Formalising Regular Language Theory with Regular Expressions, Only

Christian Urban
King's College London

joint work with Chunhan Wu and Xingyuan Zhang from the PLA University of Science and Technology in Nanjing

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Roy intertwined with my scientific life on many occasions, most notably:

- he admitted me for M.Phil. in St Andrews and made me like theory
- sent me to Cambridge for Ph.D.
- made me appreciate precision in proofs

Bob Harper
(CMU)

Frank Pfenning
(CMU)

published a proof in
ACM Transactions on Computational Logic, 2005, ∼31pp

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(I also found an **error** in my Ph.D.-thesis about cut-elimination examined by Henk Barendregt and Andy Pitts.)

Formal language theory...

in Theorem Provers

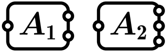
e.g. Isabelle, Coq, HOL4, ...

- automata $\Rightarrow$ graphs, matrices, functions

in Theorem Provers

e.g. Isabelle, Coq, HOL4, . . .

- automata ⇒ graphs, matrices, functions
- combining automata/graphs



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disjoint union:

$$A_1 \uplus A_2 \stackrel{\text{def}}{=} \{(1, x) \mid x \in A_1\} \cup \{(2, y) \mid y \in A_2\}$$

in Theorem Provers

e.g. Isabelle, Coq, HOL4, . . .

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Problems with definition for regularity:

$$\text{is\_regular}(A) \overset{\text{def}}{=} \exists M.\ \text{is\_dfa}(M) \wedge \mathcal{L}(M) = A$$

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A solution: use nats ⇒ state nodes

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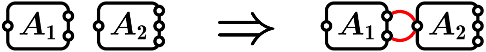

<u>A solution</u>: use nats ⇒ state nodes

You have to rename states!

in Theorem Provers

e.g. Isabelle, Coq, HOL4, ...

- Kozen's "paper" proof of Myhill-Nerode: requires absence of inaccessible states

$$\text{is\_regular}(A) \stackrel{\text{def}}{=} \exists M.\ \text{is\_dfa}(M) \wedge \mathcal{L}(M) = A$$

Definition:

A language A is regular, provided there exists a regular expression that matches all strings of A.

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Infrastructure for free. But do we lose anything?

- pumping lemma
- closure under complementation
- ~~regular expression matching~~ (⇒Brozowski'64, Owens et al '09)
- most textbooks are about automata

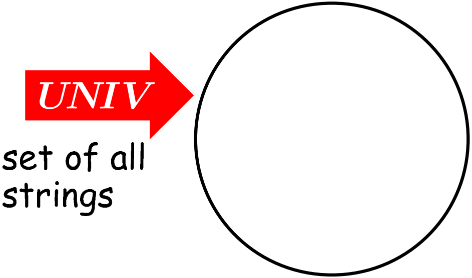
The Myhill-Nerode Theorem

- provides necessary and sufficient conditions for a language being regular
  (pumping lemma only necessary)

- key is the equivalence relation:

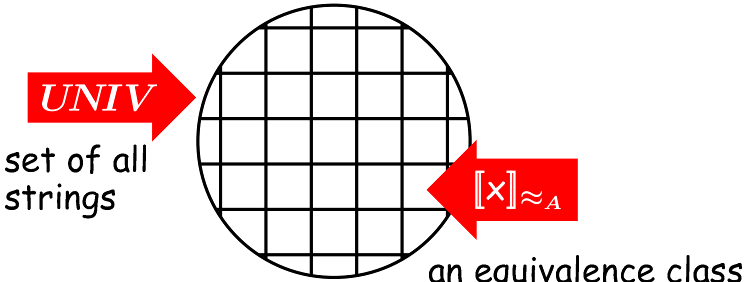
$$x \approx_A y \overset{\text{def}}{=} \forall z.\; x@z \in A \Leftrightarrow y@z \in A$$

