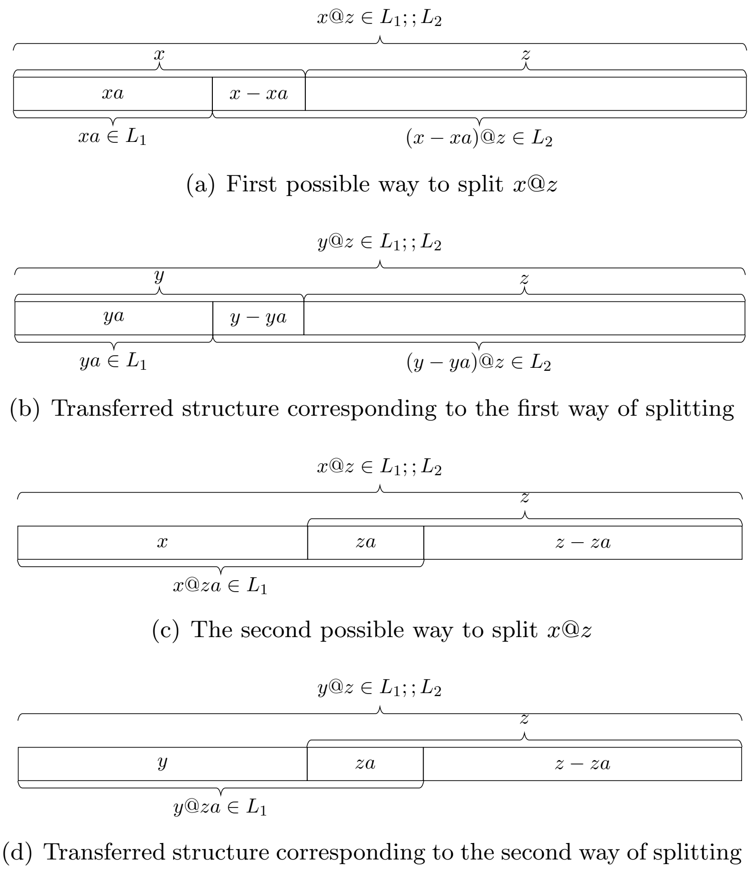
8.2.5 The inductive case for *SEQ*

For case *SEQ*, the language L is L_1 ;; L_2. Given x @ $z \in L_1$;; L_2, according to the defintion of L_1 ;; L_2, string x @ z can be splitted with the prefix in L_1 and suffix in L_2. The split point can either be in x (as shown in Fig. 1(a)), or in z (as shown in Fig. 1(c)). Whichever way it goes, the structure on x @ z cn be transfered faithfully onto y @ z (as shown in Fig. 1(b) and 1(d)) with the the help of the assumed tag equality. The following tag function *tag-str-SEQ* is such designed to facilitate such transfers and lemma *tag-str-SEQ-injI* formalizes the informal argument above. The details of structure transfer will be given their.

definition
tag-str-SEQ :: *lang* ⇒ *lang* ⇒ *string* ⇒ (*lang* × *lang set*)
where
tag-str-SEQ L1 L2 =
(λ*x*. (≈*L1* ‘‘ {*x*}, {(≈*L2* ‘‘ {*x* − *xa*}) | *xa*. *xa* ≤ *x* ∧ *xa* ∈ *L1*}))

The following is a techical lemma which helps to split the x @ $z \in L_1$;; L_2 mentioned above.

lemma *append-seq-elim*:
assumes x @ $y \in L_1$;; L_2
shows (∃ *xa* ≤ *x*. *xa* ∈ L_1 ∧ (*x* − *xa*) @ *y* ∈ L_2) ∨
(∃ *ya* ≤ *y*. (*x* @ *ya*) ∈ L_1 ∧ (*y* − *ya*) ∈ L_2)
proof−
from *assms* **obtain** s_1 s_2
where *eq-xys*: x @ y = s_1 @ s_2



(a) First possible way to split $x@z$

(b) Transferred structure corresponding to the first way of splitting

(c) The second possible way to split $x@z$

(d) Transferred structure corresponding to the second way of splitting

Figure 1: The case for SEQ

```
    and in-seq: s1 ∈ L1 ∧ s2 ∈ L2
    by (auto simp:Seq-def)
  from app-eq-dest [OF eq-xys]
  have
    (x ≤ s1 ∧ (s1 − x) @ s2 = y) ∨ (s1 ≤ x ∧ (x − s1) @ y = s2)
            (is ?Split1 ∨ ?Split2) .
  moreover have ?Split1 ⟹ ∃ ya ≤ y. (x @ ya) ∈ L1 ∧ (y − ya) ∈ L2
    using in-seq by (rule-tac x = s1 − x in exI, auto elim:prefixE)
  moreover have ?Split2 ⟹ ∃ xa ≤ x. xa ∈ L1 ∧ (x − xa) @ y ∈ L2
    using in-seq by (rule-tac x = s1 in exI, auto)
  ultimately show ?thesis by blast
qed


lemma tag-str-SEQ-injI:
  fixes v w
  assumes eq-tag: tag-str-SEQ L1 L2 v = tag-str-SEQ L1 L2 w
  shows v ≈(L1 ;; L2) w
```

