A Formalisation of the Myhill-Nerode Theorem based on Regular Expressions

Christian Urban

joint work with Chunhan Wu and Xingyuan Zhang from the PLA University of Science and Technology in Nanjing

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or, Regular Languages Done Right

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- A regular language is one where there is a DFA that recognises it.

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I can think of three reasons why this is a good definition:

- string matching via DFAs (yacc)
- pumping lemma
- closure properties of regular languages (closed under complement)

Really Bad News!

DFAs are bad news for formalisations in theorem provers. They might be represented as:

- graphs
- matrices
- partial functions

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Alexander and Tobias: "…automata theory …does not come for free …"

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All constructions are messy to reason about.

Constable et al needed (on and off) 18 months for a 3-person team to formalise automata theory in Nuprl including Myhill-Nerode. There is only very little other formalised work on regular languages I know of in Coq, Isabelle and HOL.

Really Bad News!

DFAs are bad news for formalisations in theorem provers. They might be represented as:

- graphs
- matrices
- partial functions

All constructions are messy to reason about.

typical textbook reasoning goes like: "...if M and N are any two automata with no inaccessible states ..."

Regular Expressions

...are a simple datatype:

$$
\begin{array}{lll}
\mathit{rexp} & ::= & \mathsf{NULL} \\
 & | & \mathsf{EMPTY} \\
 & | & \mathsf{CHR}\ c \\
 & | & \mathsf{ALT}\ \mathit{rexp}\ \mathit{rexp} \\
 & | & \mathsf{SEQ}\ \mathit{rexp}\ \mathit{rexp} \\
 & | & \mathsf{STAR}\ \mathit{rexp}
\end{array}
$$

Regular Expressions

...are a simple datatype:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
  & | & [] \\
  & | & c \\
  & | & r_1 + r_2 \\
  & | & r_1 \cdot r_2 \\
  & | & r^\star
\end{array}
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  & \mid & r^\star
\end{array}
$$

Induction and recursion principles come for free.

Semantics of Rexps

$$
\begin{array}{rcl}
\mathbb{L}(\mathbf{0}) & = & \varnothing \\
\mathbb{L}([]) & = & \{[]\} \\
\mathbb{L}(c) & = & \{[c]\} \\
\mathbb{L}(r_1 + r_2) & = & \mathbb{L}(r_1) \cup \mathbb{L}(r_2) \\
\mathbb{L}(r_1 \cdot r_2) & = & \mathbb{L}(r_1)\ ;\ \mathbb{L}(r_2) \\
\mathbb{L}(r^\star) & = & \mathbb{L}(r)^\star
\end{array}
$$

$$L_1 ; L_2 \quad \stackrel{\text{def}}{=} \quad \{s_1 @ s_2 \mid s_1 \in L_1 \wedge s_2 \in L_2\}$$

$$\frac{}{[] \in L^\star} \qquad\qquad \frac{s_1 \in L \quad s_2 \in L^\star}{s_1 @ s_2 \in L^\star}$$

Regular Expression Matching

- Harper in JFP'99: "Functional Pearl: Proof-Directed Debugging"
- Yi in JFP'06: "Educational Pearl: 'Proof-Directed Debugging' revisited for a first-order version"
- Owens et al in JFP'09: "Regular-expression derivatives re-examined"

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> "Unfortunately, regular expression derivatives have been lost in the sands of time, and few computer scientists are aware of them."

Demo

The Myhill-Nerode Theorem

- provides necessary and sufficient conditions for a language being regular (pumping lemma only necessary)
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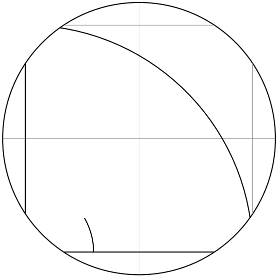
- provides necessary and sufficient conditions for a language being regular (pumping lemma only necessary)
- will help with closure properties of regular languages
- key is the equivalence relation:

$$x \approx_L y \stackrel{\text{def}}{=} \forall z.\ x@z \in L \Leftrightarrow y@z \in L$$

The Myhill-Nerode Theorem



- finite $(UNIV // \approx_L) \;\Leftrightarrow\; L$ is regular

Equivalence Classes

- $L = []$

$$\Big\{\{[]\},\ UNIV - \{[]\}\Big\}$$

- $L = [c]$

$$\Big\{\{[]\},\ \{[c]\},\ UNIV - \{[], [c]\}\Big\}$$

- $L = \varnothing$

$$\Big\{UNIV\Big\}$$

Regular Languages

- L is regular $\stackrel{\text{def}}{=}$ if there is an automaton M such that $\mathbb{L}(M) = L$

