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Contents

1 Preliminary definitions **1**

2 A slightly modified version of Arden's lemma **5**

3 Regular Expressions **6**

4 Folds for Sets **7**

5 Direction *finite partition* $\Rightarrow$ *regular language* **8**
- 5.1 The proof of this direction 12
  - 5.1.1 Basic properties 12
  - 5.1.2 Intialization 14
  - 5.1.3 Interation step 16
  - 5.1.4 Conclusion of the proof 22

6 List prefixes and postfixes **24**
- 6.1 Prefix order on lists 24
- 6.2 Basic properties of prefixes 25
- 6.3 Parallel lists 28
- 6.4 Postfix order on lists 29

7 A small theory of prefix subtraction **32**

8 Direction *regular language* $\Rightarrow$*finite partition* **33**
- 8.1 The scheme 33
- 8.2 The proof 38
  - 8.2.1 The base case for *NULL* 39
  - 8.2.2 The base case for *EMPTY* 39
  - 8.2.3 The base case for *CHAR* 39
  - 8.2.4 The inductive case for *ALT* 40
  - 8.2.5 The inductive case for *SEQ* 40
  - 8.2.6 The inductive case for *STAR* 44
  - 8.2.7 The conclusion 51

9 Preliminaries **52**
9.1 Finite automata and Myhill-Nerode theorem 52
9.2 The objective and the underlying intuition 53

10 Direction *regular language* ⇒*finite partition* **54**

```
theory Myhill-1
  imports Main
begin
```

1 Preliminary definitions

types *lang* = *string set*

Sequential composition of two languages

```
definition
  Seq :: lang ⇒ lang ⇒ lang (infixr ;; 100)
where
  A ;; B = {s₁ @ s₂ | s₁ s₂. s₁ ∈ A ∧ s₂ ∈ B}
```

Some properties of operator ;;.

```
lemma seq-add-left:
  assumes a: A = B
  shows C ;; A = C ;; B
using a by simp

lemma seq-union-distrib-right:
  shows (A ∪ B) ;; C = (A ;; C) ∪ (B ;; C)
unfolding Seq-def by auto

lemma seq-union-distrib-left:
  shows C ;; (A ∪ B) = (C ;; A) ∪ (C ;; B)
unfolding Seq-def by  auto

lemma seq-intro:
  assumes a: x ∈ A y ∈ B
  shows x @ y ∈ A ;; B
using a by (auto simp: Seq-def)

lemma seq-assoc:
  shows (A ;; B) ;; C = A ;; (B ;; C)
unfolding Seq-def
apply(auto)
apply(blast)
by (metis append-assoc)

lemma seq-empty [simp]:
  shows A ;; {[]} = A
  and   {[]} ;; A = A
```

```
by (simp-all add: Seq-def)
```

Power and Star of a language

```
fun
  pow :: lang ⇒ nat ⇒ lang (infixl ↑ 100)
where
  A ↑ 0 = {[]}
| A ↑ (Suc n) =  A ;; (A ↑ n)

definition
  Star :: lang ⇒ lang (-⋆ [101] 102)
where
  A⋆ ≡ (⋃ n. A ↑ n)


lemma star-start[intro]:
  shows [] ∈ A⋆
proof −
  have [] ∈ A ↑ 0 by auto
  then show [] ∈ A⋆ unfolding Star-def by blast
qed

lemma star-step [intro]:
  assumes a: s1 ∈ A
  and     b: s2 ∈ A⋆
  shows s1 @ s2 ∈ A⋆
proof −
  from b obtain n where s2 ∈ A ↑ n unfolding Star-def by auto
  then have s1 @ s2 ∈ A ↑ (Suc n) using a by (auto simp add: Seq-def)
  then show s1 @ s2 ∈ A⋆ unfolding Star-def by blast
qed

lemma star-induct[consumes 1, case-names start step]:
  assumes a: x ∈ A⋆
  and     b: P []
  and     c: ⋀s1 s2. ⟦s1 ∈ A; s2 ∈ A⋆; P s2⟧ ⟹ P (s1 @ s2)
  shows P x
proof −
  from a obtain n where x ∈ A ↑ n unfolding Star-def by auto
  then show P x
    by (induct n arbitrary: x)
       (auto intro!: b c simp add: Seq-def Star-def)
qed

lemma star-intro1:
  assumes a: x ∈ A⋆
  and     b: y ∈ A⋆
  shows x @ y ∈ A⋆
using a b
```

```
by (induct rule: star-induct) (auto)

lemma star-intro2:
  assumes a: y ∈ A
  shows y ∈ A⋆
proof −
  from a have y @ [] ∈ A⋆ by blast
  then show y ∈ A⋆ by simp
qed

lemma star-intro3:
  assumes a: x ∈ A⋆
  and     b: y ∈ A
  shows x @ y ∈ A⋆
using a b by (blast intro: star-intro1 star-intro2)

lemma star-cases:
  shows A⋆ =  {[]} ∪ A ;; A⋆
proof
  { fix x
    have x ∈ A⋆ ⟹ x ∈ {[]} ∪ A ;; A⋆
      unfolding Seq-def
    by (induct rule: star-induct) (auto)
  }
  then show A⋆ ⊆ {[]} ∪ A ;; A⋆ by auto
next
  show {[]} ∪ A ;; A⋆ ⊆ A⋆
    unfolding Seq-def by auto
qed

lemma star-decom:
  assumes a: x ∈ A⋆ x ≠ []
  shows ∃ a b. x = a @ b ∧ a ≠ [] ∧ a ∈ A ∧ b ∈ A⋆
using a
apply(induct rule: star-induct)
apply(simp)
apply(blast)
done

lemma
  shows seq-Union-left:  B ;; (⋃n. A ↑ n) = (⋃n. B ;; (A ↑ n))
  and   seq-Union-right: (⋃n. A ↑ n) ;; B = (⋃n. (A ↑ n) ;; B)
unfolding Seq-def by auto

lemma seq-pow-comm:
  shows A ;; (A ↑ n) = (A ↑ n) ;; A
by (induct n) (simp-all add: seq-assoc[symmetric])

lemma seq-star-comm:
```

shows A ;; $A\star = A\star$;; A
unfolding *Star-def*
unfolding *seq-Union-left*
unfolding *seq-pow-comm*
unfolding *seq-Union-right*
by *simp*

Two lemmas about the length of strings in $A \uparrow n$

lemma *pow-length*:
assumes *a*: $[] \notin A$
and *b*: $s \in A \uparrow Suc\ n$
shows $n < length\ s$
using *b*
proof (*induct n arbitrary*: *s*)
case *0*
have $s \in A \uparrow Suc\ 0$ **by** *fact*
with *a* **have** $s \neq []$ **by** *auto*
then show $0 < length\ s$ **by** *auto*
next
case (*Suc n*)
have *ih*: $\bigwedge s.\ s \in A \uparrow Suc\ n \Longrightarrow n < length\ s$ **by** *fact*
have $s \in A \uparrow Suc\ (Suc\ n)$ **by** *fact*
then obtain *s1 s2* **where** *eq*: $s = s1\ @\ s2$ **and** $*$: $s1 \in A$ **and** $**$: $s2 \in A \uparrow Suc\ n$
by (*auto simp add*: *Seq-def*)
from *ih* $**$ **have** $n < length\ s2$ **by** *simp*
moreover have $0 < length\ s1$ **using** $*$ *a* **by** *auto*
ultimately show $Suc\ n < length\ s$ **unfolding** *eq*
by (*simp only*: *length-append*)
qed

lemma *seq-pow-length*:
assumes *a*: $[] \notin A$
and *b*: $s \in B$;; $(A \uparrow Suc\ n)$
shows $n < length\ s$
proof −
from *b* **obtain** *s1 s2* **where** *eq*: $s = s1\ @\ s2$ **and** $*$: $s2 \in A \uparrow Suc\ n$
unfolding *Seq-def* **by** *auto*
from $*$ **have** $n < length\ s2$ **by** (*rule pow-length*[*OF a*])
then show $n < length\ s$ **using** *eq* **by** *simp*
qed

2 A slightly modified version of Arden's lemma

A helper lemma for Arden

lemma *ardens-helper*:
assumes *eq*: $X = X$;; $A \cup B$
shows $X = X$;; $(A \uparrow Suc\ n) \cup (\bigcup m\in\{0..n\}.\ B$;; $(A \uparrow m))$

```
proof (induct n)
  case 0
  show X = X ;; (A ↑ Suc 0) ∪ (⋃(m::nat)∈{0..0}. B ;; (A ↑ m))
    using eq by simp
next
  case (Suc n)
  have ih: X = X ;; (A ↑ Suc n) ∪ (⋃m∈{0..n}. B ;; (A ↑ m)) by fact
   also have ... = (X ;; A ∪ B) ;; (A ↑ Suc n) ∪ (⋃m∈{0..n}. B ;; (A ↑ m))
using eq by simp
  also have ... = X ;; (A ↑ Suc (Suc n)) ∪ (B ;; (A ↑ Suc n)) ∪ (⋃m∈{0..n}.
B ;; (A ↑ m))
    by (simp add: seq-union-distrib-right seq-assoc)
  also have ... = X ;; (A ↑ Suc (Suc n)) ∪ (⋃m∈{0..Suc n}. B ;; (A ↑ m))
    by (auto simp add: le-Suc-eq)
  finally show X = X ;; (A ↑ Suc (Suc n)) ∪ (⋃m∈{0..Suc n}. B ;; (A ↑ m)) .
qed

theorem ardens-revised:
  assumes nemp: [] ∉ A
  shows X = X ;; A ∪ B ⟷ X = B ;; A⋆
proof
  assume eq: X = B ;; A⋆
  have A⋆ = {[]} ∪ A⋆ ;; A
    unfolding seq-star-comm[symmetric]
    by (rule star-cases)
  then have B ;; A⋆ = B ;; ({[]} ∪ A⋆ ;; A)
    by (rule seq-add-left)
  also have ... = B ∪ B ;; (A⋆ ;; A)
    unfolding seq-union-distrib-left by simp
  also have ... = B ∪ (B ;; A⋆) ;; A
    by (simp only: seq-assoc)
  finally show X = X ;; A ∪ B
    using eq by blast
next
  assume eq: X = X ;; A ∪ B
  { fix n::nat
    have B ;; (A ↑ n) ⊆ X using ardens-helper[OF eq, of n] by auto }
  then have B ;; A⋆ ⊆ X
    unfolding Seq-def Star-def UNION-def
    by auto
  moreover
  { fix s::string
    obtain k where k = length s by auto
    then have not-in: s ∉ X ;; (A ↑ Suc k)
      using seq-pow-length[OF nemp] by blast
    assume s ∈ X
    then have s ∈ X ;; (A ↑ Suc k) ∪ (⋃m∈{0..k}. B ;; (A ↑ m))
      using ardens-helper[OF eq, of k] by auto
    then have s ∈ (⋃m∈{0..k}. B ;; (A ↑ m)) using not-in by auto
```

moreover
have $(\bigcup m \in \{0..k\}.\ B\ ;;\ (A \uparrow m)) \subseteq (\bigcup n.\ B\ ;;\ (A \uparrow n))$ **by** *auto*
ultimately
have $s \in B\ ;;\ A\star$
unfolding *seq-Union-left Star-def*
by *auto* **}**
then have $X \subseteq B\ ;;\ A\star$ **by** *auto*
ultimately
show $X = B\ ;;\ A\star$ **by** *simp*
qed

3 Regular Expressions

datatype *rexp* =
NULL
| *EMPTY*
| *CHAR char*
| *SEQ rexp rexp*
| *ALT rexp rexp*
| *STAR rexp*

The following L is an overloaded operator, where $L(x)$ evaluates to the language represented by the syntactic object x.

consts L:: $'a \Rightarrow$ *lang*

The L (*rexp*) for regular expressions.

overloading *L-rexp* $\equiv$ L:: *rexp* $\Rightarrow$ *lang*
begin
fun
L-rexp :: *rexp* $\Rightarrow$ *string set*
where
L-rexp (*NULL*) = {}
| *L-rexp* (*EMPTY*) = {[]}
| *L-rexp* (*CHAR c*) = {[c]}
| *L-rexp* (*SEQ r1 r2*) = (*L-rexp r1*) ;; (*L-rexp r2*)
| *L-rexp* (*ALT r1 r2*) = (*L-rexp r1*) $\cup$ (*L-rexp r2*)
| *L-rexp* (*STAR r*) = (*L-rexp r*)$\star$
end

4 Folds for Sets

To obtain equational system out of finite set of equivalence classes, a fold operation on finite sets *folds* is defined. The use of *SOME* makes *folds* more robust than the *fold* in the Isabelle library. The expression *folds f* makes sense when *f* is not *associative* and *commutitive*, while *fold f* does not.

definition

```
  folds :: ('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a set ⇒ 'b
where
  folds f z S ≡ SOME x. fold-graph f z S x

abbreviation
  Setalt  (⨄- [1000] 999)
where
  ⨄A == folds ALT NULL A
```

The following lemma ensures that the arbitrary choice made by the *SOME* in *folds* does not affect the *L*-value of the resultant regular expression.

```
lemma folds-alt-simp [simp]:
  assumes a: finite rs
  shows L (⨄rs) = ⋃ (L ‘ rs)
apply(rule set-eqI)
apply(simp add: folds-def)
apply(rule someI2-ex)
apply(rule-tac finite-imp-fold-graph[OF a])
apply(erule fold-graph.induct)
apply(auto)
done
```

Just a technical lemma for collections and pairs

```
lemma Pair-Collect[simp]:
  shows (x, y) ∈ {(x, y). P x y} ⟷ P x y
by simp
```

$\approx A$ is an equivalence class defined by language A.

```
definition
  str-eq-rel :: lang ⇒ (string × string) set (≈- [100] 100)
where
  ≈A ≡ {(x, y).  (∀ z. x @ z ∈ A ⟷ y @ z ∈ A)}
```

Among the equivalence clases of $\approx A$, the set *finals* A singles out those which contains the strings from A.

```
definition
  finals :: lang ⇒ lang set
where
  finals A ≡ {≈A ‘‘ {x} | x . x ∈ A}
```

The following lemma establishes the relationshipt between *finals* A and A.

```
lemma lang-is-union-of-finals:
  shows A = ⋃ finals A
unfolding finals-def
unfolding Image-def
unfolding str-eq-rel-def
apply(auto)
apply(drule-tac x = [] in spec)
```

apply(*auto*)
done

lemma *finals-in-partitions*:
shows *finals A* $\subseteq$ (*UNIV* // $\approx$*A*)
unfolding *finals-def*
unfolding *quotient-def*
by *auto*

5 Direction *finite partition* $\Rightarrow$ *regular language*

The relationship between equivalent classes can be described by an equational system. For example, in equational system (1), X_0, X_1 are equivalent classes. The first equation says every string in X_0 is obtained either by appending one b to a string in X_0 or by appending one a to a string in X_1 or just be an empty string (represented by the regular expression λ). Similary, the second equation tells how the strings inside X_1 are composed.

$$\begin{aligned} X_0 &= X_0 b + X_1 a + \lambda \\ X_1 &= X_0 a + X_1 b \end{aligned} \tag{1}$$

The summands on the right hand side is represented by the following data type *rhs-item*, mnemonic for 'right hand side item'. Generally, there are two kinds of right hand side items, one kind corresponds to pure regular expressions, like the λ in (1), the other kind corresponds to transitions from one one equivalent class to another, like the $X_0 b, X_1 a$ etc.

datatype *rhs-item* =
Lam rexp
| *Trn lang rexp*

In this formalization, pure regular expressions like λ is repsented by *Lam*(*EMPTY*), while transitions like $X_0 a$ is represented by *Trn* X_0 (*CHAR a*).