proof−
— As explained before, a pattern for just one direction needs to be dealt with:
{ **fix** *x y z*
assume *xz-in-seq*: $x @ z \in L_1 ;; L_2$
and *tag-xy*: $tag\text{-}str\text{-}SEQ\ L_1\ L_2\ x = tag\text{-}str\text{-}SEQ\ L_1\ L_2\ y$
have $y @ z \in L_1 ;; L_2$
proof−
— There are two ways to split *x@z*:
from *append-seq-elim* [*OF xz-in-seq*]
have $(\exists\ xa \le x.\ xa \in L_1 \wedge (x - xa) @ z \in L_2) \vee$
$(\exists\ za \le z.\ (x @ za) \in L_1 \wedge (z - za) \in L_2)$ **.**
— It can be shown that *?thesis* holds in either case:
moreover {
— The case for the first split:
fix *xa*
assume *h1*: $xa \le x$ **and** *h2*: $xa \in L_1$ **and** *h3*: $(x - xa) @ z \in L_2$
— The following subgoal implements the structure transfer:
obtain *ya*
where $ya \le y$
and $ya \in L_1$
and $(y - ya) @ z \in L_2$
proof −
By expanding the definition of

— $tag\text{-}str\text{-}SEQ\ L_1\ L_2\ x = tag\text{-}str\text{-}SEQ\ L_1\ L_2\ y$

and extracting the second compoent, we get:
have $\{\approx L_2\ ``\ \{x - xa\}\ |xa.\ xa \le x \wedge xa \in L_1\} =$
$\{\approx L_2\ ``\ \{y - ya\}\ |ya.\ ya \le y \wedge ya \in L_1\}$ (**is** *?Left* = *?Right*)
using *tag-xy* **unfolding** *tag-str-SEQ-def* **by** *simp*
— Since $xa \le x$ and $xa \in L_1$ hold, it is not difficult to show:
moreover have $\approx L_2\ ``\ \{x - xa\} \in$ *?Left* **using** *h1 h2* **by** *auto*
— Through tag equality, equivalent class $\approx L_2\ ``\ \{x - xa\}$
also belongs to the *?Right*:
ultimately have $\approx L_2\ ``\ \{x - xa\} \in$ *?Right* **by** *simp*
— From this, the counterpart of *xa* in *y* is obtained:
then obtain *ya*
where *eq-xya*: $\approx L_2\ ``\ \{x - xa\} = \approx L_2\ ``\ \{y - ya\}$
and *pref-ya*: $ya \le y$ **and** *ya-in*: $ya \in L_1$
by *simp blast*
— It can be proved that *ya* has the desired property:
have $(y - ya)@z \in L_2$
proof −
from *eq-xya* **have** $(x - xa)\ \approx L_2\ (y - ya)$
unfolding *Image-def str-eq-rel-def str-eq-def* **by** *auto*
with *h3* **show** *?thesis* **unfolding** *str-eq-rel-def str-eq-def* **by** *simp*
qed
— Now, *ya* has all properties to be a qualified candidate:
with *pref-ya ya-in*
show *?thesis* **using** *that* **by** *blast*

```
      qed
        — From the properties of ya, y @ z ∈ L₁ ;; L₂ is derived easily.
      hence y @ z ∈ L₁ ;; L₂ by (erule-tac prefixE, auto simp:Seq-def)
    } moreover {
      — The other case is even more simpler:
      fix za
      assume h1: za ≤ z and h2: (x @ za) ∈ L₁ and h3: z − za ∈ L₂
      have y @ za ∈ L₁
      proof−
        have ≈L₁ `` {x} = ≈L₁ `` {y}
          using tag-xy unfolding tag-str-SEQ-def by simp
        with h2 show ?thesis
          unfolding Image-def str-eq-rel-def str-eq-def by auto
      qed
      with h1 h3 have y @ z ∈ L₁ ;; L₂
        by (drule-tac A = L₁ in seq-intro, auto elim:prefixE)
    }
    ultimately show ?thesis by blast
  qed
}
— ?thesis is proved by exploiting the symmetry of eq-tag:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma quot-seq-finiteI [intro]:
  fixes L1 L2::lang
  assumes fin1: finite (UNIV // ≈L1)
  and     fin2: finite (UNIV // ≈L2)
  shows finite (UNIV // ≈(L1 ;; L2))
proof (rule-tac tag = tag-str-SEQ L1 L2 in tag-finite-imageD)
  show ⋀x y. tag-str-SEQ L1 L2 x = tag-str-SEQ L1 L2 y ⟹ x ≈(L1 ;; L2) y
    by (rule tag-str-SEQ-injI)
next
  have *: finite ((UNIV // ≈L1) × (Pow (UNIV // ≈L2)))
    using fin1 fin2 by auto
  show finite (range (tag-str-SEQ L1 L2))
    unfolding tag-str-SEQ-def
    apply(rule finite-subset[OF - *])
    unfolding quotient-def
    by auto
qed
```

