Regular Expressions & The Myhill-Nerode Theorem

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Outline

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1. Regular Expression(brief)
2. Myhill-Nerode Theorem(Intro)
3. FA to Regular Expressions
4. Proving Myhill-Nerode Theorem
   - Well-Founded iterating principle
   - Invariant predicate
   - Generating initial ES
   - Iteration step of ES
   - Final Proof

Regular Expression

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  - $\{\} \mid \{[]\} \mid \{[c]\} \mid L_1 ; L_2 \mid L_1 \cup L_2 \mid L*$

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```
definition lang_seq :: "string set ⇒ string set ⇒ string set"
  ("_ ; _" [100,100] 100)
where
   "L1 ; L2 = {s1@s2 | s1 s2. s1 ∈ L1 ∧ s2 ∈ L2}"
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inductive_set Star :: "string set ⇒ string set" ("_⋆" [101] 102)
  for L :: "string set"
where
  start[intro]: "[] ∈ L⋆"
| step[intro]:  "⟦s1 ∈ L; s2 ∈ L⋆⟧ ⟹ s1@s2 ∈ L⋆"
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 | CHAR char
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consts L:: "'a ⇒ string set"
overloading L_rexp == "L::  rexp ⇒ string set"
begin
fun L_rexp :: "rexp ⇒ string set"
where
    "L_rexp (NULL)    = {}"
  | "L_rexp (EMPTY)   = {[]}"
  | "L_rexp (CHAR c)  = {[c]}"
  | "L_rexp (SEQ r1 r2) = (L_rexp r1) ; (L_rexp r2)"
  | "L_rexp (ALT r1 r2) = (L_rexp r1) ∪ (L_rexp r2)"

  | "L_rexp (STAR r)    = (L_rexp r)⋆" end
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- In the theory of formal languages
- It provides a **necessary** & **sufficient** condition for a language to be regular
- Named after John Myhill and Anil Nerode
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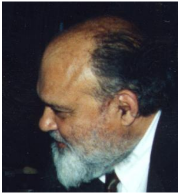

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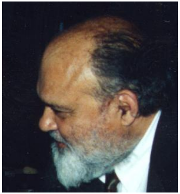

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- A equivalence relation defined by *Lang*

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- Partions of Universal Language(*UNIV*)
  - Universal Language(*UNIV*) : $\Sigma*$

$$\text{Lang} = \bigcup \{X \mid \text{UNIV Quo Lang}\}\ (\forall x{\in}X.\ x \in \text{Lang})$$

Statement of the theorem (cont.)

Theorem

Lang is **regular** *iff it has finite partitions of UNIV*

(∃fa. lang_of_fa fa = Lang) = finite (UNIV Quo Lang)

Statement of the theorem (cont.)

Theorem

Lang is **regular** *iff it has finite partitions of UNIV*

$(\exists \text{fa. lang\_of\_fa fa} = \text{Lang}) = \text{finite (UNIV Quo Lang)}$

- *lang_of_fa* is for getting language from a FA

Use and consequences

- To show a language is regular
  - prove the partition is finite
  - from $[|\lambda|]_{Lang}$ ($[|[]|]_{Lang}$)& Σ do a exhausitive search
- To show a language is not regular
  - prove the partition is infinite

$\Sigma = \{0, 1\}$ & $Lang = L(0{\cdot}(0|1)^*)$

$[|\lambda|]Lang$ $\xrightarrow{\lambda \not\equiv Lang \not\equiv 0}$ $[|\lambda|]Lang$, $[|0|]Lang$ $\xrightarrow{0 \not\equiv Lang \not\equiv 1,\ \lambda \not\equiv Lang \not\equiv 1,\ \lambda \not\equiv Lang \not\equiv 0}$ $[|\lambda|]Lang$, $[|0|]Lang$, $[|1|]Lang$

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Use and consequences

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Proof(brief.)