The Myhill-Nerode Theorem



- finite $(UNIV // \approx_A) \;\Leftrightarrow\; A$ is regular

The Myhill-Nerode Theorem



- finite $(UNIV // \approx_A)$ $\Leftrightarrow$ A is regular

The Myhill-Nerode Theorem

Two directions:

1.) finite $\Rightarrow$ regular

finite $(UNIV // \approx_A) \Rightarrow \exists r.\ A = \mathcal{L}(r)$

2.) regular $\Rightarrow$ finite

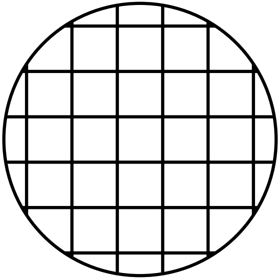
finite $(UNIV // \approx_{\mathcal{L}(r)})$

an equivalence class

- finite $(UNIV // \approx_A) \;\Leftrightarrow\; A$ is regular

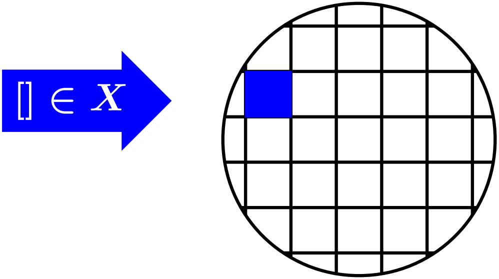
Initial and Final ~~States~~

Equivalence Classes



- finals $A \stackrel{\text{def}}{=} \{\|x\|_{\approx_A} \mid x \in A\}$
- we can prove: $A = \bigcup \text{finals } A$

Initial and Final ~~States~~ Equivalence Classes



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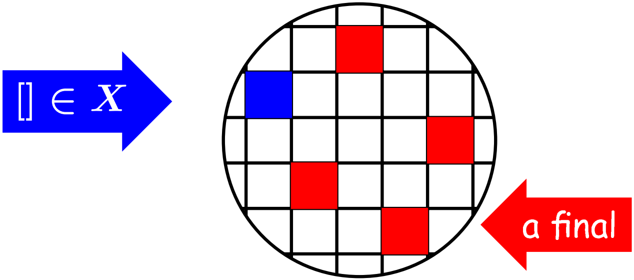

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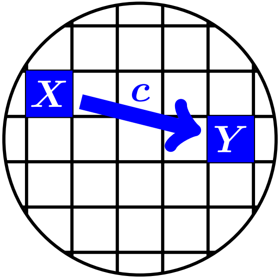
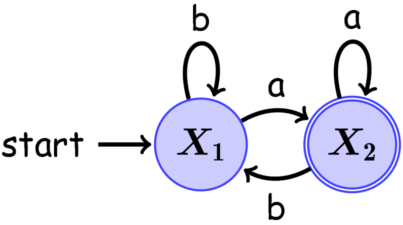
Transitions between Eq-Classes



$$X \xrightarrow{c} Y \;\stackrel{\mathsf{def}}{=}\; X;c \subseteq Y$$

Systems of Equations

Inspired by a method of Brzozowski '64:



$$X_1 = X_1; b + X_2; b$$
$$X_2 = X_1; a + X_2; a$$

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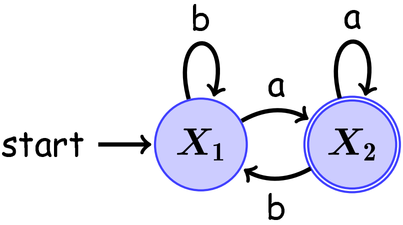

$$X_1 = X_1; b + X_2; b + \lambda; []$$
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by Arden

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$$X_1 = X_2; b \cdot b^\star + \lambda; b^\star$$
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by substitution

$$X_1 = X_1; a \cdot a^\star \cdot b \cdot b^\star + \lambda; b^\star$$
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$$X_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
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$$X_1 = X_1; b + X_2; b + \lambda; []$$
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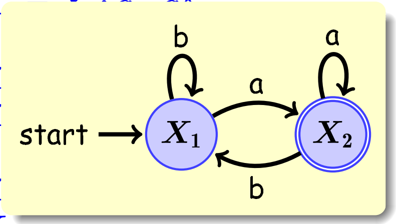