8.2.6 The inductive case for *STAR*

This turned out to be the trickiest case. The essential goal is to proved $y @ z \in L_1*$ under the assumptions that $x @ z \in L_1*$ and that x and y have the same tag. The reasoning goes as the following:

1. Since x @ $z \in$ L_1* holds, a prefix xa of x can be found such that $xa \in L_1*$ and $(x - xa)@z \in L_1*$, as shown in Fig. 2(a). Such a prefix always exists, xa = [], for example, is one.

2. There could be many but fintie many of such xa, from which we can find the longest and name it *xa-max*, as shown in Fig. 2(b).

3. The next step is to split z into za and zb such that (x − *xa-max*) @ $za \in L_1$ and $zb \in L_1*$ as shown in Fig. 2(e). Such a split always exists because:

   (a) Because (x − *x-max*) @ $z \in L_1*$, it can always be splitted into prefix a and suffix b, such that $a \in L_1$ and $b \in L_1*$, as shown in Fig. 2(c).

   (b) But the prefix a CANNOT be shorter than x − *xa-max* (as shown in Fig. 2(d)), becasue otherwise, *ma-max*@a would be in the same kind as *xa-max* but with a larger size, conflicting with the fact that *xa-max* is the longest.

4. By the assumption that x and y have the same tag, the structure on x @ z can be transferred to y @ z as shown in Fig. 2(f). The detailed steps are:

   (a) A y-prefix ya corresponding to xa can be found, which satisfies conditions: $ya \in L_1*$ and $(y - ya)@za \in L_1$.

   (b) Since we already know $zb \in L_1*$, we get $(y - ya)@za@zb \in L_1*$, and this is just $(y - ya)@z \in L_1*$.

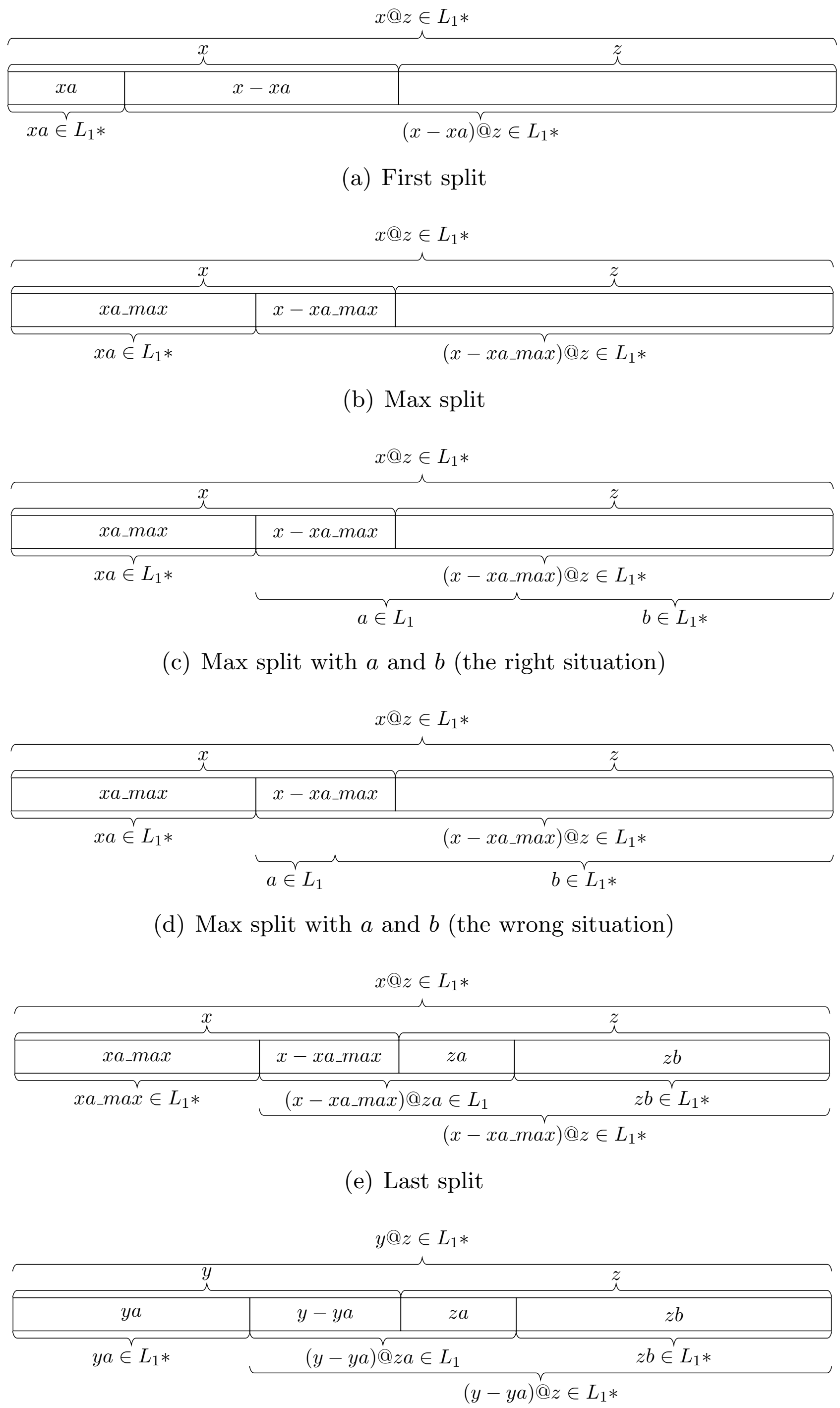
   (c) With fact $ya \in L_1*$, we finally get $y@z \in L_1*$.