- Myhill-Nerode:

  finite $\Rightarrow$ regular
  finite $(UNIV // \approx_L) \Rightarrow \exists r. L = \mathbb{L}(r)$

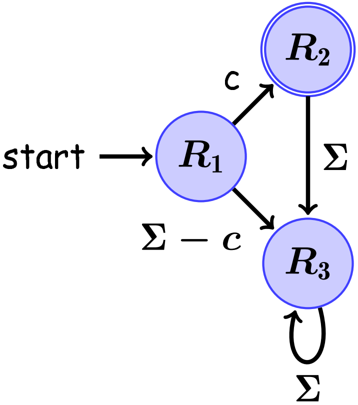
  regular $\Rightarrow$ finite
  finite $(UNIV // \approx_{\mathbb{L}(r)})$

Final States

- $\mathsf{final}_L\ X \stackrel{\mathsf{def}}{=}$
  $X \in (UNIV// \approx_L)\ \wedge\ \forall s \in X.\ s \in L$
- we can prove: $L = \bigcup\{X.\ \mathsf{final}_L\ X\}$

Transitions between Equivalence Classes

$L = \{[c]\}$



$UNIV // \approx_L$ produces

R_1: $\{[]\}$
R_2: $\{[c]\}$
R_3: $UNIV - \{[], [c]\}$

Transitions between Equivalence Classes

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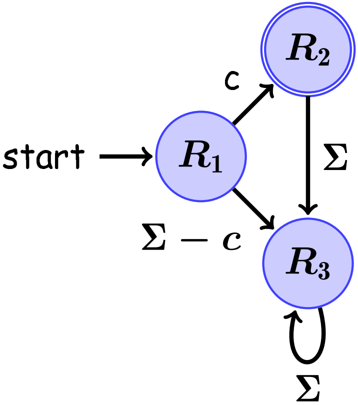


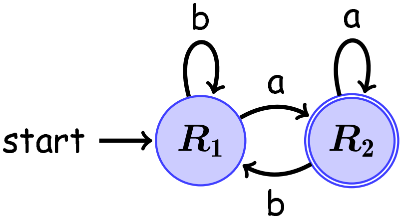
$UNIV // \approx_L$ produces

R_1: $\{[]\}$
R_2: $\{[c]\}$
R_3: $UNIV - \{[], [c]\}$

$$X \xrightarrow{c} Y \stackrel{\text{def}}{=} X ; [c] \subseteq Y$$

Systems of Equations

Inspired by a method of Brzozowski '64, we can build an equational system characterising the equivalence classes:



$$R_1 \equiv R_1; b + R_2; b$$
$$R_2 \equiv R_1; a + R_2; a$$

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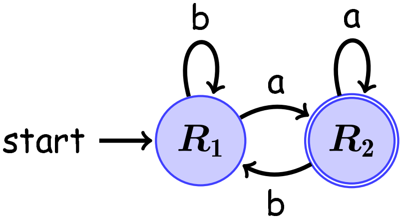

$$R_1 \equiv R_1; b + R_2; b + \lambda; []$$
$$R_2 \equiv R_1; a + R_2; a$$

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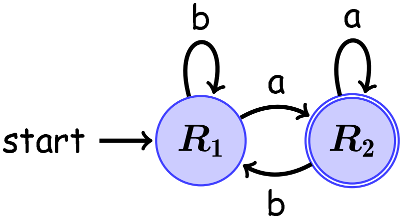

$$R_1 \equiv R_1; b + R_2; b + \lambda; []$$
$$R_2 \equiv R_1; a + R_2; a$$

we can prove
$$R_1 = R_1; \mathbb{L}(b) \;\cup\; R_2; \mathbb{L}(b) \;\cup\; \{[]\}; \{[]\}$$
$$R_2 = R_1; \mathbb{L}(a) \;\cup\; R_2; \mathbb{L}(a)$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

A Variant of Arden's Lemma

Arden's Lemma:

If $[] \notin A$ then

$$X = X; A + \text{something}$$

has the (unique) solution

$$X = \text{something}; A^{\star}$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a \cdot a^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a \cdot a^\star$$

by Arden

$$R_1 = R_2; b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a \cdot a^\star$$

by Arden

$$R_1 = R_2; b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

by substitution

$$R_1 = R_1; a \cdot a^\star \cdot b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a \cdot a^\star$$

by Arden

$$R_1 = R_2; b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

by substitution

$$R_1 = R_1; a \cdot a^\star \cdot b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

by Arden

$$R_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
$$R_2 = R_1; a \cdot a^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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$$R_1 = R_2; b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

by substitution

$$R_1 = R_1; a \cdot a^\star \cdot b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