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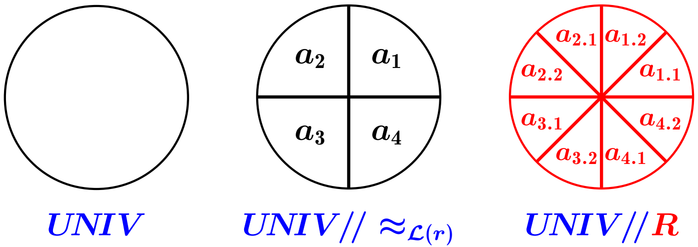
The Other Direction

One has to prove

$$\text{finite}(UNIV // \approx_{\mathcal{L}(r)})$$

by induction on r. Not trivial, but after a bit of thinking, one can find a refined relation:



Derivatives of RExps

- introduced by Brozowski '64
- a regular expressions after a character has been parsed

$$
\begin{array}{lcl}
\text{der c } \varnothing & \stackrel{\text{def}}{=} & \varnothing \\
\text{der c } [] & \stackrel{\text{def}}{=} & \varnothing \\
\text{der c d} & \stackrel{\text{def}}{=} & \text{if c} = \text{d then } [] \text{ else } \varnothing \\
\text{der c } (r_1 + r_2) & \stackrel{\text{def}}{=} & (\text{der c } r_1) + (\text{der c } r_2) \\
\text{der c } (r^\star) & \stackrel{\text{def}}{=} & (\text{der c } r) \cdot r^\star \\
\text{der c } (r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if nullable } r_1 \\
 & & \text{then } (\text{der c } r_1) \cdot r_2 + (\text{der c } r_2) \\
 & & \text{else } (\text{der c } r_1) \cdot r_2
\end{array}
$$

Derivatives of RExps

- introduced by Brozowski '64
- a regular expressions after a character has been parsed

- partial derivatives
- by Antimirov '95

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$$\text{pder c d} \stackrel{\text{def}}{=} \text{if c} = \text{d then } \{[]\} \text{ else } \{\}$$

$$\text{pder c } (r_1 + r_2) \stackrel{\text{def}}{=} (\text{pder c } r_1) \cup (\text{der c } r_2)$$

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Partial Derivatives

- pders $x\ r$ = pders $y\ r$ refines $x \approx_{\mathcal{L}(r)} y$

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Antimirov '95

- $\text{finite}(UNIV // R)$

Partial Derivatives

- $\underbrace{\text{pders } x\ r = \text{pders } y\ r}_{R}$ refines $x \approx_{\mathcal{L}(r)} y$

$$\Downarrow$$ Antimirov '95

- $\text{finite}(UNIV // R)$
- Therefore $\text{finite}(UNIV // \approx_{\mathcal{L}(r)})$. Qed.

What Have We Achieved?

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$$UNIV // \approx_A \;=\; UNIV // \approx_{\overline{A}}$$

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- non-regularity $(a^n b^n)$

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If there exists a sufficiently large set B (for example infinitely large), such that

$$\forall x, y \in B.\; x \neq y \;\Rightarrow\; x \not\approx_A y.$$

then A is not regular. $\quad (B \stackrel{\text{def}}{=} \bigcup_n a^n)$

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$$\mathit{UNIV} // \approx_A \;=\; \mathit{UNIV} // \approx_{\overline{A}}$$

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- non-regularity ($a^n b^n$)
- take **any** language; build the language of substrings
  then this language **is** regular ($a^n b^n \Rightarrow a^\star b^\star$)

Conclusion

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- We have never seen a proof of Myhill-Nerode based on regular expressions.
- great source of examples (inductions)
- no need to fight the theorem prover:
  - first direction (790 loc)
  - second direction (400 / 390 loc)

Thank you!

Questions?