The formal proof of lemma *tag-str-STAR-injI* faithfully follows this informal argument while the tagging function *tag-str-STAR* is defined to make the transfer in step **??** feasible.

definition
  tag-str-STAR :: *lang* ⇒ *string* ⇒ *lang set*
where
  tag-str-STAR L1 = (λ*x*. {≈*L1* ‘‘ {*x* − *xa*} | *xa*. *xa* < *x* ∧ *xa* ∈ *L1*⋆})

A technical lemma.

lemma *finite-set-has-max*: ⟦*finite A*; *A* ≠ {}⟧ ⟹
            (∃ *max* ∈ *A*. ∀ *a* ∈ *A*. *f a* <= (*f max* :: *nat*))
proof (*induct rule*:*finite.induct*)
  case *emptyI* **thus** *?case* **by** *simp*
next
  case (*insertI A a*)
  show *?case*
  proof (*cases A* = {})



(a) First split

(b) Max split

(c) Max split with a and b (the right situation)

(d) Max split with a and b (the wrong situation)

(e) Last split

(f) Structure transferred to y

Figure 2: The case for $STAR$

case *True* **thus** *?thesis* **by** (*rule-tac* $x = a$ **in** *bexI*, *auto*)
next
case *False*
with *insertI.hyps* **and** *False*
obtain *max*
where *h1*: $max \in A$
and *h2*: $\forall a \in A.\ f\ a \leq f\ max$ **by** *blast*
show *?thesis*
proof (*cases* $f\ a \leq f\ max$)
assume $f\ a \leq f\ max$
with *h1 h2* **show** *?thesis* **by** (*rule-tac* $x = max$ **in** *bexI*, *auto*)
next
assume $\neg (f\ a \leq f\ max)$
thus *?thesis* **using** *h2* **by** (*rule-tac* $x = a$ **in** *bexI*, *auto*)
qed
qed
qed

The following is a technical lemma.which helps to show the range finiteness of tag function.

lemma *finite-strict-prefix-set*: *finite* $\{xa.\ xa < (x::string)\}$
apply (*induct x rule*:*rev-induct*, *simp*)
apply (*subgoal-tac* $\{xa.\ xa < xs\ @\ [x]\} = \{xa.\ xa < xs\} \cup \{xs\}$)
by (*auto simp*:*strict-prefix-def*)

lemma *tag-str-STAR-injI*:
fixes *v w*
assumes *eq-tag*: *tag-str-STAR* L_1 v = *tag-str-STAR* L_1 w
shows (v::*string*) $\approx(L_1\star)$ w
proof−
— As explained before, a pattern for just one direction needs to be dealt with:
{ fix *x y z*
assume *xz-in-star*: $x\ @\ z \in L_1\star$
and *tag-xy*: *tag-str-STAR* L_1 x = *tag-str-STAR* L_1 y
have $y\ @\ z \in L_1\star$
proof(*cases* $x = []$)
— The degenerated case when x is a null string is easy to prove:
case *True*
with *tag-xy* **have** $y = []$
by (*auto simp add*: *tag-str-STAR-def strict-prefix-def*)
thus *?thesis* **using** *xz-in-star True* **by** *simp*
next
— The nontrival case:
case *False*
— Since $x\ @\ z \in L_1\star$, x can always be splitted by a prefix xa together with its suffix $x - xa$, such that both xa and $(x - xa)\ @\ z$ are in $L_1\star$, and there could be many such splittings.Therefore, the following set *?S* is nonempty, and finite as well:
let $?S = \{xa.\ xa < x \wedge xa \in L_1\star \wedge (x - xa)\ @\ z \in L_1\star\}$

