A Formalisation of the Myhill-Nerode Theorem based on Regular Expressions (Proof Pearl)

joint work with Chunhan Wu and Xingyuan Zhang from the PLA University of Science and Technology in Nanjing

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TU Munich

I want to teach students with theorem provers (especially for inductions).

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- ~~fib, even and odd~~
- formal language theory
  $\Rightarrow$ nice textbooks: Kozen, Hopcroft & Ullman...

in Nuprl

- Constable, Jackson, Naumov, Uribe
- 18 months for automata theory from Hopcroft & Ullman chapters 1–11 (including Myhill-Nerode)

in Coq

- Filliâtre, Briais, Braibant and others
- multi-year effort; a number of results in automata theory, e.g.
  - Kleene's thm. by Filliâtre ("rather big")
  - automata theory by Briais (5400 loc)
  - Braibant ATBR library, including Myhill-Nerode ($\gg$2000 loc)
  - Mirkin's partial derivative automaton construction (10600 loc)

in HOL

- automata $\Rightarrow$ graphs, matrices, functions

Formal language theory…

in HOL

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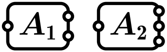
- combining automata/graphs



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disjoint union:

$$A_1 \uplus A_2 \stackrel{\text{def}}{=} \{(1, x) \mid x \in A_1\} \cup \{(2, y) \mid y \in A_2\}$$

Formal language theory...

in HOL

- automata $\Rightarrow$ graphs, matrices, functions

Problems with definition for regularity (Slind):

$$\text{is\_regular}(A) \stackrel{\text{def}}{=} \exists M.\ \text{is\_dfa}(M) \wedge \mathcal{L}(M) = A$$

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A solution: use nat $\Rightarrow$ state nodes

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You have to rename states!

in HOL

- Kozen's paper proof of Myhill-Nerode:
  requires absence of inaccessible states

$$\text{is\_regular}(A) \stackrel{\text{def}}{=} \exists M.\ \text{is\_dfa}(M) \wedge \mathcal{L}(M) = A$$

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A language A is regular, provided there exists a regular expression that matches all strings of A.

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- pumping lemma
- closure under complementation
- ~~regular expression matching~~ ($\Rightarrow$Owens et al)
- most textbooks are about automata

The Myhill-Nerode Theorem

- provides necessary and sufficient conditions for a language being regular
  (pumping lemma only necessary)
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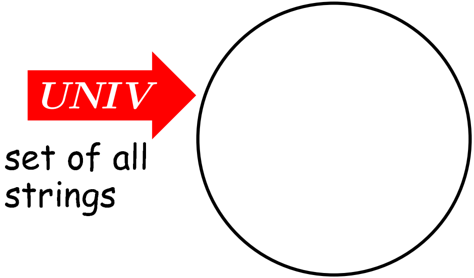
- key is the equivalence relation:

$$x \approx_A y \stackrel{\text{def}}{=} \forall z.\, x@z \in A \Leftrightarrow y@z \in A$$

The Myhill-Nerode Theorem



- finite $(UNIV // \approx_A) \;\Leftrightarrow\; A$ is regular

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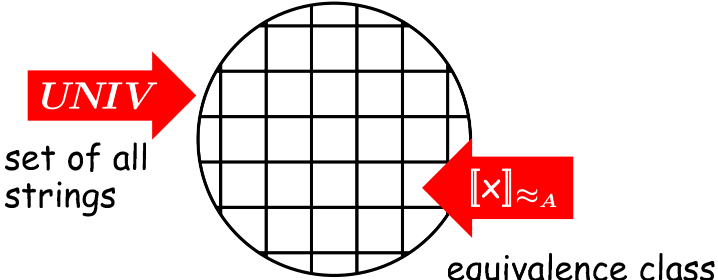

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The Myhill-Nerode Theorem

Two directions:

1.) finite $\Rightarrow$ regular

$$\text{finite}\,(UNIV // \approx_A) \Rightarrow \exists r.\ A = \mathcal{L}(r)$$

2.) regular $\Rightarrow$ finite

$$\text{finite}\,(UNIV // \approx_{\mathcal{L}(r)})$$

equivalence class

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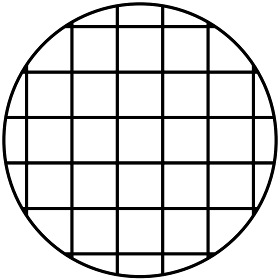
Initial and Final ~~States~~ Equivalence Classes