Every right-hand side item *itm* defines a language given by *L*(*itm*), defined as:

overloading *L-rhs-e* $\equiv$ *L*:: *rhs-item* $\Rightarrow$ *lang*
begin
fun *L-rhs-e*:: *rhs-item* $\Rightarrow$ *lang*
where
L-rhs-e (*Lam r*) = *L r*
| *L-rhs-e* (*Trn X r*) = *X* ;; *L r*
end

The right hand side of every equation is represented by a set of items. The string set defined by such a set *itms* is given by *L*(*itms*), defined as:

overloading *L-rhs* $\equiv$ *L*:: *rhs-item set* $\Rightarrow$ *lang*

begin
  fun *L-rhs*:: *rhs-item set* $\Rightarrow$ *lang*
  where
    L-rhs rhs = $\bigcup$ (*L* ' *rhs*)
end

Given a set of equivalence classes *CS* and one equivalence class *X* among *CS*, the term *init-rhs CS X* is used to extract the right hand side of the equation describing the formation of *X*. The definition of *init-rhs* is:

definition
  transition :: *lang* $\Rightarrow$ *rexp* $\Rightarrow$ *lang* $\Rightarrow$ *bool* (- $\models$-$\Rightarrow$- [*100*,*100*,*100*] *100*)
where
  $Y \models r \Rightarrow X \equiv Y$;; $(L\ r) \subseteq X$

definition
  init-rhs CS X $\equiv$
    if ($[] \in X$) *then*
      {*Lam EMPTY*} $\cup$ {*Trn Y* (*CHAR c*) | *Y c*. $Y \in CS \wedge Y \models (CHAR\ c) \Rightarrow$ *X*}
    else
      {*Trn Y* (*CHAR c*)| *Y c*. $Y \in CS \wedge Y \models (CHAR\ c) \Rightarrow X$}

In the definition of *init-rhs*, the term {*Trn Y* (*CHAR c*)| *Y c*. $Y \in CS \wedge Y$;; $\{[c]\} \subseteq X$} appearing on both branches describes the formation of strings in *X* out of transitions, while the term {*Lam*(*EMPTY*)} describes the empty string which is intrinsically contained in *X* rather than by transition. This {*Lam*(*EMPTY*)} corresponds to the λ in (1).

With the help of *init-rhs*, the equitional system descrbing the formation of every equivalent class inside *CS* is given by the following *eqs*(*CS*).

definition *eqs CS* $\equiv$ {(*X*, *init-rhs CS X*) | *X*. $X \in CS$}

The following *trns-of rhs X* returns all *X*-items in *rhs*.

definition
  trns-of rhs X $\equiv$ {*Trn X r* | *r*. *Trn X r* $\in$ *rhs*}

The following *attach-rexp rexp′ itm* attach the regular expression *rexp′* to the right of right hand side item *itm*.

fun
  attach-rexp :: *rexp* $\Rightarrow$ *rhs-item* $\Rightarrow$ *rhs-item*
where
  attach-rexp rexp′ (*Lam rexp*)   = *Lam* (*SEQ rexp rexp′*)
| *attach-rexp rexp′* (*Trn X rexp*) = *Trn X* (*SEQ rexp rexp′*)

The following *append-rhs-rexp rhs rexp* attaches *rexp* to every item in *rhs*.

definition
  append-rhs-rexp rhs rexp $\equiv$ (*attach-rexp rexp*) ' *rhs*

With the help of the two functions immediately above, Ardens' transformation on right hand side *rhs* is implemented by the following function *arden-variate X rhs*. After this transformation, the recursive occurence of X in *rhs* will be eliminated, while the string set defined by *rhs* is kept unchanged.

definition
arden-variate X rhs ≡
append-rhs-rexp (*rhs* − *trns-of rhs X*) (*STAR* (⨄ {*r*. *Trn X r* ∈ *rhs*}))

Suppose the equation defining X is $X = xrhs$, the purpose of *rhs-subst* is to substitute all occurences of X in *rhs* by *xrhs*. A litte thought may reveal that the final result should be: first append $(a_1|a_2|\ldots|a_n)$ to every item of *xrhs* and then union the result with all non-X-items of *rhs*.

definition
rhs-subst rhs X xrhs ≡
(*rhs* − (*trns-of rhs X*)) ∪ (*append-rhs-rexp xrhs* (⨄ {*r*. *Trn X r* ∈ *rhs*}))

Suppose the equation defining X is $X = xrhs$, the follwing *eqs-subst ES X xrhs* substitute *xrhs* into every equation of the equational system *ES*.

definition
eqs-subst ES X xrhs ≡ {(*Y*, *rhs-subst yrhs X xrhs*) | *Y yrhs*. (*Y*, *yrhs*) ∈ *ES*}

The computation of regular expressions for equivalence classes is accomplished using a iteration principle given by the following lemma.

lemma *wf-iter* [*rule-format*]:
  fixes *f*
  assumes *step*: ⋀ *e*. ⟦*P e*; ¬ *Q e*⟧ ⟹ (∃ *e'*. *P e'* ∧ (*f*(*e'*), *f*(*e*)) ∈ *less-than*)
  shows *pe*: *P e* ⟶ (∃ *e'*. *P e'* ∧ *Q e'*)
proof(*induct e rule*: *wf-induct*
    [*OF wf-inv-image*[*OF wf-less-than*, **where** *f* = *f*]], *clarify*)
  fix *x*
  assume *h* [*rule-format*]:
    ∀ *y*. (*y*, *x*) ∈ *inv-image less-than f* ⟶ *P y* ⟶ (∃ *e'*. *P e'* ∧ *Q e'*)
    and *px*: *P x*
  show ∃ *e'*. *P e'* ∧ *Q e'*
  proof(*cases Q x*)
    assume *Q x* **with** *px* **show** *?thesis* **by** *blast*
  next
    assume *nq*: ¬ *Q x*
    from *step* [*OF px nq*]
    obtain *e'* **where** *pe'*: *P e'* **and** *ltf*: (*f e'*, *f x*) ∈ *less-than* **by** *auto*
    show *?thesis*
    proof(*rule h*)
      from *ltf* **show** (*e'*, *x*) ∈ *inv-image less-than f*
        by (*simp add*:*inv-image-def*)
    next
      from *pe'* **show** *P e'* **.**

qed
qed
qed

The P in lemma *wf-iter* is an invariant kept throughout the iteration procedure. The particular invariant used to solve our problem is defined by function *Inv*(*ES*), an invariant over equal system *ES*. Every definition starting next till *Inv* stipulates a property to be satisfied by *ES*.

Every variable is defined at most onece in *ES*.

definition
$\textit{distinct-equas } ES \equiv$
$\forall\ X\ rhs\ rhs'.\ (X,\ rhs) \in ES \wedge (X,\ rhs') \in ES \longrightarrow rhs = rhs'$

Every equation in *ES* (represented by $(X,\ rhs)$) is valid, i.e. $(X = L\ rhs)$.

definition
$\textit{valid-eqns } ES \equiv \forall\ X\ rhs.\ (X,\ rhs) \in ES \longrightarrow (X = L\ rhs)$

The following *rhs-nonempty rhs* requires regular expressions occuring in transitional items of *rhs* does not contain empty string. This is necessary for the application of Arden's transformation to *rhs*.

definition
$\textit{rhs-nonempty } rhs \equiv (\forall\ Y\ r.\ Trn\ Y\ r \in rhs \longrightarrow [] \notin L\ r)$

The following *ardenable ES* requires that Arden's transformation is applicable to every equation of equational system *ES*.

definition
$\textit{ardenable } ES \equiv \forall\ X\ rhs.\ (X,\ rhs) \in ES \longrightarrow \textit{rhs-nonempty } rhs$

definition
$\textit{non-empty } ES \equiv \forall\ X\ rhs.\ (X,\ rhs) \in ES \longrightarrow X \neq \{\}$

The following *finite-rhs ES* requires every equation in *rhs* be finite.

definition
$\textit{finite-rhs } ES \equiv \forall\ X\ rhs.\ (X,\ rhs) \in ES \longrightarrow \textit{finite } rhs$

The following *classes-of rhs* returns all variables (or equivalent classes) occuring in *rhs*.

definition
$\textit{classes-of } rhs \equiv \{X.\ \exists\ r.\ Trn\ X\ r \in rhs\}$

The following *lefts-of ES* returns all variables defined by equational system *ES*.

definition
$\textit{lefts-of } ES \equiv \{Y \mid Y\ yrhs.\ (Y,\ yrhs) \in ES\}$

The following *self-contained ES* requires that every variable occuring on the right hand side of equations is already defined by some equation in *ES*.

```
definition
  self-contained ES ≡ ∀ (X, xrhs) ∈ ES. classes-of xrhs ⊆ lefts-of ES
```

The invariant *Inv*(*ES*) is a conjunction of all the previously defined constaints.

```
definition
  Inv ES ≡ valid-eqns ES ∧ finite ES ∧ distinct-equas ES ∧ ardenable ES ∧
              non-empty ES ∧ finite-rhs ES ∧ self-contained ES
```

5.1 The proof of this direction

5.1.1 Basic properties

The following are some basic properties of the above definitions.

```
lemma L-rhs-union-distrib:
  fixes A B::rhs-item set
  shows L A ∪ L B = L (A ∪ B)
by simp

lemma finite-Trn:
  assumes fin: finite rhs
  shows finite {r. Trn Y r ∈ rhs}
proof −
  have finite {Trn Y r | Y r. Trn Y r ∈ rhs}
    by (rule rev-finite-subset[OF fin]) (auto)
  then have finite ((λ(Y, r). Trn Y r) ‘ {(Y, r) | Y r. Trn Y r ∈ rhs})
    by (simp add: image-Collect)
  then have finite {(Y, r) | Y r. Trn Y r ∈ rhs}
    by (erule-tac finite-imageD) (simp add: inj-on-def)
  then show finite {r. Trn Y r ∈ rhs}
    by (erule-tac f=snd in finite-surj) (auto simp add: image-def)
qed

lemma finite-Lam:
  assumes fin:finite rhs
  shows finite {r. Lam r ∈ rhs}
proof −
  have finite {Lam r | r. Lam r ∈ rhs}
    by (rule rev-finite-subset[OF fin]) (auto)
  then show finite {r. Lam r ∈ rhs}
    apply(simp add: image-Collect[symmetric])
    apply(erule finite-imageD)
    apply(auto simp add: inj-on-def)
    done
qed
```

```
lemma rexp-of-empty:
  assumes finite:finite rhs
  and nonempty:rhs-nonempty rhs
  shows [] ∉ L (⨄ {r. Trn X r ∈ rhs})
using finite nonempty rhs-nonempty-def
using finite-Trn[OF finite]
by (auto)

lemma [intro!]:
  P (Trn X r) ⟹ (∃ a. (∃ r. a = Trn X r ∧ P a)) by auto

lemma lang-of-rexp-of:
  assumes finite:finite rhs
  shows L ({Trn X r| r. Trn X r ∈ rhs}) = X ;; (L (⨄{r. Trn X r ∈ rhs}))
proof -
  have finite {r. Trn X r ∈ rhs}
    by (rule finite-Trn[OF finite])
  then show ?thesis
    apply(auto simp add: Seq-def)
    apply(rule-tac x = s1 in exI, rule-tac x = s2 in exI, auto)
    apply(rule-tac x= Trn X xa in exI)
    apply(auto simp: Seq-def)
    done
qed

lemma rexp-of-lam-eq-lam-set:
  assumes fin: finite rhs
  shows L (⨄{r. Lam r ∈ rhs}) = L ({Lam r | r. Lam r ∈ rhs})
proof -
  have finite ({r. Lam r ∈ rhs}) using fin by (rule finite-Lam)
  then show ?thesis by auto
qed

lemma [simp]:
  L (attach-rexp r xb) = L xb ;; L r
apply (cases xb, auto simp: Seq-def)
apply(rule-tac x = s1 @ s1' in exI, rule-tac x = s2' in exI)
apply(auto simp: Seq-def)
done

lemma lang-of-append-rhs:
  L (append-rhs-rexp rhs r) = L rhs ;; L r
apply (auto simp:append-rhs-rexp-def image-def)
apply (auto simp:Seq-def)
apply (rule-tac x = L xb ;; L r in exI, auto simp add:Seq-def)
by (rule-tac x = attach-rexp r xb in exI, auto simp:Seq-def)

lemma classes-of-union-distrib:
  classes-of A ∪ classes-of B = classes-of (A ∪ B)
```

```
by (auto simp add:classes-of-def)

lemma lefts-of-union-distrib:
  lefts-of A ∪ lefts-of B = lefts-of (A ∪ B)
by (auto simp:lefts-of-def)
```

5.1.2 Intialization

The following several lemmas until *init-ES-satisfy-Inv* shows that the initial equational system satisfies invariant *Inv*.

```
lemma defined-by-str:
  ⟦s ∈ X; X ∈ UNIV // (≈Lang)⟧ ⟹ X = (≈Lang) `` {s}
by (auto simp:quotient-def Image-def str-eq-rel-def)

lemma every-eqclass-has-transition:
  assumes has-str: s @ [c] ∈ X
  and     in-CS:   X ∈ UNIV // (≈Lang)
  obtains Y where Y ∈ UNIV // (≈Lang) and Y ;; {[c]} ⊆ X and s ∈ Y
proof -
  def Y ≡ (≈Lang) `` {s}
  have Y ∈ UNIV // (≈Lang)
    unfolding Y-def quotient-def by auto
  moreover
  have X = (≈Lang) `` {s @ [c]}
    using has-str in-CS defined-by-str by blast
  then have Y ;; {[c]} ⊆ X
    unfolding Y-def Image-def Seq-def
    unfolding str-eq-rel-def
    by clarsimp
  moreover
  have s ∈ Y unfolding Y-def
    unfolding Image-def str-eq-rel-def by simp
  ultimately show thesis by (blast intro: that)
qed

lemma l-eq-r-in-eqs:
  assumes X-in-eqs: (X, xrhs) ∈ (eqs (UNIV // (≈Lang)))
  shows X = L xrhs
proof
  show X ⊆ L xrhs
  proof
    fix x
    assume (1): x ∈ X
    show x ∈ L xrhs
    proof (cases x = [])
      assume empty: x = []
      thus ?thesis using X-in-eqs (1)
        by (auto simp:eqs-def init-rhs-def)
    next
```

```
      assume not-empty: x ≠ []
      then obtain clist c where decom: x = clist @ [c]
        by (case-tac x rule:rev-cases, auto)
      have X ∈ UNIV // (≈Lang) using X-in-eqs by (auto simp:eqs-def)
      then obtain Y
        where Y ∈ UNIV // (≈Lang)
        and Y ;; {[c]} ⊆ X
        and clist ∈ Y
        using decom (1) every-eqclass-has-transition by blast
      hence
        x ∈ L {Trn Y (CHAR c)| Y c. Y ∈ UNIV // (≈Lang) ∧ Y ⊨(CHAR c)⇒
X}
        unfolding transition-def
        using (1) decom
        by (simp, rule-tac x = Trn Y (CHAR c) in exI, simp add:Seq-def)
      thus ?thesis using X-in-eqs (1)
        by (simp add: eqs-def init-rhs-def)
    qed
  qed
next
  show L xrhs ⊆ X using X-in-eqs
    by (auto simp:eqs-def init-rhs-def transition-def)
qed

lemma finite-init-rhs:
  assumes finite: finite CS
  shows finite (init-rhs CS X)
proof-
  have finite {Trn Y (CHAR c) |Y c. Y ∈ CS ∧ Y ;; {[c]} ⊆ X} (is finite ?A)
  proof -
    def S ≡ {(Y, c)| Y c. Y ∈ CS ∧ Y ;; {[c]} ⊆ X}
    def h ≡ λ (Y, c). Trn Y (CHAR c)
    have finite (CS × (UNIV::char set)) using finite by auto
    hence finite S using S-def
      by (rule-tac B = CS × UNIV in finite-subset, auto)
    moreover have ?A = h ` S by (auto simp: S-def h-def image-def)
    ultimately show ?thesis
      by auto
  qed
  thus ?thesis by (simp add:init-rhs-def transition-def)
qed

lemma init-ES-satisfy-Inv:
  assumes finite-CS: finite (UNIV // (≈Lang))
  shows Inv (eqs (UNIV // (≈Lang)))
proof -
  have finite (eqs (UNIV // (≈Lang))) using finite-CS
    by (simp add:eqs-def)
  moreover have distinct-equas (eqs (UNIV // (≈Lang)))
```

```
    by (simp add:distinct-equas-def eqs-def)
  moreover have ardenable (eqs (UNIV // (≈Lang)))
   by (auto simp add:ardenable-def eqs-def init-rhs-def rhs-nonempty-def del:L-rhs.simps)
  moreover have valid-eqns (eqs (UNIV // (≈Lang)))
    using l-eq-r-in-eqs by (simp add:valid-eqns-def)
  moreover have non-empty (eqs (UNIV // (≈Lang)))
    by (auto simp:non-empty-def eqs-def quotient-def Image-def str-eq-rel-def)
  moreover have finite-rhs (eqs (UNIV // (≈Lang)))
    using finite-init-rhs[OF finite-CS]
    by (auto simp:finite-rhs-def eqs-def)
  moreover have self-contained (eqs (UNIV // (≈Lang)))
    by (auto simp:self-contained-def eqs-def init-rhs-def classes-of-def lefts-of-def)
  ultimately show ?thesis by (simp add:Inv-def)
qed
```