have *finite ?S*
  by (*rule-tac* $B = \{xa.\ xa < x\}$ **in** *finite-subset*,
    auto simp:*finite-strict-prefix-set*)
moreover have $?S \neq \{\}$ **using** *False xz-in-star*
  by (*simp*, *rule-tac* $x = []$ **in** *exI*, *auto simp*:*strict-prefix-def*)
— Since *?S* is finite, we can always single out the longest and name it *xa-max*:
ultimately have $\exists$ *xa-max* $\in$ *?S*. $\forall$ *xa* $\in$ *?S*. *length xa* $\leq$ *length xa-max*
  using *finite-set-has-max* **by** *blast*
then obtain *xa-max*
  where *h1*: *xa-max* $<$ *x*
  and *h2*: *xa-max* $\in L_1\star$
  and *h3*: $(x -$ *xa-max*$)$ @ $z \in L_1\star$
  and *h4*:$\forall$ $xa < x$. $xa \in L_1\star \wedge (x - xa)$ @ $z \in L_1\star$
    $\longrightarrow$ *length xa* $\leq$ *length xa-max*
  by *blast*
— By the equality of tags, the counterpart of *xa-max* among *y*-prefixes, named *ya*, can be found:
obtain *ya*
  where *h5*: $ya < y$ **and** *h6*: $ya \in L_1\star$
  and *eq-xya*: $(x -$ *xa-max*$) \approx L_1\ (y - ya)$
proof−
  from *tag-xy* **have** $\{\approx L_1$ `` $\{x - xa\}\ |xa.\ xa < x \wedge xa \in L_1\star\} =$
    $\{\approx L_1$ `` $\{y - xa\}\ |xa.\ xa < y \wedge xa \in L_1\star\}$ (**is** *?left* = *?right*)
    by (*auto simp*:*tag-str-STAR-def*)
  moreover have $\approx L_1$ `` $\{x -$ *xa-max*$\} \in$ *?left* **using** *h1 h2* **by** *auto*
  ultimately have $\approx L_1$ `` $\{x -$ *xa-max*$\} \in$ *?right* **by** *simp*
  thus *?thesis* **using** *that*
    apply (*simp add*:*Image-def str-eq-rel-def str-eq-def*) **by** *blast*
qed
— The *?thesis*, y @ $z \in L_1\star$, is a simple consequence of the following proposition:
have $(y - ya)$ @ $z \in L_1\star$
proof−
  — The idea is to split the suffix z into *za* and *zb*, such that:
  obtain *za zb* **where** *eq-zab*: $z = za$ @ zb
    and *l-za*: $(y - ya)$@$za \in L_1$ **and** *ls-zb*: $zb \in L_1\star$
  proof −
    — Since *xa-max* $< x$, x can be splitted into a and b such that:
    from *h1* **have** $(x -$ *xa-max*$)$ @ $z \neq []$
      by (*auto simp*:*strict-prefix-def elim*:*prefixE*)
    from *star-decom* [*OF h3 this*]
    obtain a b **where** *a-in*: $a \in L_1$
      and *a-neq*: $a \neq []$ **and** *b-in*: $b \in L_1\star$
      and *ab-max*: $(x -$ *xa-max*$)$ @ $z = a$ @ b **by** *blast*
    — Now the candiates for *za* and *zb* are found:
    let $?za = a - (x -$ *xa-max*$)$ **and** $?zb = b$
    have *pfx*: $(x -$ *xa-max*$) \leq a$ (**is** *?P1*)
      and *eq-z*: $z = ?za$ @ $?zb$ (**is** *?P2*)
    proof −

```
          — Since (x − xa-max) @ z = a @ b, string (x − xa-max) @ z can
            be splitted in two ways:
          have ((x − xa-max) ≤ a ∧ (a − (x − xa-max)) @ b = z) ∨
            (a < (x − xa-max) ∧ ((x − xa-max) − a) @ z = b)
            using app-eq-dest[OF ab-max] by (auto simp:strict-prefix-def)
          moreover {
            — However, the undsired way can be refuted by absurdity:
            assume np: a < (x − xa-max)
              and b-eqs: ((x − xa-max) − a) @ z = b
            have False
            proof −
              let ?xa-max' = xa-max @ a
              have ?xa-max' < x
                using np h1 by (clarsimp simp:strict-prefix-def diff-prefix)
              moreover have ?xa-max' ∈ L1⋆
                using a-in h2 by (simp add:star-intro3)
              moreover have (x − ?xa-max') @ z ∈ L1⋆
                using b-eqs b-in np h1 by (simp add:diff-diff-appd)
              moreover have ¬ (length ?xa-max' ≤ length xa-max)
                using a-neq by simp
              ultimately show ?thesis using h4 by blast
            qed }
          — Now it can be shown that the splitting goes the way we desired.
          ultimately show ?P1 and ?P2 by auto
        qed
        hence (x − xa-max)@?za ∈ L1 using a-in by (auto elim:prefixE)
        — Now candidates ?za and ?zb have all the requred properteis.
        with eq-xya have (y − ya) @ ?za ∈ L1
          by (auto simp:str-eq-def str-eq-rel-def)
         with eq-z and b-in
        show ?thesis using that by blast
      qed
      — ?thesis can easily be shown using properties of za and zb:
      have ((y − ya) @ za) @ zb ∈ L1⋆ using l-za ls-zb by blast
      with eq-zab show ?thesis by simp
    qed
    with h5 h6 show ?thesis
      by (drule-tac star-intro1, auto simp:strict-prefix-def elim:prefixE)
  qed
}
— By instantiating the reasoning pattern just derived for both directions:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
— The thesis is proved as a trival consequence:
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma — The oringal version with less explicit details.
  fixes v w
  assumes eq-tag: tag-str-STAR L1 v = tag-str-STAR L1 w
  shows (v::string) ≈(L1⋆) w
```

