tphols-2011

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Contents

theory *Myhill*
  imports *Myhill-1*
begin

1 Direction *regular language ⇒finite partition*

1.1 The scheme

The following convenient notation $x \approx Lang\ y$ means: string x and y are equivalent with respect to language *Lang*.

definition
  str-eq :: *string* ⇒ *lang* ⇒ *string* ⇒ *bool* (- ≈- -)
where
  $x \approx Lang\ y \equiv (x, y) \in (\approx Lang)$

The main lemma (*rexp-imp-finite*) is proved by a structural induction over regular expressions. While base cases (cases for *NULL*, *EMPTY*, *CHAR*) are quite straight forward, the inductive cases are rather involved. What we have when starting to prove these inductive caes is that the partitions induced by the componet language are finite. The basic idea to show the finiteness of the partition induced by the composite language is to attach a tag $tag(x)$ to every string x. The tags are made of equivalent classes from the component partitions. Let *tag* be the tagging function and *Lang* be the composite language, it can be proved that if strings with the same tag are equivalent with respect to *Lang*, expressed as:

$$tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$$

then the partition induced by *Lang* must be finite. There are two arguments for this. The first goes as the following:

1. First, the tagging function *tag* induces an equivalent relation (=*tag*=) (defiintion of *f-eq-rel* and lemma *equiv-f-eq-rel*).

2. It is shown that: if the range of *tag* (denoted *range*(*tag*)) is finite, the partition given rise by (=*tag*=) is finite (lemma *finite-eq-f-rel*). Since tags are made from equivalent classes from component partitions, and the inductive hypothesis ensures the finiteness of these partitions, it is not difficult to prove the finiteness of *range*(*tag*).

3. It is proved that if equivalent relation *R1* is more refined than *R2* (expressed as $R1 \subseteq R2$), and the partition induced by *R1* is finite, then the partition induced by *R2* is finite as well (lemma *refined-partition-finite*).

4. The injectivity assumption $tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$ implies that (=*tag*=) is more refined than (≈*Lang*).

5. Combining the points above, we have: the partition induced by language *Lang* is finite (lemma *tag-finite-imageD*).

definition
  f-eq-rel (=-=)
where
  (=*f*=) = {(*x*, *y*) | *x y*. *f x* = *f y*}

lemma *equiv-f-eq-rel*:*equiv UNIV* (=*f*=)
  by (*auto simp*:*equiv-def f-eq-rel-def refl-on-def sym-def trans-def*)

lemma *finite-range-image*: *finite* (*range f*) ⟹ *finite* (*f* ‘ *A*)
  by (*rule-tac B* = {*y*. ∃ *x*. *y* = *f x*} **in** *finite-subset*, *auto simp*:*image-def*)

lemma *finite-eq-f-rel*:
  assumes *rng-fnt*: *finite* (*range tag*)
  shows *finite* (*UNIV* // (=*tag*=))
proof −
  let *?f* = *op* ‘ *tag* **and** *?A* = (*UNIV* // (=*tag*=))
  show *?thesis*
  proof (*rule-tac f* = *?f* **and** *A* = *?A* **in** *finite-imageD*)
    — The finiteness of *f*-image is a simple consequence of assumption *rng-fnt*:
    show *finite* (*?f* ‘ *?A*)

```
    proof -
      have ∀ X. ?f X ∈ (Pow (range tag)) by (auto simp:image-def Pow-def)
      moreover from rng-fnt have finite (Pow (range tag)) by simp
      ultimately have finite (range ?f)
        by (auto simp only:image-def intro:finite-subset)
      from finite-range-image [OF this] show ?thesis .
    qed
  next
    — The injectivity of f-image is a consequence of the definition of (=tag=):
    show inj-on ?f ?A
    proof-
      { fix X Y
        assume X-in: X ∈ ?A
          and  Y-in: Y ∈ ?A
          and  tag-eq: ?f X = ?f Y
        have X = Y
        proof -
          from X-in Y-in tag-eq
          obtain x y
            where x-in: x ∈ X and y-in: y ∈ Y and eq-tg: tag x = tag y
            unfolding quotient-def Image-def str-eq-rel-def
                                  str-eq-def image-def f-eq-rel-def
            apply simp by blast
          with X-in Y-in show ?thesis
            by (auto simp:quotient-def str-eq-rel-def str-eq-def f-eq-rel-def)
        qed
      } thus ?thesis unfolding inj-on-def by auto
    qed
  qed
qed

lemma finite-image-finite: [[∀ x ∈ A. f x ∈ B; finite B]] ⟹ finite (f ` A)
  by (rule finite-subset [of - B], auto)

lemma refined-partition-finite:
  fixes R1 R2 A
  assumes fnt: finite (A // R1)
  and refined: R1 ⊆ R2
  and eq1: equiv A R1 and eq2: equiv A R2
  shows finite (A // R2)
proof -
  let ?f = λ X. {R1 `` {x} | x. x ∈ X}
    and ?A = (A // R2) and ?B = (A // R1)
  show ?thesis
  proof(rule-tac f = ?f and A = ?A in finite-imageD)
    show finite (?f ` ?A)
    proof(rule finite-subset [of - Pow ?B])
      from fnt show finite (Pow (A // R1)) by simp
    next
```

```
      from eq2
      show  ?f ‘ A // R2 ⊆ Pow ?B
        unfolding image-def Pow-def quotient-def
        apply auto
        by (rule-tac x = xb in bexI, simp,
                 unfold equiv-def sym-def refl-on-def, blast)
    qed
  next
    show inj-on ?f ?A
    proof −
      { fix X Y
        assume X-in: X ∈ ?A and Y-in: Y ∈ ?A
          and eq-f: ?f X = ?f Y (is ?L = ?R)
        have X = Y using X-in
        proof(rule quotientE)
          fix x
          assume X = R2 ‘‘ {x} and x ∈ A with eq2
          have x-in: x ∈ X
            unfolding equiv-def quotient-def refl-on-def by auto
          with eq-f have R1 ‘‘ {x} ∈ ?R by auto
          then obtain y where
            y-in: y ∈ Y and eq-r: R1 ‘‘ {x} = R1 ‘‘{y} by auto
          have (x, y) ∈ R1
          proof −
            from x-in X-in y-in Y-in eq2
            have x ∈ A and y ∈ A
              unfolding equiv-def quotient-def refl-on-def by auto
            from eq-equiv-class-iff [OF eq1 this] and eq-r
            show ?thesis by simp
          qed
          with refined have xy-r2: (x, y) ∈ R2 by auto
          from quotient-eqI [OF eq2 X-in Y-in x-in y-in this]
          show ?thesis .
        qed
      } thus ?thesis by (auto simp:inj-on-def)
    qed
  qed
qed

lemma equiv-lang-eq: equiv UNIV (≈Lang)
  unfolding equiv-def str-eq-rel-def sym-def refl-on-def trans-def
  by blast

lemma tag-finite-imageD:
  fixes tag
  assumes rng-fnt: finite (range tag)
  — Suppose the rang of tagging fucntion tag is finite.
  and same-tag-eqvt: ⋀ m n. tag m = tag (n::string) ⟹ m ≈Lang n
  — And strings with same tag are equivalent
```

```
  shows finite (UNIV // (≈Lang))
proof -
  let ?R1 = (=tag=)
  show ?thesis
  proof(rule-tac refined-partition-finite [of - ?R1])
    from finite-eq-f-rel [OF rng-fnt]
     show finite (UNIV // =tag=) .
   next
     from same-tag-eqvt
     show (=tag=) ⊆ (≈Lang)
       by (auto simp:f-eq-rel-def str-eq-def)
   next
     from equiv-f-eq-rel
     show equiv UNIV (=tag=) by blast
   next
     from equiv-lang-eq
     show equiv UNIV (≈Lang) by blast
  qed
qed
```

