

# A Formalisation of the Myhill-Nerode Theorem based on Regular Expressions

Christian Urban

joint work with Chunhan Wu and Xingyuan Zhang  
from the PLA University of Science and  
Technology in Nanjing

# A Formalisation of the Myhill-Nerode Theorem based on Regular Expressions **or, Regular Languages Done Right**

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I can think of three reasons why this is a good definition:

- string matching via DFAs (yacc)
- pumping lemma
- closure properties of regular languages (closed under complement)

# Really Bad News!

DFAs are bad news for formalisations in theorem provers. They might be represented as:

- graphs
- matrices
- partial functions

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Alexander and Tobias: "...automata theory ... does not come for free ..."

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All constructions are messy to reason about.

Constable et al needed (on and off) 18 months for a 3-person team to formalise automata theory in Nuprl including Myhill-Nerode. There is only very little other formalised work on regular languages I know of in Coq, Isabelle and HOL.

# Really Bad News!

DFAs are bad news for formalisations in theorem provers. They might be represented as:

- graphs
- matrices
- partial functions

All constructions are messy to reason about.

typical textbook reasoning goes like: "...if  $M$  and  $N$  are any two automata with no inaccessible states ..."



# Regular Expressions

...are a simple datatype:

```
rexp ::= NULL
      | EMPTY
      | CHR c
      | ALT rexp rexp
      | SEQ rexp rexp
      | STAR rexp
```

# Regular Expressions

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$$\begin{array}{l} r ::= 0 \\ | [] \\ | c \\ | r_1 + r_2 \\ | r_1 \cdot r_2 \\ | r^* \end{array}$$

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Induction and recursion principles come for free.

# Semantics of Rexprs

$$\begin{aligned}\mathbb{L}(\mathbf{0}) &= \emptyset \\ \mathbb{L}(\square) &= \{\square\} \\ \mathbb{L}(c) &= \{[c]\} \\ \mathbb{L}(r_1 + r_2) &= \mathbb{L}(r_1) \cup \mathbb{L}(r_2) \\ \mathbb{L}(r_1 \cdot r_2) &= \mathbb{L}(r_1) ; \mathbb{L}(r_2) \\ \mathbb{L}(r^*) &= \mathbb{L}(r)^*\end{aligned}$$

$$L_1 ; L_2 \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in L_1 \wedge s_2 \in L_2\}$$

$$\frac{}{\square \in L^*} \qquad \frac{s_1 \in L \quad s_2 \in L^*}{s_1 @ s_2 \in L^*}$$

# Regular Expression Matching

- Harper in JFP'99: "Functional Pearl: Proof-Directed Debugging"
- Yi in JFP'06: "Educational Pearl: 'Proof-Directed Debugging' revisited for a first-order version"
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"Unfortunately, regular expression derivatives have been lost in the sands of time, and few computer scientists are aware of them."

# Demo

# The Myhill-Nerode Theorem

- provides necessary and sufficient conditions for a language being regular (pumping lemma only necessary)
- will help with closure properties of regular languages



# The Myhill-Nerode Theorem

- provides necessary and sufficient conditions for a language being regular (pumping lemma only necessary)
- will help with closure properties of regular languages
- key is the equivalence relation:

$$x \approx_L y \stackrel{\text{def}}{=} \forall z. x@z \in L \Leftrightarrow y@z \in L$$

# The Myhill-Nerode Theorem

- finite ( $UNIV // \approx_L$ )  $\Leftrightarrow L$  is regular

# Equivalence Classes

- $L = []$

$$\{ \{ [] \}, UNIV - \{ [] \} \}$$

- $L = [c]$

$$\{ \{ [] \}, \{ [c] \}, UNIV - \{ [], [c] \} \}$$

- $L = \emptyset$

$$\{ UNIV \}$$

# Regular Languages

- $L$  is regular  $\stackrel{\text{def}}{=}$  if there is an automaton  $M$  such that  $\mathbb{L}(M) = L$