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$$R_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
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by substitution

$$R_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
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$$R_1 = R_1; b + R_2; b + \lambda; []$$
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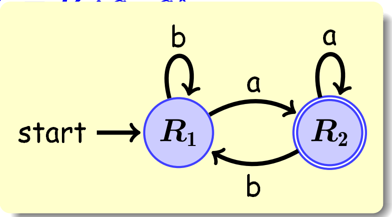
by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R$$



by Arden

$$R$$
$$R$$

by substitution

$$R$$
$$R_2 = R_1; a \cdot a$$

by Arden

$$R_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
$$R_2 = R_1; a \cdot a^\star$$

by substitution

$$R_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
$$R_2 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star \cdot a \cdot a^\star$$

The Equ's Solving Algorithm

- The algorithm must terminate: Arden makes one equation smaller; substitution deletes one variable from the right-hand sides.

- We need to maintain the invariant that Arden is applicable (if $[] \notin A$ then ...):

$$\begin{aligned} R_1 &= R_1; b + R_2; b + \lambda; [] \\ R_2 &= R_1; a + R_2; a \end{aligned}$$

by Arden

$$\begin{aligned} R_1 &= R_1; b + R_2; b + \lambda; [] \\ R_2 &= R_1; a \cdot a^\star \end{aligned}$$

The Equ's Solving Algorithm

- The algorithm is still a bit hairy to formalise because of our set-representation for equations:

$$\{(X, \{(Y_1, r_1), (Y_2, r_2), \ldots\}),$$
$$\ldots$$
$$\}$$

The Equ's Solving Algorithm

- The algorithm is still a bit hairy to formalise because of our set-representation for equations:

$$\begin{array}{l}\{(X, \{(Y_1, r_1), (Y_2, r_2), \ldots\}), \\ \quad \ldots \\ \hfill \}\end{array}$$

  they are generated from $UNIV // \approx_L$

Other Direction

One has to prove

$$\text{finite}(\boldsymbol{UNIV} /\!/ \approx_{\mathbb{L}(\boldsymbol{r})})$$

by induction on $\boldsymbol{r}$. Not trivial, but after a bit of thinking (by Chunhan), one can prove that if

$$\text{finite}(\boldsymbol{UNIV} /\!/ \approx_{\mathbb{L}(\boldsymbol{r}_1)}) \qquad \text{finite}(\boldsymbol{UNIV} /\!/ \approx_{\mathbb{L}(\boldsymbol{r}_2)})$$

then

$$\text{finite}(\boldsymbol{UNIV} /\!/ \approx_{\mathbb{L}(\boldsymbol{r}_1) \cup \mathbb{L}(\boldsymbol{r}_2)})$$

What Have We Achieved?

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$$x \approx_L y \stackrel{\text{def}}{=} \forall z.\, x@z \in L \Leftrightarrow y@z \in L$$

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$$UNIV // \approx_L \;=\; UNIV // \approx_{-L}$$
- if you want to do regular expression matching (see Scott's paper)
- I cannot yet give definite numbers

Examples

- $L \equiv \Sigma^\star 0 \Sigma$ is regular

$$\begin{aligned} A_1 &= \Sigma^\star 00 \\ A_2 &= \Sigma^\star 01 \\ A_3 &= \Sigma^\star 10 \cup \{0\} \\ A_4 &= \Sigma^\star 11 \cup \{1\} \cup \{[]\} \end{aligned}$$

- $L \equiv \{0^n 1^n \mid n \geq 0\}$ is not regular

$$\begin{aligned} B_0 &= \{0^n 1^n \mid n \geq 0\} \\ B_1 &= \{0^n 1^{(n-1)} \mid n \geq 1\} \\ B_2 &= \{0^n 1^{(n-2)} \mid n \geq 2\} \\ B_3 &= \{0^n 1^{(n-3)} \mid n \geq 3\} \\ &\vdots \end{aligned}$$

What We Have Not Achieved

- regular expressions are not good if you look for a minimal one for a language (DFAs have this notion)

What We Have Not Achieved

- regular expressions are not good if you look for a minimal one for a language (DFAs have this notion)

- Is there anything to be said about context free languages:

  A context free language is where every string can be recognised by a pushdown automaton.

Conclusion

- on balance regular expression are superior to DFAs, in my opinion
- I cannot think of a reason to not teach regular languages to students this way (!?)
- I have never ever seen a proof of Myhill-Nerode based on regular expressions
- no application, but lots of fun
- great source of examples