5.1.3 Interation step

From this point until *iteration-step*, it is proved that there exists iteration steps which keep *Inv*(*ES*) while decreasing the size of *ES*.

```
lemma arden-variate-keeps-eq:
  assumes l-eq-r: X = L rhs
  and not-empty: [] ∉ L (⨄{r. Trn X r ∈ rhs})
  and finite: finite rhs
  shows X = L (arden-variate X rhs)
proof −
  def A ≡ L (⨄{r. Trn X r ∈ rhs})
  def b ≡ rhs − trns-of rhs X
  def B ≡ L b
  have X = B ;; A⋆
  proof−
    have L rhs = L(trns-of rhs X ∪ b) by (auto simp: b-def trns-of-def)
    also have . . . = X ;; A ∪ B
      unfolding trns-of-def
      unfolding L-rhs-union-distrib[symmetric]
      by (simp only: lang-of-rexp-of finite B-def A-def)
    finally show ?thesis
      using l-eq-r not-empty
      apply(rule-tac ardens-revised[THEN iffD1])
      apply(simp add: A-def)
      apply(simp)
      done
  qed
  moreover have L (arden-variate X rhs) = (B ;; A⋆)
    by (simp only:arden-variate-def L-rhs-union-distrib lang-of-append-rhs
                  B-def A-def b-def L-rexp.simps seq-union-distrib-left)
   ultimately show ?thesis by simp
qed

lemma append-keeps-finite:
```

```
  finite rhs ⟹ finite (append-rhs-rexp rhs r)
by (auto simp:append-rhs-rexp-def)

lemma arden-variate-keeps-finite:
  finite rhs ⟹ finite (arden-variate X rhs)
by (auto simp:arden-variate-def append-keeps-finite)

lemma append-keeps-nonempty:
  rhs-nonempty rhs ⟹ rhs-nonempty (append-rhs-rexp rhs r)
apply (auto simp:rhs-nonempty-def append-rhs-rexp-def)
by (case-tac x, auto simp:Seq-def)

lemma nonempty-set-sub:
  rhs-nonempty rhs ⟹ rhs-nonempty (rhs − A)
by (auto simp:rhs-nonempty-def)

lemma nonempty-set-union:
  ⟦rhs-nonempty rhs; rhs-nonempty rhs'⟧ ⟹ rhs-nonempty (rhs ∪ rhs')
by (auto simp:rhs-nonempty-def)

lemma arden-variate-keeps-nonempty:
  rhs-nonempty rhs ⟹ rhs-nonempty (arden-variate X rhs)
by (simp only:arden-variate-def append-keeps-nonempty nonempty-set-sub)


lemma rhs-subst-keeps-nonempty:
  ⟦rhs-nonempty rhs; rhs-nonempty xrhs⟧ ⟹ rhs-nonempty (rhs-subst rhs X xrhs)
by (simp only:rhs-subst-def append-keeps-nonempty  nonempty-set-union nonempty-set-sub)

lemma rhs-subst-keeps-eq:
  assumes substor: X = L xrhs
  and finite: finite rhs
  shows L (rhs-subst rhs X xrhs) = L rhs (is ?Left = ?Right)
proof−
  def A ≡ L (rhs − trns-of rhs X)
  have ?Left = A ∪ L (append-rhs-rexp xrhs (⨄{r. Trn X r ∈ rhs}))
    unfolding rhs-subst-def
    unfolding L-rhs-union-distrib[symmetric]
    by (simp add: A-def)
  moreover have ?Right = A ∪ L ({Trn X r | r. Trn X r ∈ rhs})
  proof−
     have rhs = (rhs − trns-of rhs X) ∪ (trns-of rhs X) by (auto simp add:
trns-of-def)
    thus ?thesis
      unfolding A-def
      unfolding L-rhs-union-distrib
      unfolding trns-of-def
      by simp
  qed
```

moreover have *L* (*append-rhs-rexp xrhs* ($\biguplus$ {*r*. *Trn X r* ∈ *rhs*})) = *L* ({*Trn X r* | *r*. *Trn X r* ∈ *rhs*})
using *finite substor* **by** (*simp only*:*lang-of-append-rhs lang-of-rexp-of*)
ultimately show *?thesis* **by** *simp*
qed

lemma *rhs-subst-keeps-finite-rhs*:
⟦*finite rhs*; *finite yrhs*⟧ ⟹ *finite* (*rhs-subst rhs Y yrhs*)
by (*auto simp*:*rhs-subst-def append-keeps-finite*)

lemma *eqs-subst-keeps-finite*:
assumes *finite*:*finite* (*ES*:: (*string set* × *rhs-item set*) *set*)
shows *finite* (*eqs-subst ES Y yrhs*)
proof −
have *finite* {(*Ya*, *rhs-subst yrhsa Y yrhs*) |*Ya yrhsa*. (*Ya*, *yrhsa*) ∈ *ES*}
(**is** *finite ?A*)
proof−
def *eqns'* ≡ {((*Ya*::*string set*), *yrhsa*)| *Ya yrhsa*. (*Ya*, *yrhsa*) ∈ *ES*}
def *h* ≡ λ ((*Ya*::*string set*), *yrhsa*). (*Ya*, *rhs-subst yrhsa Y yrhs*)
have *finite* (*h* ‘ *eqns'*) **using** *finite h-def eqns'-def* **by** *auto*
moreover have *?A* = *h* ‘ *eqns'* **by** (*auto simp*:*h-def eqns'-def*)
ultimately show *?thesis* **by** *auto*
qed
thus *?thesis* **by** (*simp add*:*eqs-subst-def*)
qed

lemma *eqs-subst-keeps-finite-rhs*:
⟦*finite-rhs ES*; *finite yrhs*⟧ ⟹ *finite-rhs* (*eqs-subst ES Y yrhs*)
by (*auto intro*:*rhs-subst-keeps-finite-rhs simp add*:*eqs-subst-def finite-rhs-def*)

lemma *append-rhs-keeps-cls*:
classes-of (*append-rhs-rexp rhs r*) = *classes-of rhs*
apply (*auto simp*:*classes-of-def append-rhs-rexp-def*)
apply (*case-tac xa*, *auto simp*:*image-def*)
by (*rule-tac x* = *SEQ ra r* **in** *exI*, *rule-tac x* = *Trn x ra* **in** *bexI*, *simp*+)

lemma *arden-variate-removes-cl*:
classes-of (*arden-variate Y yrhs*) = *classes-of yrhs* − {*Y*}
apply (*simp add*:*arden-variate-def append-rhs-keeps-cls trns-of-def*)
by (*auto simp*:*classes-of-def*)

lemma *lefts-of-keeps-cls*:
lefts-of (*eqs-subst ES Y yrhs*) = *lefts-of ES*
by (*auto simp*:*lefts-of-def eqs-subst-def*)

lemma *rhs-subst-updates-cls*:
X ∉ *classes-of xrhs* ⟹
classes-of (*rhs-subst rhs X xrhs*) = *classes-of rhs* ∪ *classes-of xrhs* − {*X*}
apply (*simp only*:*rhs-subst-def append-rhs-keeps-cls*

```
                              classes-of-union-distrib[THEN sym])
by (auto simp:classes-of-def trns-of-def)

lemma eqs-subst-keeps-self-contained:
  fixes Y
  assumes sc: self-contained (ES ∪ {(Y, yrhs)}) (is self-contained ?A)
  shows self-contained (eqs-subst ES Y (arden-variate Y yrhs))
                                           (is self-contained ?B)
proof−
  { fix X xrhs'
    assume (X, xrhs') ∈ ?B
    then obtain xrhs
      where xrhs-xrhs': xrhs' = rhs-subst xrhs Y (arden-variate Y yrhs)
      and X-in: (X, xrhs) ∈ ES by (simp add:eqs-subst-def, blast)
    have classes-of xrhs' ⊆ lefts-of ?B
    proof−
      have lefts-of ?B = lefts-of ES by (auto simp add:lefts-of-def eqs-subst-def)
      moreover have classes-of xrhs' ⊆ lefts-of ES
      proof−
        have classes-of xrhs' ⊆
                   classes-of xrhs ∪ classes-of (arden-variate Y yrhs) − {Y}
        proof−
          have Y ∉ classes-of (arden-variate Y yrhs)
            using arden-variate-removes-cl by simp
          thus ?thesis using xrhs-xrhs' by (auto simp:rhs-subst-updates-cls)
        qed
        moreover have classes-of xrhs ⊆ lefts-of ES ∪ {Y} using X-in sc
          apply (simp only:self-contained-def lefts-of-union-distrib[THEN sym])
          by (drule-tac x = (X, xrhs) in bspec, auto simp:lefts-of-def)
        moreover have classes-of (arden-variate Y yrhs) ⊆ lefts-of ES ∪ {Y}
          using sc
          by (auto simp add:arden-variate-removes-cl self-contained-def lefts-of-def)
        ultimately show ?thesis by auto
      qed
      ultimately show ?thesis by simp
    qed
  } thus ?thesis by (auto simp only:eqs-subst-def self-contained-def)
qed

lemma eqs-subst-satisfy-Inv:
  assumes Inv-ES: Inv (ES ∪ {(Y, yrhs)})
  shows Inv (eqs-subst ES Y (arden-variate Y yrhs))
proof −
  have finite-yrhs: finite yrhs
    using Inv-ES by (auto simp:Inv-def finite-rhs-def)
  have nonempty-yrhs: rhs-nonempty yrhs
    using Inv-ES by (auto simp:Inv-def ardenable-def)
  have Y-eq-yrhs: Y = L yrhs
    using Inv-ES by (simp only:Inv-def valid-eqns-def, blast)
```

```
  have distinct-equas (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES
    by (auto simp:distinct-equas-def eqs-subst-def Inv-def)
  moreover have finite (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES by (simp add:Inv-def eqs-subst-keeps-finite)
  moreover have finite-rhs (eqs-subst ES Y (arden-variate Y yrhs))
  proof−
    have finite-rhs ES using Inv-ES
      by (simp add:Inv-def finite-rhs-def)
    moreover have finite (arden-variate Y yrhs)
    proof −
      have finite yrhs using Inv-ES
        by (auto simp:Inv-def finite-rhs-def)
      thus ?thesis using arden-variate-keeps-finite by simp
    qed
    ultimately show ?thesis
      by (simp add:eqs-subst-keeps-finite-rhs)
  qed
  moreover have ardenable (eqs-subst ES Y (arden-variate Y yrhs))
  proof −
    { fix X rhs
      assume (X, rhs) ∈ ES
      hence rhs-nonempty rhs  using prems Inv-ES
        by (simp add:Inv-def ardenable-def)
      with nonempty-yrhs
      have rhs-nonempty (rhs-subst rhs Y (arden-variate Y yrhs))
        by (simp add:nonempty-yrhs
               rhs-subst-keeps-nonempty arden-variate-keeps-nonempty)
    } thus ?thesis by (auto simp add:ardenable-def eqs-subst-def)
  qed
  moreover have valid-eqns (eqs-subst ES Y (arden-variate Y yrhs))
  proof−
    have Y = L (arden-variate Y yrhs)
      using Y-eq-yrhs Inv-ES finite-yrhs nonempty-yrhs
      by (rule-tac arden-variate-keeps-eq, (simp add:rexp-of-empty)+)
    thus ?thesis using Inv-ES
      by (clarsimp simp add:valid-eqns-def
              eqs-subst-def rhs-subst-keeps-eq Inv-def finite-rhs-def
                  simp del:L-rhs.simps)
  qed
  moreover have
    non-empty-subst: non-empty (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES by (auto simp:Inv-def non-empty-def eqs-subst-def)
  moreover
  have self-subst: self-contained (eqs-subst ES Y (arden-variate Y yrhs))
    using Inv-ES eqs-subst-keeps-self-contained by (simp add:Inv-def)
  ultimately show ?thesis using Inv-ES by (simp add:Inv-def)
qed
```

```
lemma eqs-subst-card-le:
  assumes finite: finite (ES::(string set × rhs-item set) set)
  shows card (eqs-subst ES Y yrhs) <= card ES
proof−
  def f ≡ λ x. ((fst x)::string set, rhs-subst (snd x) Y yrhs)
  have eqs-subst ES Y yrhs = f ‘ ES
    apply (auto simp:eqs-subst-def f-def image-def)
    by (rule-tac x = (Ya, yrhsa) in bexI, simp+)
  thus ?thesis using finite by (auto intro:card-image-le)
qed

lemma eqs-subst-cls-remains:
  (X, xrhs) ∈ ES ⟹ ∃ xrhs′. (X, xrhs′) ∈ (eqs-subst ES Y yrhs)
by (auto simp:eqs-subst-def)

lemma card-noteq-1-has-more:
  assumes card:card S ≠ 1
  and e-in: e ∈ S
  and finite: finite S
  obtains e′ where e′ ∈ S ∧ e ≠ e′
proof−
  have card (S − {e}) > 0
  proof −
    have card S > 1 using card e-in finite
      by (case-tac card S, auto)
    thus ?thesis using finite e-in by auto
  qed
  hence S − {e} ≠ {} using finite by (rule-tac notI, simp)
  thus (⋀e′. e′ ∈ S ∧ e ≠ e′ ⟹ thesis) ⟹ thesis by auto
qed

lemma iteration-step:
  assumes Inv-ES: Inv ES
  and     X-in-ES: (X, xrhs) ∈ ES
  and     not-T: card ES ≠ 1
  shows ∃ ES′. (Inv ES′ ∧ (∃ xrhs′.(X, xrhs′) ∈ ES′)) ∧
               (card ES′, card ES) ∈ less-than (is ∃ ES′. ?P ES′)
proof −
  have finite-ES: finite ES using Inv-ES by (simp add:Inv-def)
  then obtain Y yrhs
    where Y-in-ES: (Y, yrhs) ∈ ES and not-eq: (X, xrhs) ≠ (Y, yrhs)
    using not-T X-in-ES by (drule-tac card-noteq-1-has-more, auto)
  def ES′ == ES − {(Y, yrhs)}
  let ?ES′′ = eqs-subst ES′ Y (arden-variate Y yrhs)
  have ?P ?ES′′
  proof −
    have Inv ?ES′′ using Y-in-ES Inv-ES
      by (rule-tac eqs-subst-satisfy-Inv, simp add:ES′-def insert-absorb)
    moreover have ∃xrhs′. (X, xrhs′) ∈ ?ES′′  using not-eq X-in-ES
```

```
    by (rule-tac ES = ES' in eqs-subst-cls-remains, auto simp add:ES'-def)
  moreover have (card ?ES'', card ES) ∈ less-than
  proof −
    have finite ES' using finite-ES ES'-def by auto
    moreover have card ES' < card ES using finite-ES Y-in-ES
      by (auto simp:ES'-def card-gt-0-iff intro:diff-Suc-less)
    ultimately show ?thesis
      by (auto dest:eqs-subst-card-le elim:le-less-trans)
  qed
  ultimately show ?thesis by simp
 qed
 thus ?thesis by blast
qed
```