proof −

— According to the definition of $\approx$*Lang*, proving $v \approx (L_1\star)\ w$ amounts to showing: for any string u, if $v\ @\ u \in (L_1\star)$ then $w\ @\ u \in (L_1\star)$ and vice versa. The reasoning pattern for both directions are the same, as derived in the following:

{ fix $x\ y\ z$
assume *xz-in-star*: $x\ @\ z \in L_1\star$
and *tag-xy*: *tag-str-STAR* $L_1\ x$ = *tag-str-STAR* $L_1\ y$
have $y\ @\ z \in L_1\star$
proof(*cases* $x = []$)
— The degenerated case when x is a null string is easy to prove:
case *True*
with *tag-xy* **have** $y = []$
by (*auto simp*:*tag-str-STAR-def strict-prefix-def*)
thus *?thesis* **using** *xz-in-star True* **by** *simp*
next
— The case when x is not null, and $x\ @\ z$ is in $L_1\star$,
case *False*
obtain *x-max*
where *h1*: *x-max* $< x$
and *h2*: *x-max* $\in L_1\star$
and *h3*: $(x - \textit{x-max})\ @\ z \in L_1\star$
and *h4*:$\forall\ xa < x.\ xa \in L_1\star \wedge (x - xa)\ @\ z \in L_1\star$
$\longrightarrow$ *length xa* $\leq$ *length x-max*
proof −
let $?S = \{xa.\ xa < x \wedge xa \in L_1\star \wedge (x - xa)\ @\ z \in L_1\star\}$
have *finite ?S*
by (*rule-tac* $B = \{xa.\ xa < x\}$ **in** *finite-subset*,
auto simp:*finite-strict-prefix-set*)
moreover have $?S \neq \{\}$ **using** *False xz-in-star*
by (*simp*, *rule-tac* $x = []$ **in** *exI*, *auto simp*:*strict-prefix-def*)
ultimately have $\exists\ max \in ?S.\ \forall\ a \in ?S.$ *length a* $\leq$ *length max*
using *finite-set-has-max* **by** *blast*
thus *?thesis* **using** *that* **by** *blast*
qed
obtain *ya*
where *h5*: $ya < y$ **and** *h6*: $ya \in L_1\star$ **and** *h7*: $(x - \textit{x-max}) \approx L_1\ (y - ya)$
proof −
from *tag-xy* **have** $\{\approx L_1$ ‘‘ $\{x - xa\}\ |xa.\ xa < x \wedge xa \in L_1\star\} =$
$\{\approx L_1$ ‘‘ $\{y - xa\}\ |xa.\ xa < y \wedge xa \in L_1\star\}$ (**is** *?left* = *?right*)
by (*auto simp*:*tag-str-STAR-def*)
moreover have $\approx L_1$ ‘‘ $\{x - \textit{x-max}\} \in$ *?left* **using** *h1 h2* **by** *auto*
ultimately have $\approx L_1$ ‘‘ $\{x - \textit{x-max}\} \in$ *?right* **by** *simp*
with *that* **show** *?thesis* **apply**
(*simp add*:*Image-def str-eq-rel-def str-eq-def*) **by** *blast*
qed
have $(y - ya)\ @\ z \in L_1\star$
proof −
from *h3 h1* **obtain** *a b* **where** *a-in*: $a \in L_1$

```
        and a-neq: a ≠ [] and b-in: b ∈ L₁⋆
        and ab-max: (x − x-max) @ z = a @ b
        by (drule-tac star-decom, auto simp:strict-prefix-def elim:prefixE)
      have (x − x-max) ≤ a ∧ (a − (x − x-max)) @ b = z
      proof −
        have ((x − x-max) ≤ a ∧ (a − (x − x-max)) @ b = z) ∨
                      (a < (x − x-max) ∧ ((x − x-max) − a) @ z = b)
          using app-eq-dest[OF ab-max] by (auto simp:strict-prefix-def)
        moreover {
          assume np: a < (x − x-max) and b-eqs: ((x − x-max) − a) @ z = b
          have False
          proof −
            let ?x-max' = x-max @ a
            have ?x-max' < x
              using np h1 by (clarsimp simp:strict-prefix-def diff-prefix)
            moreover have ?x-max' ∈ L₁⋆
              using a-in h2 by (simp add:star-intro3)
            moreover have (x − ?x-max') @ z ∈ L₁⋆
              using b-eqs b-in np h1 by (simp add:diff-diff-appd)
            moreover have ¬ (length ?x-max' ≤ length x-max)
              using a-neq by simp
            ultimately show ?thesis using h4 by blast
          qed
        } ultimately show ?thesis by blast
      qed
      then obtain za where z-decom: z = za @ b
        and x-za: (x − x-max) @ za ∈ L₁
        using a-in by (auto elim:prefixE)
      from x-za h7 have (y − ya) @ za ∈ L₁
        by (auto simp:str-eq-def str-eq-rel-def)
      with b-in have ((y − ya) @ za) @ b ∈ L₁⋆ by blast
      with z-decom show ?thesis by auto
    qed
    with h5 h6 show ?thesis
      by (drule-tac star-intro1, auto simp:strict-prefix-def elim:prefixE)
  qed
}
— By instantiating the reasoning pattern just derived for both directions:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
— The thesis is proved as a trival consequence:
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma quot-star-finiteI [intro]:
  fixes L1::lang
  assumes finite1: finite (UNIV // ≈L1)
  shows finite (UNIV // ≈(L1⋆))
proof (rule-tac tag = tag-str-STAR L1 in tag-finite-imageD)
  show ⋀x y. tag-str-STAR L1 x = tag-str-STAR L1 y ⟹ x ≈(L1⋆) y
```