- finals $A \stackrel{\text{def}}{=} \{\|x\|_{\approx_A} \mid x \in A\}$
- we can prove: $A = \bigcup \text{finals } A$

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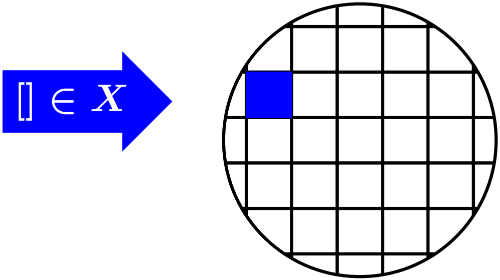

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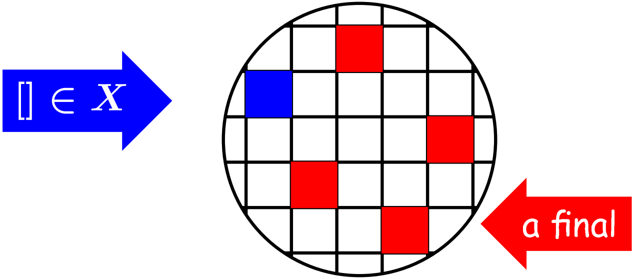

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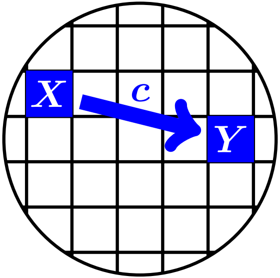
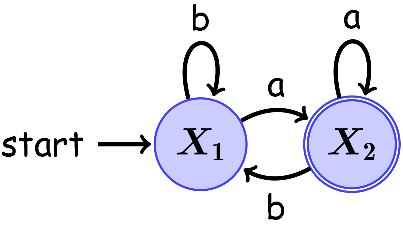
Transitions between Eq-Classes



$$X \xrightarrow{c} Y \overset{\text{def}}{=} X;c \subseteq Y$$

Systems of Equations

Inspired by a method of Brzozowski '64:



$$X_1 = X_1; b + X_2; b$$
$$X_2 = X_1; a + X_2; a$$

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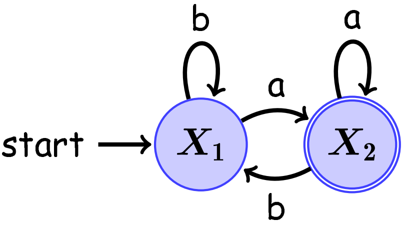

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$$X_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
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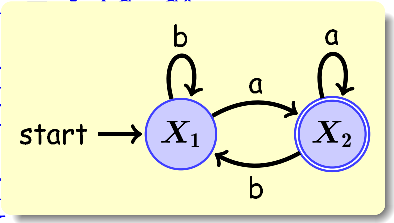

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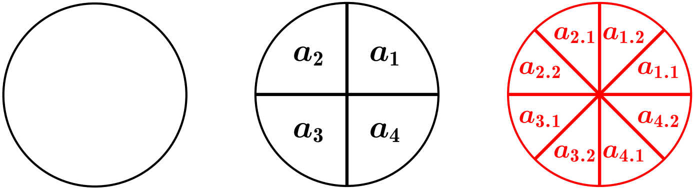
The Other Direction

One has to prove

$$\text{finite}(\boldsymbol{UNIV}/\!/ \approx_{\mathcal{L}(r)})$$

by induction on r. Not trivial, but after a bit of thinking, one can find a refined relation:



$\boldsymbol{UNIV}$ $\boldsymbol{UNIV}/\!/ \approx_{\mathcal{L}(r)}$ $\boldsymbol{UNIV}/\!/\boldsymbol{R}$

Partial Derivatives

- ...(set of) regular expressions after a string has been parsed

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Antimirov '95

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$\Downarrow$ Antimirov '95

- $\text{finite}(UNIV // R_1)$

- Therefore $\text{finite}(UNIV // \approx_{\mathcal{L}(r)})$. Qed.

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$(A \stackrel{\text{def}}{=} \bigcup_n a^n)$

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Bold Claim: (not proved!)

95% of regular language theory can be done without automata!

...and this is much more tasteful ;o)

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Thank you!

Questions?