5.1.4 Conclusion of the proof

From this point until *hard-direction*, the hard direction is proved through a simple application of the iteration principle.

```
lemma iteration-conc:
 assumes history: Inv ES
 and     X-in-ES: ∃ xrhs. (X, xrhs) ∈ ES
 shows
 ∃ ES'. (Inv ES' ∧ (∃ xrhs'. (X, xrhs') ∈ ES')) ∧ card ES' = 1
                                              (is ∃ ES'. ?P ES')
proof (cases card ES = 1)
 case True
 thus ?thesis using history X-in-ES
   by blast
next
 case False
 thus ?thesis using history iteration-step X-in-ES
   by (rule-tac f = card in wf-iter, auto)
qed

lemma last-cl-exists-rexp:
 assumes ES-single: ES = {(X, xrhs)}
 and Inv-ES: Inv ES
 shows ∃ (r::rexp). L r = X (is ∃ r. ?P r)
proof−
 def A ≡ arden-variate X xrhs
 have ?P (⨄{r. Lam r ∈ A})
 proof −
   have L (⨄{r. Lam r ∈ A}) = L ({Lam r | r. Lam r ∈  A})
   proof(rule rexp-of-lam-eq-lam-set)
     show finite A
       unfolding A-def
       using Inv-ES ES-single
       by (rule-tac arden-variate-keeps-finite)
          (auto simp add: Inv-def finite-rhs-def)
```

```
    qed
    also have ... = L A
    proof-
      have {Lam r | r. Lam r ∈ A} = A
      proof-
        have classes-of A = {} using Inv-ES ES-single
          unfolding A-def
          by (simp add:arden-variate-removes-cl
                    self-contained-def Inv-def lefts-of-def)
        thus ?thesis
          unfolding A-def
          by (auto simp only: classes-of-def, case-tac x, auto)
      qed
      thus ?thesis by simp
    qed
    also have ... = X
    unfolding A-def
    proof(rule arden-variate-keeps-eq [THEN sym])
      show X = L xrhs using Inv-ES ES-single
        by (auto simp only:Inv-def valid-eqns-def)
    next
      from Inv-ES ES-single show [] ∉ L (⨄{r. Trn X r ∈ xrhs})
        by(simp add:Inv-def ardenable-def rexp-of-empty finite-rhs-def)
    next
      from Inv-ES ES-single show finite xrhs
        by (simp add:Inv-def finite-rhs-def)
    qed
    finally show ?thesis by simp
  qed
  thus ?thesis by auto
qed

lemma every-eqcl-has-reg:
  assumes finite-CS: finite (UNIV // (≈Lang))
  and X-in-CS: X ∈ (UNIV // (≈Lang))
  shows ∃ (reg::rexp). L reg = X (is ∃ r. ?E r)
proof -
  from X-in-CS have ∃ xrhs. (X, xrhs) ∈ (eqs (UNIV // (≈Lang)))
    by (auto simp:eqs-def init-rhs-def)
  then obtain ES xrhs where Inv-ES: Inv ES
    and X-in-ES: (X, xrhs) ∈ ES
    and card-ES: card ES = 1
    using finite-CS X-in-CS init-ES-satisfy-Inv iteration-conc
    by blast
  hence ES-single-equa: ES = {(X, xrhs)}
    by (auto simp:Inv-def dest!:card-Suc-Diff1 simp:card-eq-0-iff)
  thus ?thesis using Inv-ES
    by (rule last-cl-exists-rexp)
qed
```

theorem *hard-direction*:
  assumes *finite-CS*: *finite* (*UNIV* // ≈*A*)
  shows ∃ *r*::*rexp*. *A* = *L* *r*
proof −
  have ∀ *X* ∈ (*UNIV* // ≈*A*). ∃ *reg*::*rexp*. *X* = *L* *reg*
    using *finite-CS* *every-eqcl-has-reg* **by** *blast*
  then obtain *f*
    where *f-prop*: ∀ *X* ∈ (*UNIV* // ≈*A*). *X* = *L* ((*f* *X*)::*rexp*)
    by (*auto dest*: *bchoice*)
  def *rs* ≡ *f* ‘ (*finals* *A*)
  have *A* = ⋃ (*finals* *A*) **using** *lang-is-union-of-finals* **by** *auto*
  also have ... = *L* (⨄*rs*)
  proof −
    have *finite rs*
    proof −
      have *finite* (*finals* *A*)
        using *finite-CS* *finals-in-partitions*[*of* *A*]
        by (*erule-tac finite-subset*, *simp*)
      thus *?thesis* **using** *rs-def* **by** *auto*
    qed
    thus *?thesis*
      using *f-prop* *rs-def* *finals-in-partitions*[*of* *A*] **by** *auto*
  qed
  finally show *?thesis* **by** *blast*
qed

end

6 List prefixes and postfixes

theory *List-Prefix*
imports *List Main*
begin

6.1 Prefix order on lists

instantiation *list* :: (*type*) {*order*, *bot*}
begin

definition
  prefix-def: *xs* ≤ *ys* ⟷ (∃ *zs*. *ys* = *xs* @ *zs*)

definition
  strict-prefix-def: *xs* < *ys* ⟷ *xs* ≤ *ys* ∧ *xs* ≠ (*ys*::′*a list*)

definition
  bot = []

instance proof
qed (*auto simp add*: *prefix-def strict-prefix-def bot-list-def*)

end

lemma *prefixI* [*intro?*]: *ys* = *xs* @ *zs* ==> *xs* ≤ *ys*
 unfolding *prefix-def* **by** *blast*

lemma *prefixE* [*elim?*]:
 assumes *xs* ≤ *ys*
 obtains *zs* **where** *ys* = *xs* @ *zs*
 using *assms* **unfolding** *prefix-def* **by** *blast*

lemma *strict-prefixI'* [*intro?*]: *ys* = *xs* @ *z* # *zs* ==> *xs* < *ys*
 unfolding *strict-prefix-def prefix-def* **by** *blast*

lemma *strict-prefixE'* [*elim?*]:
 assumes *xs* < *ys*
 obtains *z zs* **where** *ys* = *xs* @ *z* # *zs*
proof −
 from ‹*xs* < *ys*› **obtain** *us* **where** *ys* = *xs* @ *us* **and** *xs* ≠ *ys*
  unfolding *strict-prefix-def prefix-def* **by** *blast*
 with *that* **show** *?thesis* **by** (*auto simp add*: *neq-Nil-conv*)
qed

lemma *strict-prefixI* [*intro?*]: *xs* ≤ *ys* ==> *xs* ≠ *ys* ==> *xs* < (*ys*::'*a list*)
 unfolding *strict-prefix-def* **by** *blast*

lemma *strict-prefixE* [*elim?*]:
 fixes *xs ys* :: '*a list*
 assumes *xs* < *ys*
 obtains *xs* ≤ *ys* **and** *xs* ≠ *ys*
 using *assms* **unfolding** *strict-prefix-def* **by** *blast*

6.2 Basic properties of prefixes

theorem *Nil-prefix* [*iff*]: [] ≤ *xs*
 by (*simp add*: *prefix-def*)

theorem *prefix-Nil* [*simp*]: (*xs* ≤ []) = (*xs* = [])
 by (*induct xs*) (*simp-all add*: *prefix-def*)

lemma *prefix-snoc* [*simp*]: (*xs* ≤ *ys* @ [*y*]) = (*xs* = *ys* @ [*y*] ∨ *xs* ≤ *ys*)
proof
 assume *xs* ≤ *ys* @ [*y*]
 then obtain *zs* **where** *zs*: *ys* @ [*y*] = *xs* @ *zs* **..**
 show *xs* = *ys* @ [*y*] ∨ *xs* ≤ *ys*
  by (*metis append-Nil2 butlast-append butlast-snoc prefixI zs*)
next

assume *xs* = *ys* @ [*y*] ∨ *xs* ≤ *ys*
then show *xs* ≤ *ys* @ [*y*]
by (*metis order-eq-iff strict-prefixE strict-prefixI′ xt1*(*7*))
qed

lemma *Cons-prefix-Cons* [*simp*]: (*x* # *xs* ≤ *y* # *ys*) = (*x* = *y* ∧ *xs* ≤ *ys*)
by (*auto simp add*: *prefix-def*)

lemma *less-eq-list-code* [*code*]:
([]::′*a*::{*equal*, *ord*} *list*) ≤ *xs* ⟷ *True*
(*x*::′*a*::{*equal*, *ord*}) # *xs* ≤ [] ⟷ *False*
(*x*::′*a*::{*equal*, *ord*}) # *xs* ≤ *y* # *ys* ⟷ *x* = *y* ∧ *xs* ≤ *ys*
by *simp-all*

lemma *same-prefix-prefix* [*simp*]: (*xs* @ *ys* ≤ *xs* @ *zs*) = (*ys* ≤ *zs*)
by (*induct xs*) *simp-all*

lemma *same-prefix-nil* [*iff*]: (*xs* @ *ys* ≤ *xs*) = (*ys* = [])
by (*metis append-Nil2 append-self-conv order-eq-iff prefixI*)

lemma *prefix-prefix* [*simp*]: *xs* ≤ *ys* ==> *xs* ≤ *ys* @ *zs*
by (*metis order-le-less-trans prefixI strict-prefixE strict-prefixI*)

lemma *append-prefixD*: *xs* @ *ys* ≤ *zs* ⟹ *xs* ≤ *zs*
by (*auto simp add*: *prefix-def*)

theorem *prefix-Cons*: (*xs* ≤ *y* # *ys*) = (*xs* = [] ∨ (∃ *zs*. *xs* = *y* # *zs* ∧ *zs* ≤ *ys*))
by (*cases xs*) (*auto simp add*: *prefix-def*)

theorem *prefix-append*:
(*xs* ≤ *ys* @ *zs*) = (*xs* ≤ *ys* ∨ (∃ *us*. *xs* = *ys* @ *us* ∧ *us* ≤ *zs*))
apply (*induct zs rule*: *rev-induct*)
apply *force*
apply (*simp del*: *append-assoc add*: *append-assoc* [*symmetric*])
apply (*metis append-eq-appendI*)
done

lemma *append-one-prefix*:
xs ≤ *ys* ==> *length xs* < *length ys* ==> *xs* @ [*ys* ! *length xs*] ≤ *ys*
unfolding *prefix-def*
by (*metis Cons-eq-appendI append-eq-appendI append-eq-conv-conj*
eq-Nil-appendI nth-drop′)

theorem *prefix-length-le*: *xs* ≤ *ys* ==> *length xs* ≤ *length ys*
by (*auto simp add*: *prefix-def*)

lemma *prefix-same-cases*:
(xs_1::′*a list*) ≤ *ys* ⟹ xs_2 ≤ *ys* ⟹ xs_1 ≤ xs_2 ∨ xs_2 ≤ xs_1
unfolding *prefix-def* **by** (*metis append-eq-append-conv2*)

```
lemma set-mono-prefix: xs ≤ ys ⟹ set xs ⊆ set ys
  by (auto simp add: prefix-def)

lemma take-is-prefix: take n xs ≤ xs
  unfolding prefix-def by (metis append-take-drop-id)

lemma map-prefixI: xs ≤ ys ⟹ map f xs ≤ map f ys
  by (auto simp: prefix-def)

lemma prefix-length-less: xs < ys ⟹ length xs < length ys
  by (auto simp: strict-prefix-def prefix-def)

lemma strict-prefix-simps [simp, code]:
  xs < [] ⟷ False
  [] < x # xs ⟷ True
  x # xs < y # ys ⟷ x = y ∧ xs < ys
  by (simp-all add: strict-prefix-def cong: conj-cong)

lemma take-strict-prefix: xs < ys ⟹ take n xs < ys
  apply (induct n arbitrary: xs ys)
   apply (case-tac ys, simp-all)[1]
  apply (metis order-less-trans strict-prefixI take-is-prefix)
  done

lemma not-prefix-cases:
  assumes pfx: ¬ ps ≤ ls
  obtains
    (c1) ps ≠ [] and ls = []
  | (c2) a as x xs where ps = a#as and ls = x#xs and x = a and ¬ as ≤ xs
  | (c3) a as x xs where ps = a#as and ls = x#xs and x ≠ a
proof (cases ps)
  case Nil then show ?thesis using pfx by simp
next
  case (Cons a as)
  note c = ‹ps = a#as›
  show ?thesis
  proof (cases ls)
    case Nil then show ?thesis by (metis append-Nil2 pfx c1 same-prefix-nil)
  next
    case (Cons x xs)
    show ?thesis
    proof (cases x = a)
      case True
      have ¬ as ≤ xs using pfx c Cons True by simp
      with c Cons True show ?thesis by (rule c2)
    next
      case False
      with c Cons show ?thesis by (rule c3)
```

```
    qed
  qed
qed

lemma not-prefix-induct [consumes 1, case-names Nil Neq Eq]:
  assumes np: ¬ ps ≤ ls
    and base: ⋀x xs. P (x#xs) []
    and r1: ⋀x xs y ys. x ≠ y ⟹ P (x#xs) (y#ys)
    and r2: ⋀x xs y ys. ⟦ x = y; ¬ xs ≤ ys; P xs ys ⟧ ⟹ P (x#xs) (y#ys)
  shows P ps ls using np
proof (induct ls arbitrary: ps)
  case Nil then show ?case
    by (auto simp: neq-Nil-conv elim!: not-prefix-cases intro!: base)
next
  case (Cons y ys)
  then have npfx: ¬ ps ≤ (y # ys) by simp
  then obtain x xs where pv: ps = x # xs
    by (rule not-prefix-cases) auto
  show ?case by (metis Cons.hyps Cons-prefix-Cons npfx pv r1 r2)
qed
```