by (*rule tag-str-STAR-injI*)
next
have ∗: *finite* (*Pow* (*UNIV* // ≈*L1*))
using *finite1* **by** *auto*
show *finite* (*range* (*tag-str-STAR L1*))
unfolding *tag-str-STAR-def*
apply(*rule finite-subset*[*OF* - ∗])
unfolding *quotient-def*
by *auto*
qed

8.2.7 The conclusion

lemma *rexp-imp-finite*:
fixes *r*::*rexp*
shows *finite* (*UNIV* // ≈(*L r*))
by (*induct r*) (*auto*)

end

theory *Myhill*
imports *Myhill-2*
begin

9 Preliminaries

9.1 Finite automata and Myhill-Nerode theorem

A *determinisitc finite automata (DFA) M* is a 5-tuple $(Q, \Sigma, \delta, s, F)$, where:

1. Q is a finite set of *states*, also denoted Q_M.
2. Σ is a finite set of *alphabets*, also denoted Σ_M.
3. δ is a *transition function* of type $Q \times \Sigma \Rightarrow Q$ (a total function), also denoted δ_M.
4. $s \in Q$ is a state called *initial state*, also denoted s_M.
5. $F \subseteq Q$ is a set of states named *accepting states*, also denoted F_M.

Therefore, we have $M = (Q_M, \Sigma_M, \delta_M, s_M, F_M)$. Every DFA M can be interpreted as a function assigning states to strings, denoted $\hat{\delta}_M$, the definition of which is as the following:

$$\begin{aligned} \hat{\delta}_M([]) &\equiv s_M \\ \hat{\delta}_M(xa) &\equiv \delta_M(\hat{\delta}_M(x), a) \end{aligned} \tag{2}$$

A string x is said to be *accepted* (or *recognized*) by a DFA M if $\hat{\delta}_M(x) \in F_M$. The language recoginzed by DFA M, denoted $L(M)$, is defined as:

$$L(M) \equiv \{x \mid \hat{\delta}_M(x) \in F_M\} \tag{3}$$

The standard way of specifying a laugage $\mathcal{L}$ as *regular* is by stipulating that: $\mathcal{L} = L(M)$ for some DFA M.

For any DFA M, the DFA obtained by changing initial state to another $p \in Q_M$ is denoted M_p, which is defined as:

$$M_p \;\equiv\; (Q_M, \Sigma_M, \delta_M, p, F_M) \tag{4}$$

Two states $p, q \in Q_M$ are said to be *equivalent*, denoted $p \approx_M q$, iff.

$$L(M_p) = L(M_q) \tag{5}$$

It is obvious that $\approx_M$ is an equivalent relation over Q_M. and the partition induced by $\approx_M$ has $|Q_M|$ equivalent classes. By overloading $\approx_M$, an equivalent relation over strings can be defined:

$$x \approx_M y \;\;\equiv\;\; \hat{\delta}_M(x) \approx_M \hat{\delta}_M(y) \tag{6}$$

It can be proved that the the partition induced by $\approx_M$ also has $|Q_M|$ equivalent classes. It is also easy to show that: if $x \approx_M y$, then $x \approx_{L(M)} y$, and this means $\approx_M$ is a more refined equivalent relation than $\approx_{L(M)}$. Since partition induced by $\approx_M$ is finite, the one induced by $\approx_{L(M)}$ must also be finite, and this is one of the two directions of Myhill-Nerode theorem:

Lemma 1 (Myhill-Nerode theorem, Direction two)**.** *If a language $\mathcal{L}$ is regular (i.e. $\mathcal{L} = L(M)$ for some DFA M), then the partition induced by $\approx_{\mathcal{L}}$ is finite.*

The other direction is:

Lemma 2 (Myhill-Nerode theorem, Direction one)**.** *If the partition induced by $\approx_{\mathcal{L}}$ is finite, then $\mathcal{L}$ is regular (i.e. $\mathcal{L} = L(M)$ for some DFA M).*

The M we are seeking when prove lemma **??** can be constructed out of $\approx_{\mathcal{L}}$, denoted $M_{\mathcal{L}}$ and defined as the following:

$$Q_{M_{\mathcal{L}}} \;\equiv\; \{[\![x]\!]_{\approx_{\mathcal{L}}} \mid x \in \Sigma^*\} \tag{7a}$$

$$\Sigma_{M_{\mathcal{L}}} \;\equiv\; \Sigma_M \tag{7b}$$

$$\delta_{M_{\mathcal{L}}} \;\equiv\; (\lambda([\![x]\!]_{\approx_{\mathcal{L}}}, a).[\![xa]\!]_{\approx_{\mathcal{L}}}) \tag{7c}$$

$$s_{M_{\mathcal{L}}} \;\equiv\; [\![[]]\!]_{\approx_{\mathcal{L}}} \tag{7d}$$

$$F_{M_{\mathcal{L}}} \;\equiv\; \{[\![x]\!]_{\approx_{\mathcal{L}}} \mid x \in \mathcal{L}\} \tag{7e}$$

It can be proved that $Q_{M_{\mathcal{L}}}$ is indeed finite and $\mathcal{L} = L(M_{\mathcal{L}})$, so lemma 2 holds. It can also be proved that $M_{\mathcal{L}}$ is the minimal DFA (therefore unique) which recoginzes $\mathcal{L}$.

9.2 The objective and the underlying intuition

It is now obvious from section 9.1 that Myhill-Nerode theorem can be established easily when *reglar languages* are defined as ones recognized by finite automata. Under the context where the use of finite automata is forbiden, the situation is quite different. The theorem now has to be expressed as:

Theorem 1 (Myhill-Nerode theorem, Regular expression version)**.** *A language $\mathcal{L}$ is regular (i.e. $\mathcal{L} = L(e)$ for some* regular expression *e) iff. the partition induced by $\approx_{\mathcal{L}}$ is finite.*

The proof of this version consists of two directions (if the use of automata are not allowed):

Direction one: generating a regular expression e out of the finite partition induced by $\approx_{\mathcal{L}}$, such that $\mathcal{L} = L(e)$.

Direction two: showing the finiteness of the partition induced by $\approx_{\mathcal{L}}$, under the assmption that $\mathcal{L}$ is recognized by some regular expression e (i.e. $\mathcal{L} = L(e)$).

The development of these two directions consititutes the body of this paper.

10 Direction *regular language $\Rightarrow$finite partition*

Although not used explicitly, the notion of finite autotmata and its relationship with language partition, as outlined in section 9.1, still servers as important intuitive guides in the development of this paper. For example, *Direction one* follows the *Brzozowski algebraic method* used to convert finite autotmata to regular expressions, under the intuition that every partition member $[\![x]\!]_{\approx_{\mathcal{L}}}$ is a state in the DFA $M_{\mathcal{L}}$ constructed to prove lemma 2 of section 9.1.

The basic idea of Brzozowski method is to set aside an unknown for every DFA state and describe the state-trasition relationship by charateristic equations. By solving the equational system such obtained, regular expressions characterizing DFA states are obtained. There are choices of how DFA states can be characterized. The first is to characterize a DFA state by the set of striings leading from the state in question into accepting states. The other choice is to characterize a DFA state by the set of strings leading from initial state into the state in question. For the first choice, the lauguage recognized by a DFA can be characterized by the regular expression characterizing initial state, while in the second choice, the languaged of the DFA can be characterized by the summation of regular expressions of all accepting states.

end

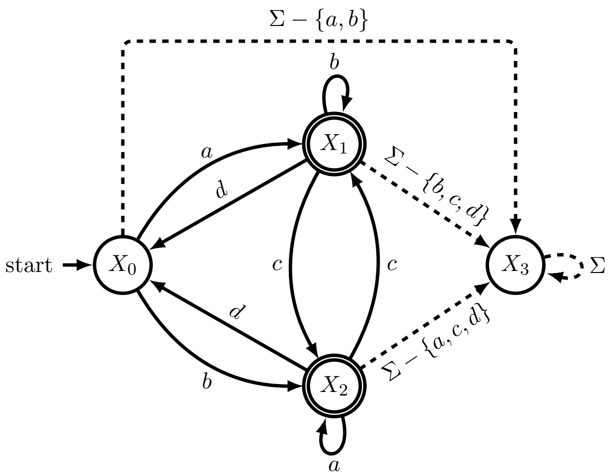


Figure 3: The relationship between automata and finite partition