A more concise, but less intelligible argument for *tag-finite-imageD* is given as the following. The basic idea is still using standard library lemma *finite-imageD*:

$$[\![\textit{finite}\ (f\ `\ A);\ \textit{inj-on}\ f\ A]\!] \Longrightarrow \textit{finite}\ A$$

which says: if the image of injective function f over set A is finite, then A must be finte, as we did in the lemmas above.

```
lemma
  fixes tag
  assumes rng-fnt: finite (range tag)
  — Suppose the rang of tagging fucntion tag is finite.
  and same-tag-eqvt: ⋀ m n. tag m = tag (n::string) ⟹ m ≈Lang n
  — And strings with same tag are equivalent
  shows finite (UNIV // (≈Lang))
  — Then the partition generated by (≈Lang) is finite.
proof -
  — The particular f and A used in finite-imageD are:
  let ?f = op ` tag and ?A = (UNIV // ≈Lang)
  show ?thesis
  proof (rule-tac f = ?f and A = ?A in finite-imageD)
    — The finiteness of f-image is a simple consequence of assumption rng-fnt:
    show finite (?f ` ?A)
    proof -
      have ∀ X. ?f X ∈ (Pow (range tag)) by (auto simp:image-def Pow-def)
      moreover from rng-fnt have finite (Pow (range tag)) by simp
      ultimately have finite (range ?f)
        by (auto simp only:image-def intro:finite-subset)
      from finite-range-image [OF this] show ?thesis .
    qed
```

```
  next
    — The injectivity of f is the consequence of assumption same-tag-eqvt:
    show inj-on ?f ?A
    proof−
      { fix X Y
        assume X-in: X ∈ ?A
          and  Y-in: Y ∈ ?A
          and  tag-eq: ?f X = ?f Y
        have X = Y
        proof −
          from X-in Y-in tag-eq
         obtain x y where x-in: x ∈ X and y-in: y ∈ Y and eq-tg: tag x = tag y
            unfolding quotient-def Image-def str-eq-rel-def str-eq-def image-def
            apply simp by blast
          from same-tag-eqvt [OF eq-tg] have x ≈Lang y .
          with X-in Y-in x-in y-in
          show ?thesis by (auto simp:quotient-def str-eq-rel-def str-eq-def)
        qed
      } thus ?thesis unfolding inj-on-def by auto
    qed
  qed
qed
```

1.2 The proof

Each case is given in a separate section, as well as the final main lemma. Detailed explainations accompanied by illustrations are given for non-trivial cases.

For ever inductive case, there are two tasks, the easier one is to show the range finiteness of of the tagging function based on the finiteness of component partitions, the difficult one is to show that strings with the same tag are equivalent with respect to the composite language. Suppose the composite language be *Lang*, tagging function be *tag*, it amounts to show:

$$tag(x) = tag(y) \Longrightarrow x \approx Lang\ y$$

expanding the definition of $\approx Lang$, it amounts to show:

$$tag(x) = tag(y) \Longrightarrow (\forall\ z.\ x@z \in Lang \longleftrightarrow y@z \in Lang)$$

Because the assumed tag equlity $tag(x) = tag(y)$ is symmetric, it is suffcient to show just one direction:

$$\bigwedge x\ y\ z.\ [\![tag(x) = tag(y);\ x@z \in Lang]\!] \Longrightarrow y@z \in Lang$$

This is the pattern followed by every inductive case.