6.3 Parallel lists

```
definition
  parallel :: 'a list => 'a list => bool  (infixl ∥ 50) where
  (xs ∥ ys) = (¬ xs ≤ ys ∧ ¬ ys ≤ xs)

lemma parallelI [intro]: ¬ xs ≤ ys ==> ¬ ys ≤ xs ==> xs ∥ ys
  unfolding parallel-def by blast

lemma parallelE [elim]:
  assumes xs ∥ ys
  obtains ¬ xs ≤ ys ∧ ¬ ys ≤ xs
  using assms unfolding parallel-def by blast

theorem prefix-cases:
  obtains xs ≤ ys | ys < xs | xs ∥ ys
  unfolding parallel-def strict-prefix-def by blast

theorem parallel-decomp:
  xs ∥ ys ==> ∃ as b bs c cs. b ≠ c ∧ xs = as @ b # bs ∧ ys = as @ c # cs
proof (induct xs rule: rev-induct)
  case Nil
  then have False by auto
  then show ?case ..
next
  case (snoc x xs)
  show ?case
  proof (rule prefix-cases)
```

```
    assume le: xs ≤ ys
    then obtain ys' where ys: ys = xs @ ys' ..
    show ?thesis
    proof (cases ys')
      assume ys' = []
      then show ?thesis by (metis append-Nil2 parallelE prefixI snoc.prems ys)
    next
      fix c cs assume ys': ys' = c # cs
      then show ?thesis
        by (metis Cons-eq-appendI eq-Nil-appendI parallelE prefixI
          same-prefix-prefix snoc.prems ys)
    qed
  next
    assume ys < xs then have ys ≤ xs @ [x] by (simp add: strict-prefix-def)
    with snoc have False by blast
    then show ?thesis ..
  next
    assume xs ∥ ys
    with snoc obtain as b bs c cs where neq: (b::'a) ≠ c
      and xs: xs = as @ b # bs and ys: ys = as @ c # cs
      by blast
    from xs have xs @ [x] = as @ b # (bs @ [x]) by simp
    with neq ys show ?thesis by blast
  qed
qed

lemma parallel-append: a ∥ b ⟹ a @ c ∥ b @ d
  apply (rule parallelI)
    apply (erule parallelE, erule conjE,
      induct rule: not-prefix-induct, simp+)+
  done

lemma parallel-appendI: xs ∥ ys ⟹ x = xs @ xs' ⟹ y = ys @ ys' ⟹ x ∥ y
  by (simp add: parallel-append)

lemma parallel-commute: a ∥ b ⟷ b ∥ a
  unfolding parallel-def by auto
```

6.4 Postfix order on lists

```
definition
  postfix :: 'a list => 'a list => bool  ((-/ >>= -) [51, 50] 50) where
  (xs >>= ys) = (∃ zs. xs = zs @ ys)

lemma postfixI [intro?]: xs = zs @ ys ==> xs >>= ys
  unfolding postfix-def by blast

lemma postfixE [elim?]:
  assumes xs >>= ys
```

```
  obtains zs where xs = zs @ ys
  using assms unfolding postfix-def by blast

lemma postfix-refl [iff]: xs >>= xs
  by (auto simp add: postfix-def)
lemma postfix-trans: ⟦xs >>= ys; ys >>= zs⟧ ⟹ xs >>= zs
  by (auto simp add: postfix-def)
lemma postfix-antisym: ⟦xs >>= ys; ys >>= xs⟧ ⟹ xs = ys
  by (auto simp add: postfix-def)

lemma Nil-postfix [iff]: xs >>= []
  by (simp add: postfix-def)
lemma postfix-Nil [simp]: ([] >>= xs) = (xs = [])
  by (auto simp add: postfix-def)

lemma postfix-ConsI: xs >>= ys ⟹ x#xs >>= ys
  by (auto simp add: postfix-def)
lemma postfix-ConsD: xs >>= y#ys ⟹ xs >>= ys
  by (auto simp add: postfix-def)

lemma postfix-appendI: xs >>= ys ⟹ zs @ xs >>= ys
  by (auto simp add: postfix-def)
lemma postfix-appendD: xs >>= zs @ ys ⟹ xs >>= ys
  by (auto simp add: postfix-def)

lemma postfix-is-subset: xs >>= ys ==> set ys ⊆ set xs
proof −
  assume xs >>= ys
  then obtain zs where xs = zs @ ys ..
  then show ?thesis by (induct zs) auto
qed

lemma postfix-ConsD2: x#xs >>= y#ys ==> xs >>= ys
proof −
  assume x#xs >>= y#ys
  then obtain zs where x#xs = zs @ y#ys ..
  then show ?thesis
    by (induct zs) (auto intro!: postfix-appendI postfix-ConsI)
qed

lemma postfix-to-prefix [code]: xs >>= ys ⟷ rev ys ≤ rev xs
proof
  assume xs >>= ys
  then obtain zs where xs = zs @ ys ..
  then have rev xs = rev ys @ rev zs by simp
  then show rev ys <= rev xs ..
next
  assume rev ys <= rev xs
  then obtain zs where rev xs = rev ys @ zs ..
```

```
  then have rev (rev xs) = rev zs @ rev (rev ys) by simp
  then have xs = rev zs @ ys by simp
  then show xs >>= ys ..
qed

lemma distinct-postfix: distinct xs ⟹ xs >>= ys ⟹ distinct ys
  by (clarsimp elim!: postfixE)

lemma postfix-map: xs >>= ys ⟹ map f xs >>= map f ys
  by (auto elim!: postfixE intro: postfixI)

lemma postfix-drop: as >>= drop n as
  unfolding postfix-def
  apply (rule exI [where x = take n as])
  apply simp
  done

lemma postfix-take: xs >>= ys ⟹ xs = take (length xs − length ys) xs @ ys
  by (clarsimp elim!: postfixE)

lemma parallelD1: x ∥ y ⟹ ¬ x ≤ y
  by blast

lemma parallelD2: x ∥ y ⟹ ¬ y ≤ x
  by blast

lemma parallel-Nil1 [simp]: ¬ x ∥ []
  unfolding parallel-def by simp

lemma parallel-Nil2 [simp]: ¬ [] ∥ x
  unfolding parallel-def by simp

lemma Cons-parallelI1: a ≠ b ⟹ a # as ∥ b # bs
  by auto

lemma Cons-parallelI2: ⟦ a = b; as ∥ bs ⟧ ⟹ a # as ∥ b # bs
  by (metis Cons-prefix-Cons parallelE parallelI)

lemma not-equal-is-parallel:
  assumes neq: xs ≠ ys
    and len: length xs = length ys
  shows xs ∥ ys
  using len neq
proof (induct rule: list-induct2)
  case Nil
  then show ?case by simp
next
  case (Cons a as b bs)
  have ih: as ≠ bs ⟹ as ∥ bs by fact
```

```
  show ?case
  proof (cases a = b)
    case True
    then have as ≠ bs using Cons by simp
    then show ?thesis by (rule Cons-parallelI2 [OF True ih])
  next
    case False
    then show ?thesis by (rule Cons-parallelI1)
  qed
qed

end

theory Prefix-subtract
  imports Main List-Prefix
begin
```

7 A small theory of prefix subtraction

The notion of *prefix-subtract* is need to make proofs more readable.

```
fun prefix-subtract :: 'a list ⇒ 'a list ⇒ 'a list (infix − 51)
where
  prefix-subtract []      xs      = []
| prefix-subtract (x#xs) []       = x#xs
| prefix-subtract (x#xs) (y#ys) = (if x = y then prefix-subtract xs ys else (x#xs))

lemma [simp]: (x @ y) − x = y
apply (induct x)
by (case-tac y, simp+)

lemma [simp]: x − x = []
by (induct x, auto)

lemma [simp]: x = xa @ y ⟹ x − xa = y
by (induct x, auto)

lemma [simp]: x − [] = x
by (induct x, auto)

lemma [simp]: (x − y = []) ⟹ (x ≤ y)
proof−
  have ∃ xa. x = xa @ (x − y) ∧ xa ≤ y
    apply (rule prefix-subtract.induct[of - x y], simp+)
    by (clarsimp, rule-tac x = y # xa in exI, simp+)
  thus (x − y = []) ⟹ (x ≤ y) by simp
qed

lemma diff-prefix:
```

$[\![c \le a - b;\ b \le a]\!] \Longrightarrow b\ @\ c \le a$
by (*auto elim*:*prefixE*)

lemma *diff-diff-appd*:
$[\![c < a - b;\ b < a]\!] \Longrightarrow (a - b) - c = a - (b\ @\ c)$
apply (*clarsimp simp*:*strict-prefix-def*)
by (*drule diff-prefix*, *auto elim*:*prefixE*)

lemma *app-eq-cases*[*rule-format*]:
$\forall\ x\ .\ x\ @\ y = m\ @\ n \longrightarrow (x \le m \lor m \le x)$
apply (*induct y*, *simp*)
apply (*clarify*, *drule-tac* $x = x\ @\ [a]$ **in** *spec*)
by (*clarsimp*, *auto simp*:*prefix-def*)

lemma *app-eq-dest*:
$x\ @\ y = m\ @\ n \Longrightarrow$
$(x \le m \land (m - x)\ @\ n = y) \lor (m \le x \land (x - m)\ @\ y = n)$
by (*frule-tac app-eq-cases*, *auto elim*:*prefixE*)

end

theory *Myhill-2*
imports *Myhill-1 List-Prefix Prefix-subtract*
begin

8 Direction *regular language* ⇒*finite partition*

8.1 The scheme

The following convenient notation $x \approx A\ y$ means: string x and y are equivalent with respect to language A.

definition
str-eq :: *string* ⇒ *lang* ⇒ *string* ⇒ *bool* (- ≈- -)
where
$x \approx A\ y \equiv (x,\ y) \in (\approx A)$

The main lemma (*rexp-imp-finite*) is proved by a structural induction over regular expressions. While base cases (cases for *NULL*, *EMPTY*, *CHAR*) are quite straight forward, the inductive cases are rather involved. What we have when starting to prove these inductive caes is that the partitions induced by the componet language are finite. The basic idea to show the finiteness of the partition induced by the composite language is to attach a tag $tag(x)$ to every string x. The tags are made of equivalent classes from the component partitions. Let *tag* be the tagging function and *Lang* be the composite language, it can be proved that if strings with the same tag are equivalent with respect to *Lang*, expressed as:

$$tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$$

then the partition induced by *Lang* must be finite. There are two arguments for this. The first goes as the following:

1. First, the tagging function *tag* induces an equivalent relation (=*tag*=) (defiintion of *f-eq-rel* and lemma *equiv-f-eq-rel*).

2. It is shown that: if the range of *tag* (denoted *range*(*tag*)) is finite, the partition given rise by (=*tag*=) is finite (lemma *finite-eq-f-rel*). Since tags are made from equivalent classes from component partitions, and the inductive hypothesis ensures the finiteness of these partitions, it is not difficult to prove the finiteness of *range*(*tag*).

3. It is proved that if equivalent relation *R1* is more refined than *R2* (expressed as $R1 \subseteq R2$), and the partition induced by *R1* is finite, then the partition induced by *R2* is finite as well (lemma *refined-partition-finite*).

4. The injectivity assumption $tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$ implies that (=*tag*=) is more refined than (≈*Lang*).

5. Combining the points above, we have: the partition induced by language *Lang* is finite (lemma *tag-finite-imageD*).

definition
  f-eq-rel (=-=)
where
  $(=f=) = \{(x, y) \mid x\ y.\ f\ x = f\ y\}$

lemma *equiv-f-eq-rel*:*equiv UNIV* (=*f*=)
  by (*auto simp*:*equiv-def f-eq-rel-def refl-on-def sym-def trans-def*)

lemma *finite-range-image*: *finite* (*range f*) ⟹ *finite* (*f* ‘ *A*)
  by (*rule-tac* $B = \{y.\ \exists x.\ y = f\ x\}$ **in** *finite-subset*, *auto simp*:*image-def*)

lemma *finite-eq-f-rel*:
  assumes *rng-fnt*: *finite* (*range tag*)
  shows *finite* (*UNIV* // (=*tag*=))
proof −
  let *?f* = *op* ‘ *tag* **and** *?A* = (*UNIV* // (=*tag*=))
  show *?thesis*
  proof (*rule-tac f* = *?f* **and** *A* = *?A* **in** *finite-imageD*)
    — The finiteness of *f*-image is a simple consequence of assumption *rng-fnt*:
    show *finite* (*?f* ‘ *?A*)
    proof −
      have ∀ *X*. *?f X* ∈ (*Pow* (*range tag*)) **by** (*auto simp*:*image-def Pow-def*)
      moreover from *rng-fnt* **have** *finite* (*Pow* (*range tag*)) **by** *simp*
      ultimately have *finite* (*range ?f*)
        by (*auto simp only*:*image-def intro*:*finite-subset*)
      from *finite-range-image* [*OF this*] **show** *?thesis* **.**
    qed

```
  next
    — The injectivity of f-image is a consequence of the definition of (=tag=):
    show inj-on ?f ?A
    proof−
      { fix X Y
        assume X-in: X ∈ ?A
          and  Y-in: Y ∈ ?A
          and  tag-eq: ?f X = ?f Y
        have X = Y
        proof −
          from X-in Y-in tag-eq
          obtain x y
            where x-in: x ∈ X and y-in: y ∈ Y and eq-tg: tag x = tag y
            unfolding quotient-def Image-def str-eq-rel-def
                              str-eq-def image-def f-eq-rel-def
            apply simp by blast
          with X-in Y-in show ?thesis
            by (auto simp:quotient-def str-eq-rel-def str-eq-def f-eq-rel-def)
        qed
      } thus ?thesis unfolding inj-on-def by auto
    qed
  qed
qed

lemma finite-image-finite: ⟦∀ x ∈ A. f x ∈ B; finite B⟧ ⟹ finite (f ‘ A)
  by (rule finite-subset [of - B], auto)

lemma refined-partition-finite:
  fixes R1 R2 A
  assumes fnt: finite (A // R1)
  and refined: R1 ⊆ R2
  and eq1: equiv A R1 and eq2: equiv A R2
  shows finite (A // R2)
proof −
  let ?f = λ X. {R1 “ {x} | x. x ∈ X}
    and ?A = (A // R2) and ?B = (A // R1)
  show ?thesis
  proof(rule-tac f = ?f and A = ?A in finite-imageD)
    show finite (?f ‘ ?A)
    proof(rule finite-subset [of - Pow ?B])
      from fnt show finite (Pow (A // R1)) by simp
    next
      from eq2
      show  ?f ‘ A // R2 ⊆ Pow ?B
        unfolding image-def Pow-def quotient-def
        apply auto
        by (rule-tac x = xb in bexI, simp,
                unfold equiv-def sym-def refl-on-def, blast)
    qed
```

```
  next
    show inj-on ?f ?A
    proof -
      { fix X Y
        assume X-in: X ∈ ?A and Y-in: Y ∈ ?A
          and eq-f: ?f X = ?f Y (is ?L = ?R)
        have X = Y using X-in
        proof(rule quotientE)
          fix x
          assume X = R2 `` {x} and x ∈ A with eq2
          have x-in: x ∈ X
            unfolding equiv-def quotient-def refl-on-def by auto
          with eq-f have R1 `` {x} ∈ ?R by auto
          then obtain y where
            y-in: y ∈ Y and eq-r: R1 `` {x} = R1 ``{y} by auto
          have (x, y) ∈ R1
          proof -
            from x-in X-in y-in Y-in eq2
            have x ∈ A and y ∈ A
              unfolding equiv-def quotient-def refl-on-def by auto
            from eq-equiv-class-iff [OF eq1 this] and eq-r
            show ?thesis by simp
          qed
          with refined have xy-r2: (x, y) ∈ R2 by auto
          from quotient-eqI [OF eq2 X-in Y-in x-in y-in this]
          show ?thesis .
        qed
      } thus ?thesis by (auto simp:inj-on-def)
    qed
  qed
qed

lemma equiv-lang-eq: equiv UNIV (≈Lang)
  unfolding equiv-def str-eq-rel-def sym-def refl-on-def trans-def
  by blast

lemma tag-finite-imageD:
  fixes tag
  assumes rng-fnt: finite (range tag)
  — Suppose the rang of tagging fucntion tag is finite.
  and same-tag-eqvt: ⋀ m n. tag m = tag (n::string) ⟹ m ≈Lang n
  — And strings with same tag are equivalent
  shows finite (UNIV // (≈Lang))
proof -
  let ?R1 = (=tag=)
  show ?thesis
  proof(rule-tac refined-partition-finite [of - ?R1])
    from finite-eq-f-rel [OF rng-fnt]
     show finite (UNIV // =tag=) .
```

```
  next
    from same-tag-eqvt
    show (=tag=) ⊆ (≈Lang)
      by (auto simp:f-eq-rel-def str-eq-def)
  next
    from equiv-f-eq-rel
    show equiv UNIV (=tag=) by blast
  next
    from equiv-lang-eq
    show equiv UNIV (≈Lang) by blast
  qed
qed
```