Finite partitions $\longrightarrow$ Regular

- Exists a k, where k partitions (equiv-classes)
- We can get a DFA (Q,Σ,δ,q_0,F)
  - $Q = UNIV\ Quo\ Lang$
  - $\delta(p, a) = q$ iff exists a word $x \in p$ such that $x@a \in q$
  - $q_0 = [|[]|]Lang$
  - $q \in F$ iff exists a word $x \in q$ such that $x \in Lang$
  - δ is a function because:

    If $x \approx Lang \approx y$, then $(x@a) \approx Lang \approx (y@a)$
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Regular $\longrightarrow$ Finite partitions

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Outline

FA to Reg Exps

State Removal method

- How to do?
- Characters

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- How to do?
  1. Identifies patterns within the graph
  2. Removes states
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  1. Easy to visualize
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Equation-System (0)

$R_1 = \lambda$

$R_2 = R_1; 0 + R_2; (0|1)$

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$+$ is actually $\cup$

ES Subst (1)

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Arden's Lemma (revised.)

Given an equation of the form $X = X; A + B$ when $[] \notin A$ the equation has the solution $X = B; A*$

Brzozowski Algebraic method (revised.)

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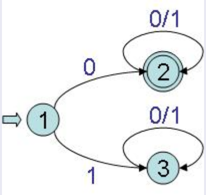
DFA



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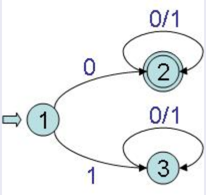

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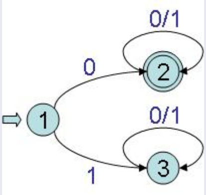

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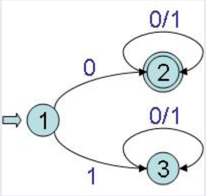

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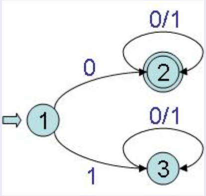

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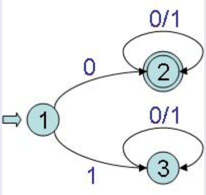

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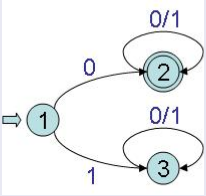

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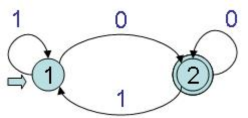

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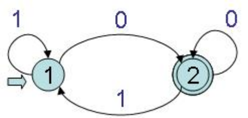

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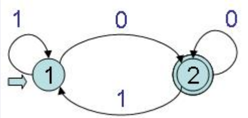

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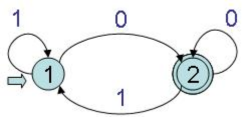

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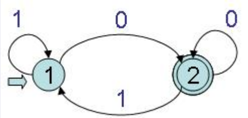

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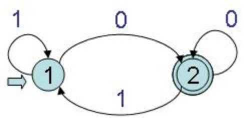

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  - Generate initial ES derived from *Lang*
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      6. *Lang* is a set of equiv-class
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- Target: finite (UNIV Quo Lang) $\Longrightarrow$ $\exists$ reg. Lang = L reg
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WF-iter

- Elimination of ES can be abstracted as

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    2. ES is finite

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    3. Each equiv-class has only one equation

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definition
  distinct_rhs :: "t_equa_rhs ⇒ bool"
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Outline

1. Regular Expression(brief)
2. Myhill-Nerode Theorem(Intro)
3. FA to Regular Expressions
4. Proving Myhill-Nerode Theorem
   - Well-Founded iterating principle
   - Invariant predicate
   - Generating initial ES
   - Iteration step of ES
   - Final Proof

Generating Initial Equation-System

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definition
  CT :: "string set ⇒ char ⇒ string set ⇒ bool" ("_-_→_" [99,99]99)
where "X-c→Y ≡ ((X;{[c]}) ⊆ Y)"

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Equation-System (0)