1.2.1 The base case for *NULL*

```
lemma quot-null-eq:
  shows (UNIV // ≈{}) = ({UNIV}::lang set)
  unfolding quotient-def Image-def str-eq-rel-def by auto

lemma quot-null-finiteI [intro]:
  shows finite ((UNIV // ≈{})::lang set)
unfolding quot-null-eq by simp
```

1.2.2 The base case for *EMPTY*

```
lemma quot-empty-subset:
  UNIV // (≈{[]}) ⊆ {{[]}, UNIV − {[]}}
proof
  fix x
  assume x ∈ UNIV // ≈{[]}
  then obtain y where h: x = {z. (y, z) ∈ ≈{[]}}
    unfolding quotient-def Image-def by blast
  show x ∈ {{[]}, UNIV − {[]}}
  proof (cases y = [])
    case True with h
    have x = {[]} by (auto simp: str-eq-rel-def)
    thus ?thesis by simp
  next
    case False with h
    have x = UNIV − {[]} by (auto simp: str-eq-rel-def)
    thus ?thesis by simp
  qed
qed

lemma quot-empty-finiteI [intro]:
  shows finite (UNIV // (≈{[]}))
by (rule finite-subset[OF quot-empty-subset]) (simp)
```

1.2.3 The base case for *CHAR*

```
lemma quot-char-subset:
  UNIV // (≈{[c]}) ⊆ {{[]},{[c]}, UNIV − {[], [c]}}
proof
  fix x
  assume x ∈ UNIV // ≈{[c]}
  then obtain y where h: x = {z. (y, z) ∈ ≈{[c]}}
    unfolding quotient-def Image-def by blast
  show x ∈ {{[]},{[c]}, UNIV − {[], [c]}}
  proof −
    { assume y = [] hence x = {[]} using h
        by (auto simp:str-eq-rel-def)
    } moreover {
      assume y = [c] hence x = {[c]} using h
```

by (*auto dest*!:*spec*[**where** x = []] *simp*:*str-eq-rel-def*)
} moreover {
assume $y \neq []$ **and** $y \neq [c]$
hence $\forall\ z.\ (y\ @\ z) \neq [c]$ **by** (*case-tac y*, *auto*)
moreover have $\bigwedge p.\ (p \neq [] \wedge p \neq [c]) = (\forall\ q.\ p\ @\ q \neq [c])$
by (*case-tac p*, *auto*)
ultimately have $x = UNIV - \{[],[c]\}$ **using** h
by (*auto simp add*:*str-eq-rel-def*)
} ultimately show *?thesis* **by** *blast*
qed
qed

lemma *quot-char-finiteI* [*intro*]:
shows *finite* ($UNIV$ // ($\approx\{[c]\}$))
by (*rule finite-subset*[*OF quot-char-subset*]) (*simp*)

1.2.4 The inductive case for *ALT*

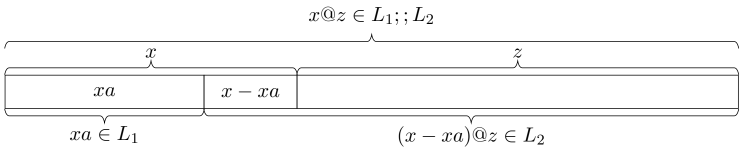
definition
tag-str-ALT :: *lang* ⇒ *lang* ⇒ *string* ⇒ (*lang* × *lang*)
where
tag-str-ALT L1 L2 = ($\lambda x.$ (≈*L1* `` $\{x\}$, ≈*L2* `` $\{x\}$))

lemma *quot-union-finiteI* [*intro*]:
fixes *L1 L2*::*lang*
assumes *finite1*: *finite* (*UNIV* // ≈*L1*)
and *finite2*: *finite* (*UNIV* // ≈*L2*)
shows *finite* (*UNIV* // ≈(*L1* ∪ *L2*))
proof (*rule-tac tag* = *tag-str-ALT L1 L2* **in** *tag-finite-imageD*)
show ⋀*x y*. *tag-str-ALT L1 L2 x* = *tag-str-ALT L1 L2 y* ⟹ *x* ≈(*L1* ∪ *L2*) *y*
unfolding *tag-str-ALT-def*
unfolding *str-eq-def*
unfolding *Image-def*
unfolding *str-eq-rel-def*
by *auto*
next
have ∗: *finite* ((*UNIV* // ≈*L1*) × (*UNIV* // ≈*L2*))
using *finite1 finite2* **by** *auto*
show *finite* (*range* (*tag-str-ALT L1 L2*))
unfolding *tag-str-ALT-def*
apply(*rule finite-subset*[*OF* - ∗])
unfolding *quotient-def*
by *auto*
qed

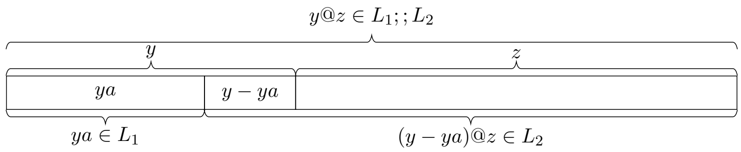
1.2.5 The inductive case for *SEQ*

For case *SEQ*, the language L is L_1 ;; L_2. Given $x\ @\ z \in L_1$;; L_2, according to the defintion of L_1 ;; L_2, string $x\ @\ z$ can be splitted with the prefix in

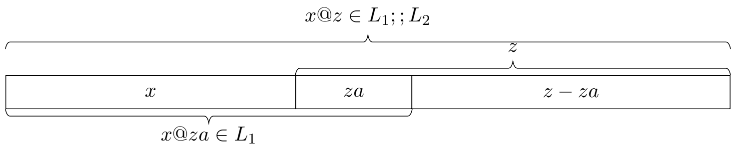
L_1 and suffix in L_2. The split point can either be in x (as shown in Fig. 1(a)), or in z (as shown in Fig. 1(c)). Whichever way it goes, the structure on x @ z cn be transfered faithfully onto y @ z (as shown in Fig. 1(b) and 1(d)) with the the help of the assumed tag equality. The following tag function *tag-str-SEQ* is such designed to facilitate such transfers and lemma *tag-str-SEQ-injI* formalizes the informal argument above. The details of structure transfer will be given their.