A more concise, but less intelligible argument for *tag-finite-imageD* is given as the following. The basic idea is still using standard library lemma *finite-imageD*:

$$[\![finite\ (f\ `\ A);\ inj\text{-}on\ f\ A]\!] \Longrightarrow finite\ A$$

which says: if the image of injective function f over set A is finite, then A must be finte, as we did in the lemmas above.

```
lemma
  fixes tag
  assumes rng-fnt: finite (range tag)
  — Suppose the rang of tagging fucntion tag is finite.
  and same-tag-eqvt: ⋀ m n. tag m = tag (n::string) ⟹ m ≈Lang n
  — And strings with same tag are equivalent
  shows finite (UNIV // (≈Lang))
  — Then the partition generated by (≈Lang) is finite.
proof –
  — The particular f and A used in finite-imageD are:
  let ?f = op ` tag and ?A = (UNIV // ≈Lang)
  show ?thesis
  proof (rule-tac f = ?f and A = ?A in finite-imageD)
    — The finiteness of f-image is a simple consequence of assumption rng-fnt:
    show finite (?f ` ?A)
    proof –
      have ∀ X. ?f X ∈ (Pow (range tag)) by (auto simp:image-def Pow-def)
      moreover from rng-fnt have finite (Pow (range tag)) by simp
      ultimately have finite (range ?f)
        by (auto simp only:image-def intro:finite-subset)
      from finite-range-image [OF this] show ?thesis .
    qed
  next
    — The injectivity of f is the consequence of assumption same-tag-eqvt:
    show inj-on ?f ?A
    proof–
      { fix X Y
        assume X-in: X ∈ ?A
          and  Y-in: Y ∈ ?A
```

```
        and  tag-eq: ?f X = ?f Y
      have X = Y
      proof -
        from X-in Y-in tag-eq
        obtain x y where x-in: x ∈ X and y-in: y ∈ Y and eq-tg: tag x = tag y
          unfolding quotient-def Image-def str-eq-rel-def str-eq-def image-def
          apply simp by blast
        from same-tag-eqvt [OF eq-tg] have x ≈Lang y .
        with X-in Y-in x-in y-in
        show ?thesis by (auto simp:quotient-def str-eq-rel-def str-eq-def)
      qed
    } thus ?thesis unfolding inj-on-def by auto
  qed
 qed
qed
```

8.2 The proof

Each case is given in a separate section, as well as the final main lemma. Detailed explainations accompanied by illustrations are given for non-trivial cases.

For ever inductive case, there are two tasks, the easier one is to show the range finiteness of of the tagging function based on the finiteness of component partitions, the difficult one is to show that strings with the same tag are equivalent with respect to the composite language. Suppose the composite language be *Lang*, tagging function be *tag*, it amounts to show:

$$tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$$

expanding the definition of $\approx Lang$, it amounts to show:

$$tag(x) = tag(y) \Longrightarrow (\forall\ z.\ x@z \in Lang \longleftrightarrow y@z \in Lang)$$

Because the assumed tag equlity $tag(x) = tag(y)$ is symmetric, it is suffcient to show just one direction:

$$\bigwedge x\ y\ z.\ [\![tag(x) = tag(y);\ x@z \in Lang]\!] \Longrightarrow y@z \in Lang$$

This is the pattern followed by every inductive case.

8.2.1 The base case for *NULL*

```
lemma quot-null-eq:
  shows (UNIV // ≈{}) = ({UNIV}::lang set)
  unfolding quotient-def Image-def str-eq-rel-def by auto

lemma quot-null-finiteI [intro]:
  shows finite ((UNIV // ≈{})::lang set)
unfolding quot-null-eq by simp
```

8.2.2 The base case for *EMPTY*

lemma *quot-empty-subset*:
  $UNIV\ //\ (\approx\{[]\}) \subseteq \{\{[]\},\ UNIV - \{[]\}\}$
proof
  fix x
  assume $x \in UNIV\ //\ \approx\{[]\}$
  then obtain y **where** h: $x = \{z.\ (y, z) \in \approx\{[]\}\}$
    unfolding *quotient-def Image-def* **by** *blast*
  show $x \in \{\{[]\},\ UNIV - \{[]\}\}$
  proof (*cases* $y = []$)
    case *True* **with** h
    have $x = \{[]\}$ **by** (*auto simp*: *str-eq-rel-def*)
    thus *?thesis* **by** *simp*
  next
    case *False* **with** h
    have $x = UNIV - \{[]\}$ **by** (*auto simp*: *str-eq-rel-def*)
    thus *?thesis* **by** *simp*
  qed
qed

lemma *quot-empty-finiteI* [*intro*]:
  shows *finite* $(UNIV\ //\ (\approx\{[]\}))$
by (*rule finite-subset*[*OF quot-empty-subset*]) (*simp*)

8.2.3 The base case for *CHAR*

lemma *quot-char-subset*:
  $UNIV\ //\ (\approx\{[c]\}) \subseteq \{\{[]\},\{[c]\},\ UNIV - \{[],\ [c]\}\}$
proof
  fix x
  assume $x \in UNIV\ //\ \approx\{[c]\}$
  then obtain y **where** h: $x = \{z.\ (y, z) \in \approx\{[c]\}\}$
    unfolding *quotient-def Image-def* **by** *blast*
  show $x \in \{\{[]\},\{[c]\},\ UNIV - \{[],\ [c]\}\}$
  proof −
    { assume $y = []$ **hence** $x = \{[]\}$ **using** h
        by (*auto simp*:*str-eq-rel-def*)
    } moreover {
      assume $y = [c]$ **hence** $x = \{[c]\}$ **using** h
        by (*auto dest*!:*spec*[**where** $x = []$] *simp*:*str-eq-rel-def*)
    } moreover {
      assume $y \neq []$ **and** $y \neq [c]$
      hence $\forall\ z.\ (y\ @\ z) \neq [c]$ **by** (*case-tac* y, *auto*)
      moreover have $\bigwedge p.\ (p \neq [] \wedge p \neq [c]) = (\forall\ q.\ p\ @\ q \neq [c])$
        by (*case-tac* p, *auto*)
      ultimately have $x = UNIV - \{[],[c]\}$ **using** h
        by (*auto simp add*:*str-eq-rel-def*)
    } ultimately show *?thesis* **by** *blast*
  qed

qed

lemma *quot-char-finiteI* [*intro*]:
  shows *finite* (*UNIV* // ($\approx$\{[*c*]\}))
by (*rule finite-subset*[*OF quot-char-subset*]) (*simp*)

8.2.4 The inductive case for *ALT*

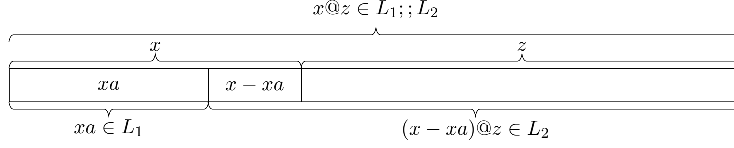
definition
  tag-str-ALT :: *lang* $\Rightarrow$ *lang* $\Rightarrow$ *string* $\Rightarrow$ (*lang* $\times$ *lang*)
where
  tag-str-ALT L1 L2 = (λx. ($\approx$*L1* ‘‘ \{x\}, $\approx$*L2* ‘‘ \{x\}))

lemma *quot-union-finiteI* [*intro*]:
  fixes *L1 L2*::*lang*
  assumes *finite1*: *finite* (*UNIV* // $\approx$*L1*)
  and *finite2*: *finite* (*UNIV* // $\approx$*L2*)
  shows *finite* (*UNIV* // $\approx$(*L1* $\cup$ *L2*))
proof (*rule-tac tag* = *tag-str-ALT L1 L2* **in** *tag-finite-imageD*)
  show $\bigwedge x\ y$. *tag-str-ALT L1 L2 x* = *tag-str-ALT L1 L2 y* $\Longrightarrow$ x $\approx$(*L1* $\cup$ *L2*) y
    unfolding *tag-str-ALT-def*
    unfolding *str-eq-def*
    unfolding *Image-def*
    unfolding *str-eq-rel-def*
    by *auto*
next
  have $*$: *finite* ((*UNIV* // $\approx$*L1*) $\times$ (*UNIV* // $\approx$*L2*))
    using *finite1 finite2* **by** *auto*
  show *finite* (*range* (*tag-str-ALT L1 L2*))
    unfolding *tag-str-ALT-def*
    apply(*rule finite-subset*[*OF* - $*$])
    unfolding *quotient-def*
    by *auto*
qed

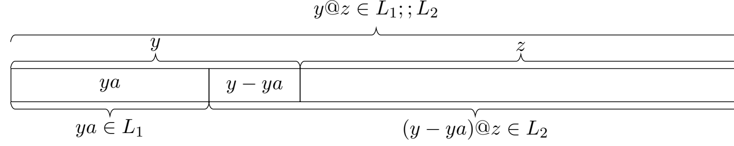
8.2.5 The inductive case for *SEQ*

For case *SEQ*, the language L is L_1 ;; L_2. Given x @ $z \in L_1$;; L_2, according to the defintion of L_1 ;; L_2, string x @ z can be splitted with the prefix in L_1 and suffix in L_2. The split point can either be in x (as shown in Fig. 1(a)), or in z (as shown in Fig. 1(c)). Whichever way it goes, the structure on x @ z cn be transfered faithfully onto y @ z (as shown in Fig. 1(b) and 1(d)) with the the help of the assumed tag equality. The following tag function *tag-str-SEQ* is such designed to facilitate such transfers and lemma *tag-str-SEQ-injI* formalizes the informal argument above. The details of structure transfer will be given their.

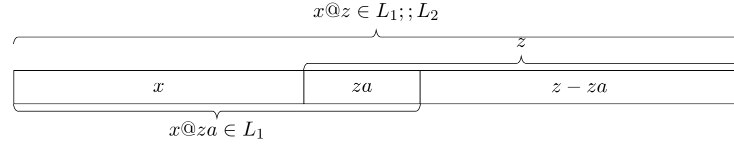
definition
  tag-str-SEQ :: *lang* $\Rightarrow$ *lang* $\Rightarrow$ *string* $\Rightarrow$ (*lang* $\times$ *lang set*)



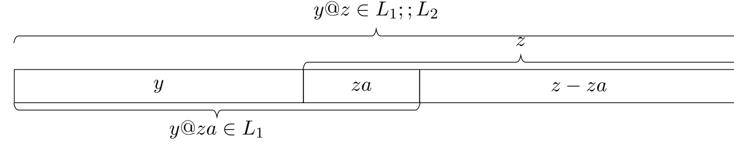
(a) First possible way to split $x@z$



(b) Transferred structure corresponding to the first way of splitting



(c) The second possible way to split $x@z$



(d) Transferred structure corresponding to the second way of splitting

Figure 1: The case for SEQ

where
  tag-str-SEQ L1 L2 =
    $(\lambda x.\ (\approx L1\ ``\ \{x\},\ \{(\approx L2\ ``\ \{x - xa\})\ |\ xa.\ \ xa \leq x \wedge xa \in L1\}))$

The following is a techical lemma which helps to split the $x\ @\ z \in L_1\ ;;\ L_2$ mentioned above.

lemma *append-seq-elim*:
  assumes $x\ @\ y \in L_1\ ;;\ L_2$
  shows $(\exists\ xa \leq x.\ xa \in L_1 \wedge (x - xa)\ @\ y \in L_2) \vee$
        $(\exists\ ya \leq y.\ (x\ @\ ya) \in L_1 \wedge (y - ya) \in L_2)$
proof−
  from *assms* **obtain** $s_1\ s_2$
    where *eq-xys*: $x\ @\ y = s_1\ @\ s_2$
    and *in-seq*: $s_1 \in L_1 \wedge s_2 \in L_2$
    by (*auto simp*:*Seq-def*)
  from *app-eq-dest* [*OF eq-xys*]
  have
    $(x \leq s_1 \wedge (s_1 - x)\ @\ s_2 = y) \vee (s_1 \leq x \wedge (x - s_1)\ @\ y = s_2)$

(**is** *?Split1* $\vee$ *?Split2*) **.**
moreover have *?Split1* $\Longrightarrow \exists\ ya \leq y.\ (x\ @\ ya) \in L_1 \wedge (y - ya) \in L_2$
using *in-seq* **by** (*rule-tac* $x = s_1 - x$ **in** *exI*, *auto elim:prefixE*)
moreover have *?Split2* $\Longrightarrow \exists\ xa \leq x.\ xa \in L_1 \wedge (x - xa)\ @\ y \in L_2$
using *in-seq* **by** (*rule-tac* $x = s_1$ **in** *exI*, *auto*)
ultimately show *?thesis* **by** *blast*
qed

lemma *tag-str-SEQ-injI*:
fixes $v\ w$
assumes *eq-tag*: *tag-str-SEQ* $L_1\ L_2\ v =$ *tag-str-SEQ* $L_1\ L_2\ w$
shows $v \approx(L_1\ ;;\ L_2)\ w$
proof−
— As explained before, a pattern for just one direction needs to be dealt with:
{ fix $x\ y\ z$
assume *xz-in-seq*: $x\ @\ z \in L_1\ ;;\ L_2$
and *tag-xy*: *tag-str-SEQ* $L_1\ L_2\ x =$ *tag-str-SEQ* $L_1\ L_2\ y$
have$y\ @\ z \in L_1\ ;;\ L_2$
proof−
— There are two ways to split $x@z$:
from *append-seq-elim* [*OF xz-in-seq*]
have $(\exists\ xa \leq x.\ xa \in L_1 \wedge (x - xa)\ @\ z \in L_2) \vee$
$(\exists\ za \leq z.\ (x\ @\ za) \in L_1 \wedge (z - za) \in L_2)$ **.**
— It can be shown that *?thesis* holds in either case:
moreover {
— The case for the first split:
fix xa
assume *h1*: $xa \leq x$ **and** *h2*: $xa \in L_1$ **and** *h3*: $(x - xa)\ @\ z \in L_2$
— The following subgoal implements the structure transfer:
obtain ya
where $ya \leq y$
and $ya \in L_1$
and $(y - ya)\ @\ z \in L_2$
proof −
By expanding the definition of