$R_1 = \lambda$

$R_2 = R_1; 0 + R_2; (0|1)$

$R_3 = R_1; 1 + R_3; (0|1)$

Generating Initial Equation-System

```
definition
  CT :: "string set ⇒ char ⇒ string set ⇒ bool" ("_-_→_" [99,99]99)
where "X-c→Y ≡ ((X;{[c]}) ⊆ Y)"

 types t_equa_rhs = "(string set × rexp) set"
types t_equa = "(string set × t_equa_rhs)"
types t_equas = "t_equa set"

 definition
  empty_rhs :: "string set ⇒ t_equa_rhs"
where "empty_rhs X ≡ if ([] ∈ X) then {({[]}, EMPTY)} else {}"
 definition
  folds :: "('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a set ⇒ 'b"
where "folds f z S ≡ SOME x. fold_graph f z S x"

 definition
  equation_rhs :: "(string set) set ⇒ string set ⇒ t_equa_rhs"
where "equation_rhs CS X ≡ if (X = {[]}) then {({[]}, EMPTY)}
                         else {(S, folds ALT NULL {CHAR c| c. S-c→X} )| S.
S ∈ CS} ∪
                            empty_rhs X"

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  equations :: "(string set) set ⇒ t_equas"

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Outline

eliminating one equation

- Not the equation of target equiv-class
- Approach: substitution
  1. Well-formed substitutor equation
     - substitutor = an equiv-class
     - rhs should not contain itself
     - if not, use Arden's Lemma to reform itself
  2. Substituting
     - if substitutor is empty-string itself
     - then do nothing
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     - merging
  3. Delete substitutor equation

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formalization

```
definition
  seq_rhs_r :: "t_equa_rhs ⇒ rexp ⇒ t_equa_rhs"
where "seq_rhs_r rhs r ≡ (λ(X, reg). (X, SEQ reg r)) ‘ rhs"
definition
  del_x_paired :: "('a × 'b) set ⇒ 'a ⇒ ('a × 'b) set"
where  "del_x_paired S x ≡ S - {X. X ∈ S ∧ fst X = x}"
definition
  merge_rhs :: "t_equa_rhs ⇒ t_equa_rhs ⇒ t_equa_rhs"
where "merge_rhs rhs rhs' ≡ {(X, r). (∃ r1 r2. (X,r1) ∈ rhs ∧ (X, r2) ∈ rhs' ∧ r = ALT r1
r2) ∨
                    (∃ r1. (X, r1) ∈ rhs ∧ (¬ (∃ r2. (X, r2) ∈ rhs')) ∧ r = r1)     ∨
                    (∃ r2. (X, r2) ∈ rhs' ∧ (¬ (∃ r1. (X, r1) ∈ rhs)) ∧ r = r2)     }"

definition
  arden_variate :: "string set ⇒ rexp ⇒ t_equa_rhs ⇒ t_equa_rhs"
where "arden_variate X r rhs ≡ seq_rhs_r (del_x_paired rhs X) (STAR r)"

definition
  rhs_subst :: "t_equa_rhs ⇒ string set ⇒ t_equa_rhs ⇒ rexp ⇒ t_equa_rhs"
where "rhs_subst rhs X xrhs r ≡ merge_rhs (del_x_paired rhs X) (seq_rhs_r xrhs r)"

definition
  equas_subst_f :: "string set ⇒ t_equa_rhs ⇒ t_equa ⇒ t_equa"
where "equas_subst_f X xrhs equa ≡ let (Y, rhs) = equa in
   if (∃ r. (X, r) ∈ rhs) then (Y, rhs_subst rhs X xrhs (SOME r. (X, r) ∈ rhs)) else equa"
definition
  equas_subst :: "t_equas ⇒ string set ⇒ t_equa_rhs ⇒ t_equas"
where "equas_subst ES X xrhs ≡ del_x_paired (equas_subst_f X xrhs ‘ ES) X"
```

Outline

1. Regular Expression(brief)
2. Myhill-Nerode Theorem(Intro)
3. FA to Regular Expressions
4. Proving Myhill-Nerode Theorem
   - Well-Founded iterating principle
   - Invariant predicate
   - Generating initial ES
   - Iteration step of ES
   - Final Proof

WF-iter Usage

```
lemma iteration_step:
  assumes Inv_ES: "Inv X ES" and not_T: "¬ TCon ES"
  shows "(∃ ES'. Inv X ES' ∧ (card ES', card ES) ∈ less_than)"
proof -
  from Inv_ES not_T have another: "∃Y yrhs. (Y, yrhs) ∈ ES ∧ X ≠ Y" unfolding Inv_def
    by (clarify, rule_tac exist_another_equa[where X = X], auto)
  then obtain Y yrhs where subst: "(Y, yrhs) ∈ ES" and not_X: " X ≠ Y" by blast
  show ?thesis (is "∃ ES'. ?P ES'")
  proof (cases "Y = {[]}")
    case True — in this situation, we pick a λ equation, thus directly remove it
    have "?P (ES - {(Y, yrhs)})"    next
    case False — first use arden's lemma, then do the substitution
    hence "?P (equas_subst ES Y yrhs')"
  qed
qed