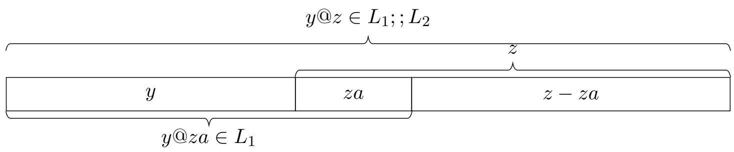
(a) First possible way to split $x@z$



(b) Transferred structure corresponding to the first way of splitting



(c) The second possible way to split $x@z$



(d) Transferred structure corresponding to the second way of splitting

Figure 1: The case for *SEQ*

definition
  tag-str-SEQ :: *lang* ⇒ *lang* ⇒ *string* ⇒ (*lang* × *lang set*)
where
  tag-str-SEQ L1 L2 =
    (λ*x*. (≈*L1* ‘‘ {*x*}, {(≈*L2* ‘‘ {*x* − *xa*}) | *xa*. *xa* ≤ *x* ∧ *xa* ∈ *L1*}))

The following is a techical lemma which helps to split the x @ $z \in L_1$;; L_2 mentioned above.

lemma *append-seq-elim*:
  assumes x @ $y \in L_1$;; L_2

shows $(\exists\ xa \le x.\ xa \in L_1 \wedge (x - xa)\ @\ y \in L_2) \vee$
$(\exists\ ya \le y.\ (x\ @\ ya) \in L_1 \wedge (y - ya) \in L_2)$
proof−
from *assms* **obtain** $s_1\ s_2$
where *eq-xys*: $x\ @\ y = s_1\ @\ s_2$
and *in-seq*: $s_1 \in L_1 \wedge s_2 \in L_2$
by (*auto simp*:*Seq-def*)
from *app-eq-dest* [*OF eq-xys*]
have
$(x \le s_1 \wedge (s_1 - x)\ @\ s_2 = y) \vee (s_1 \le x \wedge (x - s_1)\ @\ y = s_2)$
(**is** *?Split1* ∨ *?Split2*) **.**
moreover have *?Split1* $\Longrightarrow \exists\ ya \le y.\ (x\ @\ ya) \in L_1 \wedge (y - ya) \in L_2$
using *in-seq* **by** (*rule-tac* $x = s_1 - x$ **in** *exI*, *auto elim*:*prefixE*)
moreover have *?Split2* $\Longrightarrow \exists\ xa \le x.\ xa \in L_1 \wedge (x - xa)\ @\ y \in L_2$
using *in-seq* **by** (*rule-tac* $x = s_1$ **in** *exI*, *auto*)
ultimately show *?thesis* **by** *blast*
qed

lemma *tag-str-SEQ-injI*:
fixes $v\ w$
assumes *eq-tag*: *tag-str-SEQ* $L_1\ L_2\ v =$ *tag-str-SEQ* $L_1\ L_2\ w$
shows $v \approx (L_1\ ;;\ L_2)\ w$
proof−
— As explained before, a pattern for just one direction needs to be dealt with:
{ fix $x\ y\ z$
assume *xz-in-seq*: $x\ @\ z \in L_1\ ;;\ L_2$
and *tag-xy*: *tag-str-SEQ* $L_1\ L_2\ x =$ *tag-str-SEQ* $L_1\ L_2\ y$
have$y\ @\ z \in L_1\ ;;\ L_2$
proof−
— There are two ways to split $x@z$:
from *append-seq-elim* [*OF xz-in-seq*]
have $(\exists\ xa \le x.\ xa \in L_1 \wedge (x - xa)\ @\ z \in L_2) \vee$
$(\exists\ za \le z.\ (x\ @\ za) \in L_1 \wedge (z - za) \in L_2)$ **.**
— It can be shown that *?thesis* holds in either case:
moreover {
— The case for the first split:
fix *xa*
assume *h1*: $xa \le x$ **and** *h2*: $xa \in L_1$ **and** *h3*: $(x - xa)\ @\ z \in L_2$
— The following subgoal implements the structure transfer:
obtain *ya*
where $ya \le y$
and $ya \in L_1$
and $(y - ya)\ @\ z \in L_2$
proof −
By expanding the definition of

— *tag-str-SEQ* $L_1\ L_2\ x =$ *tag-str-SEQ* $L_1\ L_2\ y$

and extracting the second compoent, we get:

```
        have {≈L₂ `` {x - xa} |xa. xa ≤ x ∧ xa ∈ L₁} =
                {≈L₂ `` {y - ya} |ya. ya ≤ y ∧ ya ∈ L₁} (is ?Left = ?Right)
          using tag-xy unfolding tag-str-SEQ-def by simp
          — Since xa ≤ x and xa ∈ L₁ hold, it is not difficult to show:
        moreover have ≈L₂ `` {x - xa} ∈ ?Left using h1 h2 by auto
          — Through tag equality, equivalent class ≈L₂ `` {x - xa}
            also belongs to the ?Right:
        ultimately have ≈L₂ `` {x - xa} ∈ ?Right by simp
          — From this, the counterpart of xa in y is obtained:
        then obtain ya
          where eq-xya: ≈L₂ `` {x - xa} = ≈L₂ `` {y - ya}
          and pref-ya: ya ≤ y and ya-in: ya ∈ L₁
          by simp blast
        — It can be proved that ya has the desired property:
        have (y - ya)@z ∈ L₂
        proof -
          from eq-xya have (x - xa) ≈L₂ (y - ya)
            unfolding Image-def str-eq-rel-def str-eq-def by auto
          with h3 show ?thesis unfolding str-eq-rel-def str-eq-def by simp
        qed
        — Now, ya has all properties to be a qualified candidate:
        with pref-ya ya-in
        show ?thesis using that by blast
      qed
        — From the properties of ya, y @ z ∈ L₁ ;; L₂ is derived easily.
      hence y @ z ∈ L₁ ;; L₂ by (erule-tac prefixE, auto simp:Seq-def)
    } moreover {
      — The other case is even more simpler:
      fix za
      assume h1: za ≤ z and h2: (x @ za) ∈ L₁ and h3: z - za ∈ L₂
      have y @ za ∈ L₁
      proof-
        have ≈L₁ `` {x} = ≈L₁ `` {y}
          using tag-xy unfolding tag-str-SEQ-def by simp
        with h2 show ?thesis
          unfolding Image-def str-eq-rel-def str-eq-def by auto
      qed
      with h1 h3 have y @ z ∈ L₁ ;; L₂
        by (drule-tac A = L₁ in seq-intro, auto elim:prefixE)
    }
    ultimately show ?thesis by blast
  qed
}
— ?thesis is proved by exploiting the symmetry of eq-tag:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma quot-seq-finiteI [intro]:
```