— *tag-str-SEQ* $L_1\ L_2\ x =$ *tag-str-SEQ* $L_1\ L_2\ y$

and extracting the second compoent, we get:
have $\{\approx L_2\ ``\ \{x - xa\}\ |xa.\ xa \leq x \wedge xa \in L_1\} =$
$\{\approx L_2\ ``\ \{y - ya\}\ |ya.\ ya \leq y \wedge ya \in L_1\}$ (**is** *?Left* = *?Right*)
using *tag-xy* **unfolding** *tag-str-SEQ-def* **by** *simp*
— Since $xa \leq x$ and $xa \in L_1$ hold, it is not difficult to show:
moreover have $\approx L_2\ ``\ \{x - xa\} \in$ *?Left* **using** *h1 h2* **by** *auto*
— Through tag equality, equivalent class $\approx L_2\ ``\ \{x - xa\}$ also belongs to the *?Right*:
ultimately have $\approx L_2\ ``\ \{x - xa\} \in$ *?Right* **by** *simp*
— From this, the counterpart of xa in y is obtained:
then obtain ya

```
        where eq-xya: ≈L₂ `` {x - xa} = ≈L₂ `` {y - ya}
        and pref-ya: ya ≤ y and ya-in: ya ∈ L₁
        by simp blast
      — It can be proved that ya has the desired property:
      have (y - ya)@z ∈ L₂
      proof -
        from eq-xya have (x - xa) ≈L₂ (y - ya)
          unfolding Image-def str-eq-rel-def str-eq-def by auto
        with h3 show ?thesis unfolding str-eq-rel-def str-eq-def by simp
      qed
      — Now, ya has all properties to be a qualified candidate:
      with pref-ya ya-in
      show ?thesis using that by blast
    qed
      — From the properties of ya, y @ z ∈ L₁ ;; L₂ is derived easily.
    hence y @ z ∈ L₁ ;; L₂ by (erule-tac prefixE, auto simp:Seq-def)
  } moreover {
    — The other case is even more simpler:
    fix za
    assume h1: za ≤ z and h2: (x @ za) ∈ L₁ and h3: z - za ∈ L₂
    have y @ za ∈ L₁
    proof-
      have ≈L₁ `` {x} = ≈L₁ `` {y}
        using tag-xy unfolding tag-str-SEQ-def by simp
      with h2 show ?thesis
        unfolding Image-def str-eq-rel-def str-eq-def by auto
    qed
    with h1 h3 have y @ z ∈ L₁ ;; L₂
      by (drule-tac A = L₁ in seq-intro, auto elim:prefixE)
  }
  ultimately show ?thesis by blast
 qed
}
— ?thesis is proved by exploiting the symmetry of eq-tag:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma quot-seq-finiteI [intro]:
  fixes L1 L2::lang
  assumes fin1: finite (UNIV // ≈L1)
  and     fin2: finite (UNIV // ≈L2)
  shows finite (UNIV // ≈(L1 ;; L2))
proof (rule-tac tag = tag-str-SEQ L1 L2 in tag-finite-imageD)
  show ⋀x y. tag-str-SEQ L1 L2 x = tag-str-SEQ L1 L2 y ⟹ x ≈(L1 ;; L2) y
    by (rule tag-str-SEQ-injI)
next
  have *: finite ((UNIV // ≈L1) × (Pow (UNIV // ≈L2)))
    using fin1 fin2 by auto
```

show *finite* (*range* (*tag-str-SEQ L1 L2*))
  unfolding *tag-str-SEQ-def*
  apply(*rule finite-subset*[*OF* - ∗])
  unfolding *quotient-def*
  by *auto*
qed

8.2.6 The inductive case for *STAR*

This turned out to be the trickiest case. The essential goal is to proved y @ $z \in L_1{*}$ under the assumptions that x @ $z \in L_1{*}$ and that x and y have the same tag. The reasoning goes as the following:

1. Since x @ $z \in L_1{*}$ holds, a prefix xa of x can be found such that $xa \in L_1{*}$ and $(x - xa)@z \in L_1{*}$, as shown in Fig. 2(a). Such a prefix always exists, $xa = []$, for example, is one.

2. There could be many but fintie many of such xa, from which we can find the longest and name it *xa-max*, as shown in Fig. 2(b).

3. The next step is to split z into za and zb such that $(x - xa\text{-}max)$ @ $za \in L_1$ and $zb \in L_1{*}$ as shown in Fig. 2(e). Such a split always exists because:

   (a) Because $(x - x\text{-}max)$ @ $z \in L_1{*}$, it can always be splitted into prefix a and suffix b, such that $a \in L_1$ and $b \in L_1{*}$, as shown in Fig. 2(c).

   (b) But the prefix a CANNOT be shorter than $x - xa\text{-}max$ (as shown in Fig. 2(d)), becasue otherwise, $ma\text{-}max@a$ would be in the same kind as *xa-max* but with a larger size, conflicting with the fact that *xa-max* is the longest.

4. By the assumption that x and y have the same tag, the structure on x @ z can be transferred to y @ z as shown in Fig. 2(f). The detailed steps are:

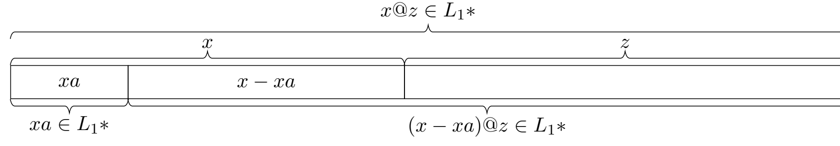
   (a) A y-prefix ya corresponding to xa can be found, which satisfies conditions: $ya \in L_1{*}$ and $(y - ya)@za \in L_1$.

   (b) Since we already know $zb \in L_1{*}$, we get $(y - ya)@za@zb \in L_1{*}$, and this is just $(y - ya)@z \in L_1{*}$.

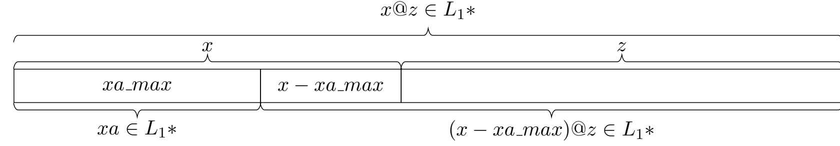
   (c) With fact $ya \in L_1{*}$, we finally get $y@z \in L_1{*}$.

The formal proof of lemma *tag-str-STAR-injI* faithfully follows this informal argument while the tagging function *tag-str-STAR* is defined to make the transfer in step **??** feasible.

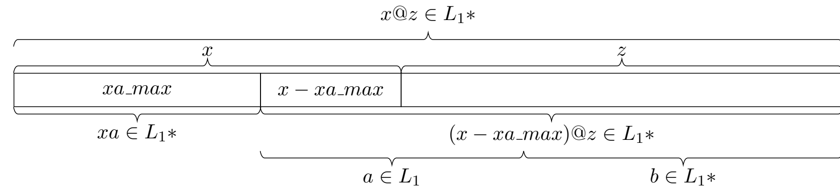
definition



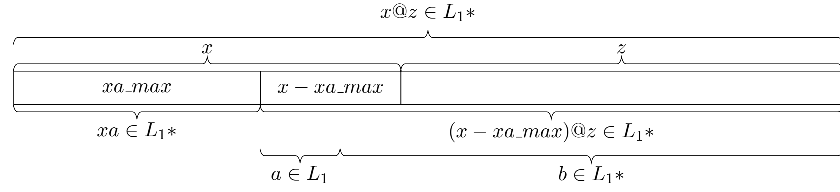
(a) First split



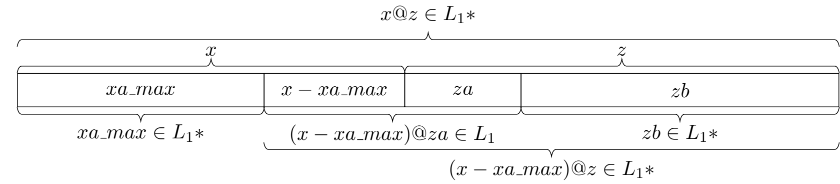
(b) Max split



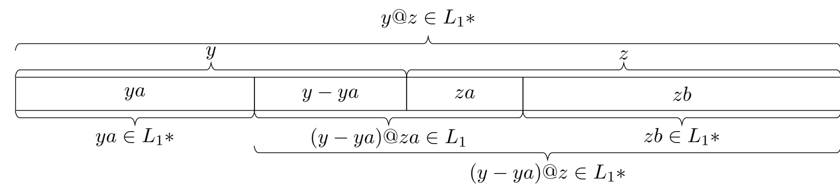
(c) Max split with a and b (the right situation)



(d) Max split with a and b (the wrong situation)



(e) Last split



(f) Structure transferred to y

Figure 2: The case for $STAR$

tag-str-STAR :: *lang* ⇒ *string* ⇒ *lang set*
where
tag-str-STAR L1 = (λ*x*. {≈*L1* ‘‘ {*x* − *xa*} | *xa*. *xa* < *x* ∧ *xa* ∈ *L1*⋆})

A technical lemma.

lemma *finite-set-has-max*: ⟦*finite A*; *A* ≠ {}⟧ ⟹
(∃ *max* ∈ *A*. ∀ *a* ∈ *A*. *f a* <= (*f max* :: *nat*))
proof (*induct rule*:*finite.induct*)
case *emptyI* **thus** *?case* **by** *simp*
next
case (*insertI A a*)
show *?case*
proof (*cases A* = {})
case *True* **thus** *?thesis* **by** (*rule-tac x* = *a* **in** *bexI*, *auto*)
next
case *False*
with *insertI.hyps* **and** *False*
obtain *max*
where *h1*: *max* ∈ *A*
and *h2*: ∀ *a*∈*A*. *f a* ≤ *f max* **by** *blast*
show *?thesis*
proof (*cases f a* ≤ *f max*)
assume *f a* ≤ *f max*
with *h1 h2* **show** *?thesis* **by** (*rule-tac x* = *max* **in** *bexI*, *auto*)
next
assume ¬ (*f a* ≤ *f max*)
thus *?thesis* **using** *h2* **by** (*rule-tac x* = *a* **in** *bexI*, *auto*)
qed
qed
qed

The following is a technical lemma.which helps to show the range finiteness of tag function.

lemma *finite-strict-prefix-set*: *finite* {*xa*. *xa* < (*x*::*string*)}
apply (*induct x rule*:*rev-induct*, *simp*)
apply (*subgoal-tac* {*xa*. *xa* < *xs* @ [*x*]} = {*xa*. *xa* < *xs*} ∪ {*xs*})
by (*auto simp*:*strict-prefix-def*)

lemma *tag-str-STAR-injI*:
fixes *v w*
assumes *eq-tag*: *tag-str-STAR* L_1 *v* = *tag-str-STAR* L_1 *w*
shows (*v*::*string*) ≈(L_1⋆) *w*
proof−
— As explained before, a pattern for just one direction needs to be dealt with:
{ **fix** *x y z*
assume *xz-in-star*: *x* @ *z* ∈ L_1⋆
and *tag-xy*: *tag-str-STAR* L_1 *x* = *tag-str-STAR* L_1 *y*
have *y* @ *z* ∈ L_1⋆