- Myhill-Nerode:

finite  $\Rightarrow$  regular

$$\text{finite}(UNIV// \approx_L) \Rightarrow \exists r. L = \mathbb{L}(r)$$

regular  $\Rightarrow$  finite

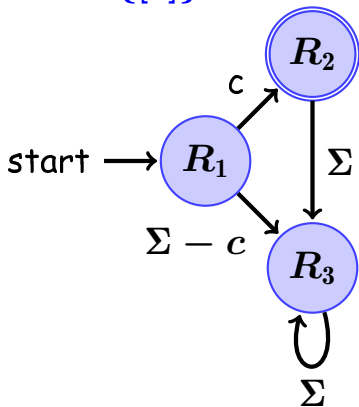
$$\text{finite}(UNIV// \approx_{\mathbb{L}(r)})$$

# Final States

- $\text{final}_L X \stackrel{\text{def}}{=} X \in (\text{UNIV} // \approx_L) \wedge \forall s \in X. s \in L$
- we can prove:  $L = \bigcup \{X. \text{final}_L X\}$

# Transitions between Equivalence Classes

$$L = \{[c]\}$$



$UNIV // \approx_L$  produces

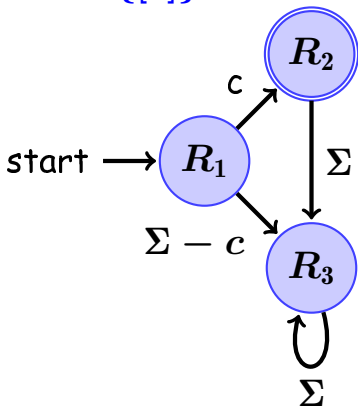
$R_1: \{[]\}$

$R_2: \{[c]\}$

$R_3: UNIV - \{[], [c]\}$

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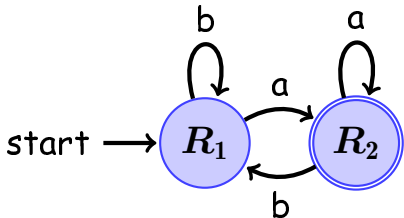
$$R_2: \{[c]\}$$

$$R_3: UNIV - \{[], [c]\}$$

$$X \xrightarrow{c} Y \stackrel{\text{def}}{=} X; [c] \subseteq Y$$

# Systems of Equations

Inspired by a method of Brzozowski '64, we can build an equational system characterising the equivalence classes:



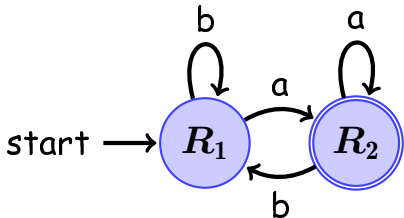
$$R_1 \equiv R_1; b + R_2; b$$

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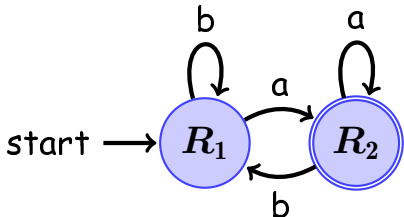


$$R_1 \equiv R_1; b + R_2; b + \lambda; []$$

$$R_2 \equiv R_1; a + R_2; a$$

# Systems of Equations

Inspired by a method of Brzozowski '64, we can build an equational system characterising the equivalence classes:



$$R_1 \equiv R_1; b + R_2; b + \lambda; \square$$

$$R_2 \equiv R_1; a + R_2; a$$

we can prove  $R_1 = R_1; \mathbb{L}(b) \cup R_2; \mathbb{L}(b) \cup \{\square\}; \{\square\}$   
 $R_2 = R_1; \mathbb{L}(a) \cup R_2; \mathbb{L}(a)$

$$R_1 = R_1; b + R_2; b + \lambda; []$$

$$R_2 = R_1; a + R_2; a$$

# A Variant of Arden's Lemma

**Arden's Lemma:**

If  $\lambda \notin A$  then

$$X = X; A + \text{something}$$

has the (unique) solution

$$X = \text{something}; A^*$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$

$$R_2 = R_1; a + R_2; a$$

$$\begin{aligned}R_1 &= R_1; b + R_2; b + \lambda; [] \\R_2 &= R_1; a + R_2; a\end{aligned}$$