fixes *L1 L2::lang*
assumes *fin1*: *finite* (*UNIV* // ≈*L1*)
and *fin2*: *finite* (*UNIV* // ≈*L2*)
shows *finite* (*UNIV* // ≈(*L1* ;; *L2*))
proof (*rule-tac tag* = *tag-str-SEQ L1 L2* **in** *tag-finite-imageD*)
show $\bigwedge x\ y$. *tag-str-SEQ L1 L2 x* = *tag-str-SEQ L1 L2 y* ⟹ *x* ≈(*L1* ;; *L2*) *y*
by (*rule tag-str-SEQ-injI*)
next
have ∗: *finite* ((*UNIV* // ≈*L1*) × (*Pow* (*UNIV* // ≈*L2*)))
using *fin1 fin2* **by** *auto*
show *finite* (*range* (*tag-str-SEQ L1 L2*))
unfolding *tag-str-SEQ-def*
apply(*rule finite-subset*[*OF* - ∗])
unfolding *quotient-def*
by *auto*
qed

1.2.6 The inductive case for *STAR*

This turned out to be the trickiest case. The essential goal is to proved y @ $z \in L_1*$ under the assumptions that x @ $z \in L_1*$ and that x and y have the same tag. The reasoning goes as the following:

1. Since x @ $z \in L_1*$ holds, a prefix xa of x can be found such that $xa \in L_1*$ and $(x - xa)@z \in L_1*$, as shown in Fig. 2(a). Such a prefix always exists, $xa = []$, for example, is one.

2. There could be many but fintie many of such xa, from which we can find the longest and name it *xa-max*, as shown in Fig. 2(b).

3. The next step is to split z into za and zb such that (x − *xa-max*) @ $za \in L_1$ and $zb \in L_1*$ as shown in Fig. 2(e). Such a split always exists because:

   (a) Because (x − *x-max*) @ $z \in L_1*$, it can always be splitted into prefix a and suffix b, such that $a \in L_1$ and $b \in L_1*$, as shown in Fig. 2(c).

   (b) But the prefix a CANNOT be shorter than x − *xa-max* (as shown in Fig. 2(d)), becasue otherwise, *ma-max*@a would be in the same kind as *xa-max* but with a larger size, conflicting with the fact that *xa-max* is the longest.

4. By the assumption that x and y have the same tag, the structure on x @ z can be transferred to y @ z as shown in Fig. 2(f). The detailed steps are:

   (a) A y-prefix ya corresponding to xa can be found, which satisfies conditions: $ya \in L_1*$ and $(y - ya)@za \in L_1$.

(b) Since we already know $zb \in L_1*$, we get $(y - ya)@za@zb \in L_1*$, and this is just $(y - ya)@z \in L_1*$.

(c) With fact $ya \in L_1*$, we finally get $y@z \in L_1*$.

The formal proof of lemma *tag-str-STAR-injI* faithfully follows this informal argument while the tagging function *tag-str-STAR* is defined to make the transfer in step **??** feasible.

definition
  tag-str-STAR :: *lang* ⇒ *string* ⇒ *lang set*
where
  $\textit{tag-str-STAR}\ L1 = (\lambda x.\ \{\approx L1\ ``\ \{x - xa\} \mid xa.\ xa < x \wedge xa \in L1\star\})$

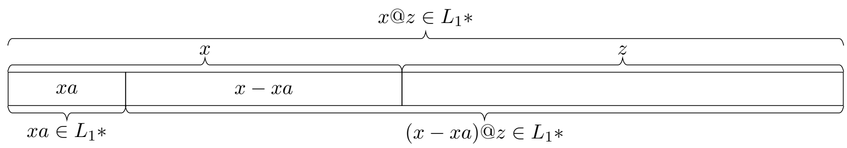
A technical lemma.

lemma *finite-set-has-max*: ⟦*finite A*; $A \neq \{\}$⟧ ⟹
  $(\exists\ max \in A.\ \forall\ a \in A.\ f\ a <= (f\ max :: nat))$
proof (*induct rule*:*finite.induct*)
  case *emptyI* **thus** *?case* **by** *simp*
next
  case (*insertI A a*)
  show *?case*
  proof (*cases* $A = \{\}$)
    case *True* **thus** *?thesis* **by** (*rule-tac* $x = a$ **in** *bexI*, *auto*)
  next
    case *False*
    with *insertI.hyps* **and** *False*
    obtain *max*
      where *h1*: $max \in A$
      and *h2*: $\forall a{\in}A.\ f\ a \leq f\ max$ **by** *blast*
    show *?thesis*
    proof (*cases* $f\ a \leq f\ max$)
      assume $f\ a \leq f\ max$
      with *h1 h2* **show** *?thesis* **by** (*rule-tac* $x = max$ **in** *bexI*, *auto*)
    next
      assume $\neg\ (f\ a \leq f\ max)$
      thus *?thesis* **using** *h2* **by** (*rule-tac* $x = a$ **in** *bexI*, *auto*)
    qed
  qed
qed