lemma iteration_conc:
  assumes history: "Inv X ES"
  shows "∃ ES'. Inv X ES' ∧ TCon ES'" (is "∃ ES'. ?P ES'")
proof (cases "TCon ES")
  case True    hence "?P ES" using history by simp
  thus ?thesis by blast
next
  case False
  thus ?thesis using history iteration_step
    by (rule_tac f = card in wf_iter, simp_all)
qed
```

proof: every equiv-class has a Reg Exp.

```
lemma every_eqcl_has_reg:
  assumes finite_CS: "finite (UNIV Quo Lang)" and X_in_CS: "X ∈ (UNIV Quo Lang)"
  shows "∃ (reg::rexp). L reg = X" (is "∃ r. ?E r")
proof-
  have "∃ES'. Inv X ES' ∧ TCon ES'" using finite_CS X_in_CS
    by (auto intro:init_ES_satisfy_Inv iteration_conc)   have "∃ rhs. ES' = {(X, rhs)}"
by (auto dest!:card_Suc_Diff1 simp:card_eq_0_iff)
  then obtain rhs where ES'_single_equa: "ES' = {(X, rhs)}" ..
  hence X_ardenable: "ardenable (X, rhs)" using Inv_ES'
    by (simp add:Inv_def)  show ?thesis
  proof (cases "X = {[]}")
    case True hence "?E EMPTY" by simp
    thus ?thesis by blast
  next
    case False with  X_ardenable
    have "∃ rhs'. X = L rhs' ∧ rhs_eq_cls rhs' = rhs_eq_cls rhs - {X} ∧ distinct_rhs rhs'"
      by (drule_tac ardenable_prop, auto)
    then obtain rhs' where X_eq_rhs': "X = L rhs'"
      and rhs'_eq_cls: "rhs_eq_cls rhs' = rhs_eq_cls rhs - {X}"
      and rhs'_dist : "distinct_rhs rhs'" by blast
    hence "rhs_eq_cls rhs' = {{[]}}" using X_not_empty X_eq_rhs'
      by (auto simp:rhs_eq_cls_def)
    hence "∃ r. rhs' = {({[]}, r)}"
    then obtain r where "rhs' = {({[]}, r)}" ..
    hence "?E r" using X_eq_rhs' by (auto simp add:lang_seq_def)
    thus ?thesis by blast

  qed qed
```

proof: Myhill-Nerode(one direction)

```
theorem myhill_nerode:
  assumes finite_CS: "finite (UNIV Quo Lang)"
  shows   "∃ (reg::rexp). Lang = L reg" (is "∃ r. ?P r")
proof -
  have has_r_each: "∀C∈{X ∈ UNIV Quo Lang. ∀x∈X. x ∈ Lang}. ∃(r::rexp). C = L r"
    using finite_CS
    by (auto dest:every_eqcl_has_reg)
  have "∃ (rS::rexp set). finite rS ∧
                         (∀ C ∈ {X ∈ UNIV Quo Lang. ∀x∈X. x ∈ Lang}. ∃ r ∈ rS. C = L r) ∧
                         (∀ r ∈ rS. ∃ C ∈ {X ∈ UNIV Quo Lang. ∀x∈X. x ∈ Lang}. C = L r)"
then obtain rS where finite_rS : "finite rS"
    and r_each': "∀ C ∈ {X ∈ UNIV Quo Lang. ∀x∈X. x ∈ Lang}. ∃ r ∈ (rS::rexp set). C = L
r"
    and cl_each: "∀ r ∈ (rS::rexp set). ∃ C ∈ {X ∈ UNIV Quo Lang. ∀x∈X. x ∈ Lang}. C = L
r"
    by blast
  have "?P (folds ALT NULL rS)"
  proof
    show "Lang ⊆ L (folds ALT NULL rS)"      apply (clarsimp simp:fold_alt_null_eqs) by
blast
  next
    show "L (folds ALT NULL rS) ⊆ Lang"      by (clarsimp simp:fold_alt_null_eqs)
  qed
  thus ?thesis by blast

qed
```