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$$\begin{aligned}R_1 &= R_2; b \cdot b^* + \lambda; b^* \\ R_2 &= R_1; a \cdot a^*\end{aligned}$$



$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

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$$R_1 = R_2; b \cdot b^* + \lambda; b^*$$
$$R_2 = R_1; a \cdot a^*$$

by substitution

$$R_1 = R_1; a \cdot a^* \cdot b \cdot b^* + \lambda; b^*$$
$$R_2 = R_1; a \cdot a^*$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

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$$R_1 = R_1; a \cdot a^* \cdot b \cdot b^* + \lambda; b^*$$
$$R_2 = R_1; a \cdot a^*$$

by Arden

$$R_1 = \lambda; b^* \cdot (a \cdot a^* \cdot b \cdot b^*)^*$$
$$R_2 = R_1; a \cdot a^*$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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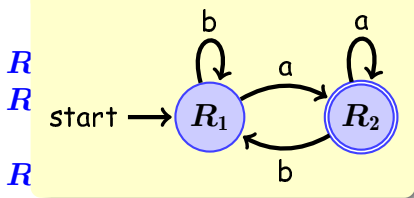
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# The Equ's Solving Algorithm

- The algorithm must terminate: Arden makes one equation smaller; substitution deletes one variable from the right-hand sides.
- We need to maintain the invariant that Arden is applicable (if  $\square \notin A$  then ...):

$$R_1 = R_1; b + R_2; b + \lambda; \square$$

$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; \square$$

$$R_2 = R_1; a \cdot a^*$$

# The Equ's Solving Algorithm

- The algorithm is still a bit hairy to formalise because of our set-representation for equations:

$$\left\{ (X, \{(Y_1, r_1), (Y_2, r_2), \dots\}), \dots \right\}$$

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$$\left\{ (X, \{(Y_1, r_1), (Y_2, r_2), \dots\}), \dots \right\}$$

they are generated from  $UNIV // \approx_L$

# Other Direction

One has to prove

$$\text{finite}(UNIV// \approx_{\mathbb{L}(r)})$$

by induction on  $r$ . Not trivial, but after a bit of thinking (by Chunhan), one can prove that if

$$\text{finite}(UNIV// \approx_{\mathbb{L}(r_1)}) \quad \text{finite}(UNIV// \approx_{\mathbb{L}(r_2)})$$

then

$$\text{finite}(UNIV// \approx_{\mathbb{L}(r_1) \cup \mathbb{L}(r_2)})$$



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- if you want to do regular expression matching (see Scott's paper)
- I cannot yet give definite numbers

# Examples

- $L \equiv \Sigma^*0\Sigma$  is regular

$$A_1 = \Sigma^*00$$

$$A_2 = \Sigma^*01$$

$$A_3 = \Sigma^*10 \cup \{0\}$$

$$A_4 = \Sigma^*11 \cup \{1\} \cup \{\epsilon\}$$

- $L \equiv \{0^n1^n \mid n \geq 0\}$  is not regular

$$B_0 = \{0^n1^n \mid n \geq 0\}$$

$$B_1 = \{0^n1^{(n-1)} \mid n \geq 1\}$$

$$B_2 = \{0^n1^{(n-2)} \mid n \geq 2\}$$

$$B_3 = \{0^n1^{(n-3)} \mid n \geq 3\}$$

⋮

# What We Have Not Achieved

- regular expressions are not good if you look for a minimal one for a language (DFAs have this notion)

# What We Have Not Achieved

- regular expressions are not good if you look for a minimal one for a language (DFAs have this notion)
- Is there anything to be said about context free languages:

A context free language is where every string can be recognised by a pushdown automaton.

# Conclusion

- on balance regular expressions are superior to DFAs, in my opinion
- I cannot think of a reason to not teach regular languages to students this way (!?)
- I have never ever seen a proof of Myhill-Nerode based on regular expressions
- no application, but lots of fun
- great source of examples