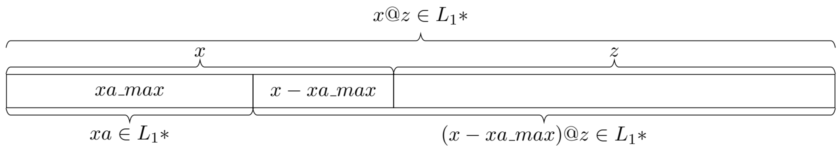
The following is a technical lemma.which helps to show the range finiteness of tag function.

lemma *finite-strict-prefix-set*: *finite* $\{xa.\ xa < (x{::}string)\}$
apply (*induct x rule*:*rev-induct*, *simp*)
apply (*subgoal-tac* $\{xa.\ xa < xs\ @\ [x]\} = \{xa.\ xa < xs\} \cup \{xs\}$)
by (*auto simp*:*strict-prefix-def*)

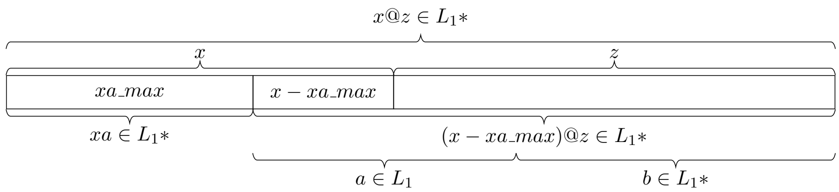
lemma *tag-str-STAR-injI*:



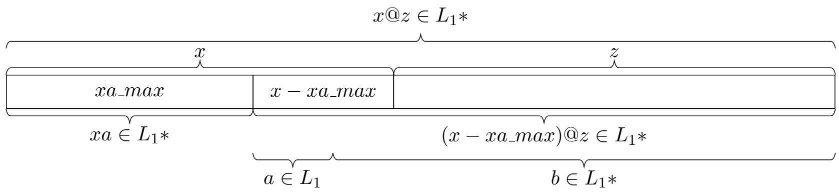
(a) First split



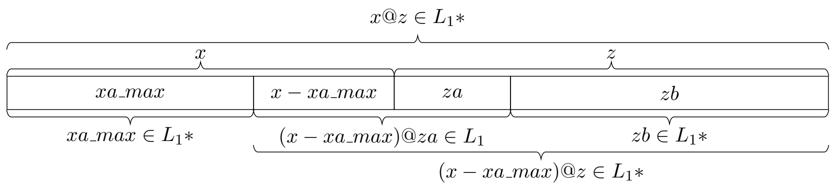
(b) Max split



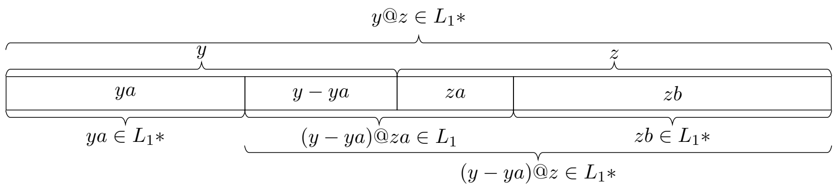
(c) Max split with a and b (the right situation)



(d) Max split with a and b (the wrong situation)