proof(*cases* $x = []$)
— The degenerated case when x is a null string is easy to prove:
case *True*
with *tag-xy* **have** $y = []$
by (*auto simp add*: *tag-str-STAR-def strict-prefix-def*)
thus *?thesis* **using** *xz-in-star True* **by** *simp*
next
— The nontrival case:
case *False*
— Since $x @ z \in L_1\star$, x can always be splitted by a prefix xa together with its suffix $x - xa$, such that both xa and $(x - xa) @ z$ are in $L_1\star$, and there could be many such splittings.Therefore, the following set *?S* is nonempty, and finite as well:
let $?S = \{xa.\ xa < x \wedge xa \in L_1\star \wedge (x - xa) @ z \in L_1\star\}$
have *finite ?S*
by (*rule-tac* $B = \{xa.\ xa < x\}$ **in** *finite-subset*,
auto simp:*finite-strict-prefix-set*)
moreover have $?S \neq \{\}$ **using** *False xz-in-star*
by (*simp*, *rule-tac* $x = []$ **in** *exI*, *auto simp*:*strict-prefix-def*)
— Since *?S* is finite, we can always single out the longest and name it *xa-max*:
ultimately have $\exists\ xa\text{-}max \in ?S.\ \forall\ xa \in ?S.\ length\ xa \leq length\ xa\text{-}max$
using *finite-set-has-max* **by** *blast*
then obtain *xa-max*
where *h1*: $xa\text{-}max < x$
and *h2*: $xa\text{-}max \in L_1\star$
and *h3*: $(x - xa\text{-}max) @ z \in L_1\star$
and *h4*:$\forall\ xa < x.\ xa \in L_1\star \wedge (x - xa) @ z \in L_1\star$
$\longrightarrow length\ xa \leq length\ xa\text{-}max$
by *blast*
— By the equality of tags, the counterpart of *xa-max* among *y*-prefixes, named *ya*, can be found:
obtain *ya*
where *h5*: $ya < y$ **and** *h6*: $ya \in L_1\star$
and *eq-xya*: $(x - xa\text{-}max) \approx L_1\ (y - ya)$
proof−
from *tag-xy* **have** $\{\approx L_1\ ``\ \{x - xa\}\ |xa.\ xa < x \wedge xa \in L_1\star\} =$
$\{\approx L_1\ ``\ \{y - xa\}\ |xa.\ xa < y \wedge xa \in L_1\star\}$ (**is** *?left = ?right*)
by (*auto simp*:*tag-str-STAR-def*)
moreover have $\approx L_1\ ``\ \{x - xa\text{-}max\} \in ?left$ **using** *h1 h2* **by** *auto*
ultimately have $\approx L_1\ ``\ \{x - xa\text{-}max\} \in ?right$ **by** *simp*
thus *?thesis* **using** *that*
apply (*simp add*:*Image-def str-eq-rel-def str-eq-def*) **by** *blast*
qed
— The *?thesis*, $y @ z \in L_1\star$, is a simple consequence of the following proposition:
have $(y - ya) @ z \in L_1\star$
proof−
— The idea is to split the suffix z into za and zb, such that:
obtain *za zb* **where** *eq-zab*: $z = za @ zb$
and *l-za*: $(y - ya)@za \in L_1$ **and** *ls-zb*: $zb \in L_1\star$

```
proof –
  — Since xa-max < x, x can be splitted into a and b such that:
  from h1 have (x − xa-max) @ z ≠ []
    by (auto simp:strict-prefix-def elim:prefixE)
  from star-decom [OF h3 this]
  obtain a b where a-in: a ∈ L1
    and a-neq: a ≠ [] and b-in: b ∈ L1⋆
    and ab-max: (x − xa-max) @ z = a @ b by blast
  — Now the candiates for za and zb are found:
  let ?za = a − (x − xa-max) and ?zb = b
  have pfx: (x − xa-max) ≤ a (is ?P1)
    and eq-z: z = ?za @ ?zb (is ?P2)
  proof –
    — Since (x − xa-max) @ z = a @ b, string (x − xa-max) @ z can
      be splitted in two ways:
    have ((x − xa-max) ≤ a ∧ (a − (x − xa-max)) @ b = z) ∨
      (a < (x − xa-max) ∧ ((x − xa-max) − a) @ z = b)
      using app-eq-dest[OF ab-max] by (auto simp:strict-prefix-def)
    moreover {
      — However, the undsired way can be refuted by absurdity:
      assume np: a < (x − xa-max)
        and b-eqs: ((x − xa-max) − a) @ z = b
      have False
      proof –
        let ?xa-max' = xa-max @ a
        have ?xa-max' < x
          using np h1 by (clarsimp simp:strict-prefix-def diff-prefix)
        moreover have ?xa-max' ∈ L1⋆
          using a-in h2 by (simp add:star-intro3)
        moreover have (x − ?xa-max') @ z ∈ L1⋆
          using b-eqs b-in np h1 by (simp add:diff-diff-appd)
        moreover have ¬ (length ?xa-max' ≤ length xa-max)
          using a-neq by simp
        ultimately show ?thesis using h4 by blast
      qed }
    — Now it can be shown that the splitting goes the way we desired.
    ultimately show ?P1 and ?P2 by auto
  qed
  hence (x − xa-max)@?za ∈ L1 using a-in by (auto elim:prefixE)
  — Now candidates ?za and ?zb have all the requred properteis.
  with eq-xya have (y − ya) @ ?za ∈ L1
    by (auto simp:str-eq-def str-eq-rel-def)
   with eq-z and b-in
  show ?thesis using that by blast
qed
— ?thesis can easily be shown using properties of za and zb:
have ((y − ya) @ za) @ zb ∈ L1⋆ using  l-za ls-zb by blast
with eq-zab show ?thesis by simp
qed
```

```
    with h5 h6 show ?thesis
      by (drule-tac star-intro1, auto simp:strict-prefix-def elim:prefixE)
  qed
}
— By instantiating the reasoning pattern just derived for both directions:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
— The thesis is proved as a trival consequence:
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma — The oringal version with less explicit details.
  fixes v w
  assumes eq-tag: tag-str-STAR L₁ v = tag-str-STAR L₁ w
  shows (v::string) ≈(L₁⋆) w
proof−
  — According to the definition of ≈Lang, proving v ≈(L₁⋆) w amounts
    to showing: for any string u, if v @ u ∈ (L₁⋆) then w @ u ∈ (L₁⋆)
    and vice versa. The reasoning pattern for both directions are the
    same, as derived in the following:
  { fix x y z
    assume xz-in-star: x @ z ∈ L₁⋆
      and tag-xy: tag-str-STAR L₁ x = tag-str-STAR L₁ y
    have y @ z ∈ L₁⋆
    proof(cases x = [])
      — The degenerated case when x is a null string is easy to prove:
      case True
      with tag-xy have y = []
        by (auto simp:tag-str-STAR-def strict-prefix-def)
      thus ?thesis using xz-in-star True by simp
    next
      — The case when x is not null, and x @ z is in L₁⋆,
      case False
      obtain x-max
        where h1: x-max < x
        and h2: x-max ∈ L₁⋆
        and h3: (x − x-max) @ z ∈ L₁⋆
        and h4:∀ xa < x. xa ∈ L₁⋆ ∧ (x − xa) @ z ∈ L₁⋆
                               ⟶ length xa ≤ length x-max
      proof−
        let ?S = {xa. xa < x ∧ xa ∈ L₁⋆ ∧ (x − xa) @ z ∈ L₁⋆}
        have finite ?S
          by (rule-tac B = {xa. xa < x} in finite-subset,
                            auto simp:finite-strict-prefix-set)
        moreover have ?S ≠ {} using False xz-in-star
          by (simp, rule-tac x = [] in exI, auto simp:strict-prefix-def)
        ultimately have ∃ max ∈ ?S. ∀ a ∈ ?S. length a ≤ length max
          using finite-set-has-max by blast
        thus ?thesis using that by blast
      qed
```

```
obtain ya
  where h5: ya < y and h6: ya ∈ L₁⋆ and h7: (x − x-max) ≈L₁ (y − ya)
proof−
  from tag-xy have {≈L₁ ‘‘ {x − xa} |xa. xa < x ∧ xa ∈ L₁⋆} =
    {≈L₁ ‘‘ {y − xa} |xa. xa < y ∧ xa ∈ L₁⋆} (is ?left = ?right)
    by (auto simp:tag-str-STAR-def)
  moreover have ≈L₁ ‘‘ {x − x-max} ∈ ?left using h1 h2 by auto
  ultimately have ≈L₁ ‘‘ {x − x-max} ∈ ?right by simp
  with that show ?thesis apply
    (simp add:Image-def str-eq-rel-def str-eq-def) by blast
qed
have (y − ya) @ z ∈ L₁⋆
proof−
  from h3 h1 obtain a b where a-in: a ∈ L₁
    and a-neq: a ≠ [] and b-in: b ∈ L₁⋆
    and ab-max: (x − x-max) @ z = a @ b
    by (drule-tac star-decom, auto simp:strict-prefix-def elim:prefixE)
  have (x − x-max) ≤ a ∧ (a − (x − x-max)) @ b = z
  proof −
    have ((x − x-max) ≤ a ∧ (a − (x − x-max)) @ b = z) ∨
                  (a < (x − x-max) ∧ ((x − x-max) − a) @ z = b)
      using app-eq-dest[OF ab-max] by (auto simp:strict-prefix-def)
    moreover {
      assume np: a < (x − x-max) and b-eqs:  ((x − x-max) − a) @ z = b
      have False
      proof −
        let ?x-max' = x-max @ a
        have ?x-max' < x
          using np h1 by (clarsimp simp:strict-prefix-def diff-prefix)
        moreover have ?x-max' ∈ L₁⋆
          using a-in h2 by (simp add:star-intro3)
        moreover have (x − ?x-max') @ z ∈ L₁⋆
          using b-eqs b-in np h1 by (simp add:diff-diff-appd)
        moreover have ¬ (length ?x-max' ≤ length x-max)
          using a-neq by simp
        ultimately show ?thesis using h4 by blast
      qed
    } ultimately show ?thesis by blast
  qed
  then obtain za where z-decom: z = za @ b
    and x-za: (x − x-max) @ za ∈ L₁
    using a-in by (auto elim:prefixE)
  from x-za h7 have (y − ya) @ za ∈ L₁
    by (auto simp:str-eq-def str-eq-rel-def)
  with b-in have ((y − ya) @ za) @ b ∈ L₁⋆ by blast
  with z-decom show ?thesis by auto
qed
with h5 h6 show ?thesis
  by (drule-tac star-intro1, auto simp:strict-prefix-def elim:prefixE)
```

qed
}
— By instantiating the reasoning pattern just derived for both directions:
from *this* [*OF* - *eq-tag*] **and** *this* [*OF* - *eq-tag* [*THEN sym*]]
— The thesis is proved as a trival consequence:
show *?thesis* **unfolding** *str-eq-def str-eq-rel-def* **by** *blast*
qed

lemma *quot-star-finiteI* [*intro*]:
fixes *L1*::*lang*
assumes *finite1*: *finite* (*UNIV* // $\approx$*L1*)
shows *finite* (*UNIV* // $\approx$(*L1*$\star$))
proof (*rule-tac tag* = *tag-str-STAR L1* **in** *tag-finite-imageD*)
show $\bigwedge$*x y*. *tag-str-STAR L1 x* = *tag-str-STAR L1 y* $\Longrightarrow$ *x* $\approx$(*L1*$\star$) *y*
by (*rule tag-str-STAR-injI*)
next
have $*$: *finite* (*Pow* (*UNIV* // $\approx$*L1*))
using *finite1* **by** *auto*
show *finite* (*range* (*tag-str-STAR L1*))
unfolding *tag-str-STAR-def*
apply(*rule finite-subset*[*OF* - $*$])
unfolding *quotient-def*
by *auto*
qed

8.2.7 The conclusion

lemma *rexp-imp-finite*:
fixes *r*::*rexp*
shows *finite* (*UNIV* // $\approx$(*L r*))
by (*induct r*) (*auto*)

end

theory *Myhill*
imports *Myhill-2*
begin

9 Preliminaries

9.1 Finite automata and Myhill-Nerode theorem

A *determinisitc finite automata (DFA) M* is a 5-tuple $(Q, \Sigma, \delta, s, F)$, where:

1. Q is a finite set of *states*, also denoted Q_M.
2. Σ is a finite set of *alphabets*, also denoted Σ_M.
3. δ is a *transition function* of type $Q \times \Sigma \Rightarrow Q$ (a total function), also denoted δ_M.

4. $s \in Q$ is a state called *initial state*, also denoted s_M.

5. $F \subseteq Q$ is a set of states named *accepting states*, also denoted F_M.

Therefore, we have $M = (Q_M, \Sigma_M, \delta_M, s_M, F_M)$. Every DFA M can be interpreted as a function assigning states to strings, denoted $\hat{\delta}_M$, the definition of which is as the following:

$$\begin{aligned} \hat{\delta}_M([]) &\equiv s_M \\ \hat{\delta}_M(xa) &\equiv \delta_M(\hat{\delta}_M(x), a) \end{aligned} \tag{2}$$

A string x is said to be *accepted* (or *recognized*) by a DFA M if $\hat{\delta}_M(x) \in F_M$. The language recoginzed by DFA M, denoted $L(M)$, is defined as:

$$L(M) \equiv \{x \mid \hat{\delta}_M(x) \in F_M\} \tag{3}$$

The standard way of specifying a laugage $\mathcal{L}$ as *regular* is by stipulating that: $\mathcal{L} = L(M)$ for some DFA M.
For any DFA M, the DFA obtained by changing initial state to another $p \in Q_M$ is denoted M_p, which is defined as:

$$M_p \;\equiv\; (Q_M, \Sigma_M, \delta_M, p, F_M) \tag{4}$$

Two states $p, q \in Q_M$ are said to be *equivalent*, denoted $p \approx_M q$, iff.

$$L(M_p) = L(M_q) \tag{5}$$

It is obvious that $\approx_M$ is an equivalent relation over Q_M. and the partition induced by $\approx_M$ has $|Q_M|$ equivalent classes. By overloading $\approx_M$, an equivalent relation over strings can be defined:

$$x \approx_M y \;\;\equiv\;\; \hat{\delta}_M(x) \approx_M \hat{\delta}_M(y) \tag{6}$$

It can be proved that the the partition induced by $\approx_M$ also has $|Q_M|$ equivalent classes. It is also easy to show that: if $x \approx_M y$, then $x \approx_{L(M)} y$, and this means $\approx_M$ is a more refined equivalent relation than $\approx_{L(M)}$. Since partition induced by $\approx_M$ is finite, the one induced by $\approx_{L(M)}$ must also be finite, and this is one of the two directions of Myhill-Nerode theorem:

Lemma 1 (Myhill-Nerode theorem, Direction two). *If a language $\mathcal{L}$ is regular (i.e. $\mathcal{L} = L(M)$ for some DFA M), then the partition induced by $\approx_{\mathcal{L}}$ is finite.*

The other direction is:

Lemma 2 (Myhill-Nerode theorem, Direction one). *If the partition induced by $\approx_{\mathcal{L}}$ is finite, then $\mathcal{L}$ is regular (i.e. $\mathcal{L} = L(M)$ for some DFA M).*

The M we are seeking when prove lemma **??** can be constructed out of $\approx_{\mathcal{L}}$, denoted $M_{\mathcal{L}}$ and defined as the following:

$$
\begin{aligned}
Q_{M_{\mathcal{L}}} &\equiv \{[\![x]\!]_{\approx_{\mathcal{L}}} \mid x \in \Sigma^*\} && \text{(7a)}\\
\Sigma_{M_{\mathcal{L}}} &\equiv \Sigma_M && \text{(7b)}\\
\delta_{M_{\mathcal{L}}} &\equiv (\lambda([\![x]\!]_{\approx_{\mathcal{L}}}, a).[\![xa]\!]_{\approx_{\mathcal{L}}}) && \text{(7c)}\\
s_{M_{\mathcal{L}}} &\equiv [\![[]]\!]_{\approx_{\mathcal{L}}} && \text{(7d)}\\
F_{M_{\mathcal{L}}} &\equiv \{[\![x]\!]_{\approx_{\mathcal{L}}} \mid x \in \mathcal{L}\} && \text{(7e)}
\end{aligned}
$$

It can be proved that $Q_{M_{\mathcal{L}}}$ is indeed finite and $\mathcal{L} = L(M_{\mathcal{L}})$, so lemma 2 holds. It can also be proved that $M_{\mathcal{L}}$ is the minimal DFA (therefore unique) which recoginzes $\mathcal{L}$.

9.2 The objective and the underlying intuition

It is now obvious from section 9.1 that Myhill-Nerode theorem can be established easily when *reglar languages* are defined as ones recognized by finite automata. Under the context where the use of finite automata is forbiden, the situation is quite different. The theorem now has to be expressed as:

Theorem 1 (Myhill-Nerode theorem, Regular expression version)**.** *A language $\mathcal{L}$ is regular (i.e. $\mathcal{L} = L(e)$ for some* regular expression *e) iff. the partition induced by $\approx_{\mathcal{L}}$ is finite.*

The proof of this version consists of two directions (if the use of automata are not allowed):

Direction one: generating a regular expression e out of the finite partition induced by $\approx_{\mathcal{L}}$, such that $\mathcal{L} = L(e)$.

Direction two: showing the finiteness of the partition induced by $\approx_{\mathcal{L}}$, under the assmption that $\mathcal{L}$ is recognized by some regular expression e (i.e. $\mathcal{L} = L(e)$).

The development of these two directions consititutes the body of this paper.

10 Direction *regular language ⇒finite partition*

Although not used explicitly, the notion of finite autotmata and its relationship with language partition, as outlined in section 9.1, still servers as important intuitive guides in the development of this paper. For example, *Direction one* follows the *Brzozowski algebraic method* used to convert finite autotmata to regular expressions, under the intuition that every partition

member $[\![x]\!]_{\approx_{\mathcal{L}}}$ is a state in the DFA $M_{\mathcal{L}}$ constructed to prove lemma 2 of section 9.1.

The basic idea of Brzozowski method is to set aside an unknown for every DFA state and describe the state-trasition relationship by charateristic equations. By solving the equational system such obtained, regular expressions characterizing DFA states are obtained. There are choices of how DFA states can be characterized. The first is to characterize a DFA state by the set of striings leading from the state in question into accepting states. The other choice is to characterize a DFA state by the set of strings leading from initial state into the state in question. For the first choice, the lauguage recognized by a DFA can be characterized by the regular expression characterizing initial state, while in the second choice, the languaged of the DFA can be characterized by the summation of regular expressions of all accepting states.

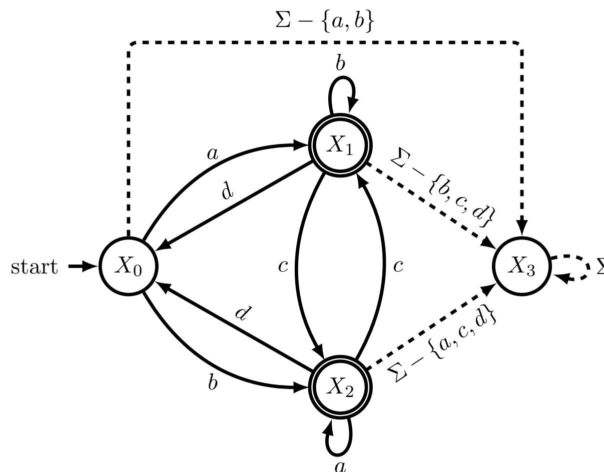


Figure 3: The relationship between automata and finite partition

end