(e) Last split



(f) Structure transferred to y

Figure 2: The case for $STAR$

fixes *v w*
assumes *eq-tag*: *tag-str-STAR* L_1 *v* = *tag-str-STAR* L_1 *w*
shows (*v*::*string*) ≈(L_1⋆) *w*
proof−
— As explained before, a pattern for just one direction needs to be dealt with:
{ **fix** *x y z*
assume *xz-in-star*: *x* @ *z* ∈ L_1⋆
and *tag-xy*: *tag-str-STAR* L_1 *x* = *tag-str-STAR* L_1 *y*
have *y* @ *z* ∈ L_1⋆
proof(*cases x* = [])
— The degenerated case when *x* is a null string is easy to prove:
case *True*
with *tag-xy* **have** *y* = []
by (*auto simp add*: *tag-str-STAR-def strict-prefix-def*)
thus *?thesis* **using** *xz-in-star True* **by** *simp*
next
— The nontrival case:
case *False*
— Since *x* @ *z* ∈ L_1⋆, *x* can always be splitted by a prefix *xa* together with its suffix *x* − *xa*, such that both *xa* and (*x* − *xa*) @ *z* are in L_1⋆, and there could be many such splittings.Therefore, the following set *?S* is nonempty, and finite as well:
let *?S* = {*xa*. *xa* < *x* ∧ *xa* ∈ L_1⋆ ∧ (*x* − *xa*) @ *z* ∈ L_1⋆}
have *finite ?S*
by (*rule-tac B* = {*xa*. *xa* < *x*} **in** *finite-subset*,
auto simp:*finite-strict-prefix-set*)
moreover have *?S* ≠ {} **using** *False xz-in-star*
by (*simp*, *rule-tac x* = [] **in** *exI*, *auto simp*:*strict-prefix-def*)
— Since *?S* is finite, we can always single out the longest and name it *xa-max*:
ultimately have ∃ *xa-max* ∈ *?S*. ∀ *xa* ∈ *?S*. *length xa* ≤ *length xa-max*
using *finite-set-has-max* **by** *blast*
then obtain *xa-max*
where *h1*: *xa-max* < *x*
and *h2*: *xa-max* ∈ L_1⋆
and *h3*: (*x* − *xa-max*) @ *z* ∈ L_1⋆
and *h4*:∀ *xa* < *x*. *xa* ∈ L_1⋆ ∧ (*x* − *xa*) @ *z* ∈ L_1⋆
⟶ *length xa* ≤ *length xa-max*
by *blast*
— By the equality of tags, the counterpart of *xa-max* among *y*-prefixes, named *ya*, can be found:
obtain *ya*
where *h5*: *ya* < *y* **and** *h6*: *ya* ∈ L_1⋆
and *eq-xya*: (*x* − *xa-max*) ≈L_1 (*y* − *ya*)
proof−
from *tag-xy* **have** {≈L_1 `` {*x* − *xa*} |*xa*. *xa* < *x* ∧ *xa* ∈ L_1⋆} =
{≈L_1 `` {*y* − *xa*} |*xa*. *xa* < *y* ∧ *xa* ∈ L_1⋆} (**is** *?left* = *?right*)
by (*auto simp*:*tag-str-STAR-def*)
moreover have ≈L_1 `` {*x* − *xa-max*} ∈ *?left* **using** *h1 h2* **by** *auto*
ultimately have ≈L_1 `` {*x* − *xa-max*} ∈ *?right* **by** *simp*
thus *?thesis* **using** *that*

```
    apply (simp add:Image-def str-eq-rel-def str-eq-def) by blast
qed
— The ?thesis, y @ z ∈ L₁⋆, is a simple consequence of the following
  proposition:
have (y − ya) @ z ∈ L₁⋆
proof−
  — The idea is to split the suffix z into za and zb, such that:
  obtain za zb where eq-zab: z = za @ zb
    and l-za: (y − ya)@za ∈ L₁ and ls-zb: zb ∈ L₁⋆
  proof −
    — Since xa-max < x, x can be splitted into a and b such that:
    from h1 have (x − xa-max) @ z ≠ []
      by (auto simp:strict-prefix-def elim:prefixE)
    from star-decom [OF h3 this]
    obtain a b where a-in: a ∈ L₁
      and a-neq: a ≠ [] and b-in: b ∈ L₁⋆
      and ab-max: (x − xa-max) @ z = a @ b by blast
    — Now the candiates for za and zb are found:
    let ?za = a − (x − xa-max) and ?zb = b
    have pfx: (x − xa-max) ≤ a (is ?P1)
      and eq-z: z = ?za @ ?zb (is ?P2)
    proof −
      — Since (x − xa-max) @ z = a @ b, string (x − xa-max) @ z can
        be splitted in two ways:
      have ((x − xa-max) ≤ a ∧ (a − (x − xa-max)) @ b = z) ∨
        (a < (x − xa-max) ∧ ((x − xa-max) − a) @ z = b)
        using app-eq-dest[OF ab-max] by (auto simp:strict-prefix-def)
      moreover {
        — However, the undsired way can be refuted by absurdity:
        assume np: a < (x − xa-max)
          and b-eqs: ((x − xa-max) − a) @ z = b
        have False
        proof −
          let ?xa-max′ = xa-max @ a
          have ?xa-max′ < x
            using np h1 by (clarsimp simp:strict-prefix-def diff-prefix)
          moreover have ?xa-max′ ∈ L₁⋆
            using a-in h2 by (simp add:star-intro3)
          moreover have (x − ?xa-max′) @ z ∈ L₁⋆
            using b-eqs b-in np h1 by (simp add:diff-diff-appd)
          moreover have ¬ (length ?xa-max′ ≤ length xa-max)
            using a-neq by simp
          ultimately show ?thesis using h4 by blast
        qed }
      — Now it can be shown that the splitting goes the way we desired.
      ultimately show ?P1 and ?P2 by auto
    qed
    hence (x − xa-max)@?za ∈ L₁ using a-in by (auto elim:prefixE)
    — Now candidates ?za and ?zb have all the requred properteis.
    with eq-xya have (y − ya) @ ?za ∈ L₁
```

by (*auto simp*:*str-eq-def str-eq-rel-def*)
with *eq-z* **and** *b-in*
show *?thesis* **using** *that* **by** *blast*
qed
— *?thesis* can easily be shown using properties of *za* and *zb*:
have $((y - ya) @ za) @ zb \in L_1\star$ **using** *l-za ls-zb* **by** *blast*
with *eq-zab* **show** *?thesis* **by** *simp*
qed
with *h5 h6* **show** *?thesis*
by (*drule-tac star-intro1*, *auto simp*:*strict-prefix-def elim*:*prefixE*)
qed
}
— By instantiating the reasoning pattern just derived for both directions:
from *this* [*OF* - *eq-tag*] **and** *this* [*OF* - *eq-tag* [*THEN sym*]]
— The thesis is proved as a trival consequence:
show *?thesis* **unfolding** *str-eq-def str-eq-rel-def* **by** *blast*
qed

lemma — The oringal version with less explicit details.
fixes *v w*
assumes *eq-tag*: *tag-str-STAR* L_1 *v* = *tag-str-STAR* L_1 *w*
shows (*v*::*string*) $\approx(L_1\star)$ *w*
proof−
— According to the definition of ≈*Lang*, proving $v \approx(L_1\star)\ w$ amounts to showing: for any string *u*, if $v @ u \in (L_1\star)$ then $w @ u \in (L_1\star)$ and vice versa. The reasoning pattern for both directions are the same, as derived in the following:
{ fix *x y z*
assume *xz-in-star*: $x @ z \in L_1\star$
and *tag-xy*: *tag-str-STAR* L_1 *x* = *tag-str-STAR* L_1 *y*
have $y @ z \in L_1\star$
proof(*cases* *x* = [])
— The degenerated case when *x* is a null string is easy to prove:
case *True*
with *tag-xy* **have** *y* = []
by (*auto simp*:*tag-str-STAR-def strict-prefix-def*)
thus *?thesis* **using** *xz-in-star True* **by** *simp*
next
— The case when *x* is not null, and $x @ z$ is in $L_1\star$,
case *False*
obtain *x-max*
where *h1*: *x-max* < *x*
and *h2*: $x\text{-}max \in L_1\star$
and *h3*: $(x - x\text{-}max) @ z \in L_1\star$
and *h4*:$\forall\ xa < x.\ xa \in L_1\star \wedge (x - xa) @ z \in L_1\star$
$\longrightarrow$ *length xa* ≤ *length x-max*
proof−
let $?S = \{xa.\ xa < x \wedge xa \in L_1\star \wedge (x - xa) @ z \in L_1\star\}$
have *finite ?S*

by (*rule-tac* $B = \{xa.\ xa < x\}$ **in** *finite-subset*,
auto simp:*finite-strict-prefix-set*)
moreover have *?S* $\neq \{\}$ **using** *False xz-in-star*
by (*simp*, *rule-tac* $x = []$ **in** *exI*, *auto simp*:*strict-prefix-def*)
ultimately have $\exists$ *max* $\in$ *?S*. $\forall$ *a* $\in$ *?S*. *length a* $\leq$ *length max*
using *finite-set-has-max* **by** *blast*
thus *?thesis* **using** *that* **by** *blast*
qed
obtain *ya*
where *h5*: $ya < y$ **and** *h6*: $ya \in L_1\star$ **and** *h7*: $(x - x\text{-}max) \approx L_1\ (y - ya)$
proof−
from *tag-xy* **have** $\{\approx L_1$ `` $\{x - xa\}\ |xa.\ xa < x \wedge xa \in L_1\star\}$ =
$\{\approx L_1$ `` $\{y - xa\}\ |xa.\ xa < y \wedge xa \in L_1\star\}$ (**is** *?left* = *?right*)
by (*auto simp*:*tag-str-STAR-def*)
moreover have $\approx L_1$ `` $\{x - x\text{-}max\} \in$ *?left* **using** *h1 h2* **by** *auto*
ultimately have $\approx L_1$ `` $\{x - x\text{-}max\} \in$ *?right* **by** *simp*
with *that* **show** *?thesis* **apply**
(*simp add*:*Image-def str-eq-rel-def str-eq-def*) **by** *blast*
qed
have $(y - ya)\ @\ z \in L_1\star$
proof−
from *h3 h1* **obtain** *a b* **where** *a-in*: $a \in L_1$
and *a-neq*: $a \neq []$ **and** *b-in*: $b \in L_1\star$
and *ab-max*: $(x - x\text{-}max)\ @\ z = a\ @\ b$
by (*drule-tac star-decom*, *auto simp*:*strict-prefix-def elim*:*prefixE*)
have $(x - x\text{-}max) \leq a \wedge (a - (x - x\text{-}max))\ @\ b = z$
proof −
have $((x - x\text{-}max) \leq a \wedge (a - (x - x\text{-}max))\ @\ b = z) \vee$
$(a < (x - x\text{-}max) \wedge ((x - x\text{-}max) - a)\ @\ z = b)$
using *app-eq-dest*[*OF ab-max*] **by** (*auto simp*:*strict-prefix-def*)
moreover **{**
assume *np*: $a < (x - x\text{-}max)$ **and** *b-eqs*: $((x - x\text{-}max) - a)\ @\ z = b$
have *False*
proof −
let $?x\text{-}max' = x\text{-}max\ @\ a$
have $?x\text{-}max' < x$
using *np h1* **by** (*clarsimp simp*:*strict-prefix-def diff-prefix*)
moreover have $?x\text{-}max' \in L_1\star$
using *a-in h2* **by** (*simp add*:*star-intro3*)
moreover have $(x - ?x\text{-}max')\ @\ z \in L_1\star$
using *b-eqs b-in np h1* **by** (*simp add*:*diff-diff-appd*)
moreover have $\neg$ (*length* $?x\text{-}max' \leq$ *length x-max*)
using *a-neq* **by** *simp*
ultimately show *?thesis* **using** *h4* **by** *blast*
qed
} ultimately show *?thesis* **by** *blast*
qed
then obtain *za* **where** *z-decom*: $z = za\ @\ b$
and *x-za*: $(x - x\text{-}max)\ @\ za \in L_1$

```
        using a-in by (auto elim:prefixE)
      from x-za h7 have (y − ya) @ za ∈ L₁
        by (auto simp:str-eq-def str-eq-rel-def)
      with b-in have ((y − ya) @ za) @ b ∈ L₁⋆ by blast
      with z-decom show ?thesis by auto
    qed
    with h5 h6 show ?thesis
      by (drule-tac star-intro1, auto simp:strict-prefix-def elim:prefixE)
  qed
}
— By instantiating the reasoning pattern just derived for both directions:
from this [OF - eq-tag] and this [OF - eq-tag [THEN sym]]
— The thesis is proved as a trival consequence:
  show ?thesis unfolding str-eq-def str-eq-rel-def by blast
qed

lemma quot-star-finiteI [intro]:
  fixes L1::lang
  assumes finite1: finite (UNIV // ≈L1)
  shows finite (UNIV // ≈(L1⋆))
proof (rule-tac tag = tag-str-STAR L1 in tag-finite-imageD)
  show ⋀x y. tag-str-STAR L1 x = tag-str-STAR L1 y ⟹ x ≈(L1⋆) y
    by (rule tag-str-STAR-injI)
next
  have ∗: finite (Pow (UNIV // ≈L1))
    using finite1 by auto
  show finite (range (tag-str-STAR L1))
    unfolding tag-str-STAR-def
    apply(rule finite-subset[OF - ∗])
    unfolding quotient-def
    by auto
qed
```

1.2.7 The conclusion

```
lemma rexp-imp-finite:
  fixes r::rexp
  shows finite (UNIV // ≈(L r))
by (induct r) (auto)

end
```