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A game-theoretic characterization of Boolean grammars☆,☆☆

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ABSTRACT

We obtain a simple, purely game-theoretic characterization of Boolean grammars [A. Okhotin, Boolean grammars, Information and Computation, 194(1) (2004) 19–48]. In particular, we propose a two-player infinite game of perfect information for Boolean grammars, which is equivalent to their well-founded semantics. The game is directly applicable to the simpler classes of conjunctive and context-free grammars, and offers a promising new connection between game theory and formal languages.

1. Introduction

Boolean grammars were recently proposed by Okhotin [10] as a generalization of context-free grammars. The main characteristic of Boolean grammars is that they allow conjunction and negation to appear in the right hand sides of rules. The resulting formalism has proven to be a very expressive one (see for example [12]), while retaining to a large extent the efficient parsing properties of context-free grammars.

The theory of Boolean grammars is presently under rapid development. However, there exist many fundamental questions that still remain unanswered (see [11] for an exposition of the basic open problems of the field). The area appears to be quite an intriguing one, since many of the problems remain unanswered even for the negation-free class (namely, for the class of conjunctive grammars [9]).

In this paper we contribute to this area of research by providing a simple, purely game-theoretic characterization of the semantics of these type of grammars. More specifically, we propose a two-player infinite game of perfect information for Boolean grammars, which is equivalent to their well-founded semantics [3].

The game characterization we propose has the advantage of being very simple to understand and present, due to its anthropomorphic flavor. In this respect, it appears to be easier to use than the corresponding well-founded approach of [3]. We believe that the new approach will offer more benefits when used in order to prove the correctness of transformations on Boolean grammars, while the well-founded approach will be more appropriate for computing the meaning of specific grammars. Finally, it should be mentioned that the game is also applicable to the simpler classes of *conjunctive* [9] and *context-free* grammars, and therefore it also provides an alternative equivalent way to define the semantics for these type of grammars.

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The game we present has been inspired from a recent game-theoretic characterization of logic programs with negation [1]. Actually, the present game is more complicated than the one in [1] since it involves string manipulation by the two-players. A contribution of our work is that it gives an alternative proof technique than the one derived in [1]. More specifically, the proof in [1] proceeds in two steps: it first establishes the determinacy of the logic programming game by using certain deep notions from infinite-game theory (namely, the theory of *Borel sets* [6] and Martin's *Borel Determinacy Theorem* [5]); subsequently, based on the determinacy result, it establishes the equivalence of the game-semantics to the well-founded semantics of logic programs. The proof in [1] also uses an intermediate game (called the *refined-game*) as-well-as a refined version of the well-founded semantics [13]. Our present proof establishes *at the same time* both the determinacy of the game *and* its equivalence to the well-founded semantics (avoiding completely the use of Borel sets, Martin's theorem, the refined game and the refined well-founded semantics used in [1]).

The key idea of the new proof can be outlined as follows. The well-founded model is first used as a "guide" in order to define a strategy for Player I and a corresponding one for Player II of the infinite game we propose. It is then demonstrated that these two strategies are optimal, i.e., they ensure the best possible outcome for the two players. Based on this fact, it is shown that the game has the same value as that computed by the well-founded construction.

The rest of the paper is organized as follows: Section 2 presents preliminary material; in particular, it gives a self-contained presentation of the well-founded semantics for Boolean grammars. Section 3 gives an informal explanation of the game and illustrates it by examples. Section 4 gives a precise formalization of the new game. Section 5 proves the equivalence of the game to the well-founded semantics of Boolean grammars. Section 6 contains pointers to future work.

2. Preliminaries

In [9,10] Okhotin introduced the classes of conjunctive and Boolean grammars respectively.[1] Formally:

Definition 1. A *Boolean grammar* is a quadruple $G = (\Sigma, N, P, S)$, where Σ and N are disjoint finite nonempty sets of terminal and nonterminal symbols respectively, P is a finite set of rules, each of the form

$$C \to \alpha_1 \& \cdots \& \alpha_m \& \neg\beta_1 \& \cdots \& \neg\beta_n \quad (m + n \geq 1, C \in N, \alpha_i, \beta_j \in (\Sigma \cup N)^*)$$

and $S \in N$ is the start symbol of the grammar. We will call the α_i's *positive conjuncts* and the $\neg\beta_j$'s *negative*. A Boolean grammar is called *conjunctive* if all its rules contain only positive conjuncts.

The semantics of Boolean grammars is not straightforward due to the fact that the nonterminals of the grammar may depend on each other in a circular way that involves negation. To circumvent this problem, it has been proposed [3,4] that the correct mathematical formulation of the meaning of Boolean grammars should be based on *three-valued formal languages*. Intuitively, given a three-valued language L and a string w over the alphabet of L, there are three cases: either $w \in L$ (i.e., $L(w) = 1$), or $w \notin L$ (i.e., $L(w) = 0$), or finally, the membership of w in L is unclear (i.e., $L(w) = \frac{1}{2}$). Given this extended notion of language, it is now possible to interpret Boolean grammars with circularities that involve negation. For example, the meaning of the grammar $S \to \neg S$ is the language which assigns to every string the value $\frac{1}{2}$. These ideas are formalized in the rest of this section (our presentation follows [8,4]).

Definition 2. Let Σ be a finite non-empty set of symbols. Then, a (three-valued) language over Σ is a function from Σ^* to the set $\left\{0, \frac{1}{2}, 1\right\}$.

Based on the above definition, we can generalize the usual set-theoretic notion of *subset* as well as that of the *empty language*:

Definition 3. Let L, L' be three-valued languages over Σ. Then, we write $L \subseteq L'$ if and only if for every $w \in \Sigma^*, L(w) \leq L'(w)$. The *empty* three-valued language is the language L such that for every $w \in \Sigma^*, L(w) = 0$.

We will also need a second subset relation (the *Fitting subset relation*) which compares the *degree of information* of two languages:

Definition 4. Let L, L' be three-valued languages over Σ. Then, we write $L \subseteq_F L'$ if and only if for every $w \in \Sigma^*$, if $L(w) \neq \frac{1}{2}$ then $L(w) = L'(w)$. The *Fitting-empty* three-valued language is the language L such that for every $w \in \Sigma^*, L(w) = \frac{1}{2}$.

The following definition, which generalizes the familiar notion of concatenation of languages, is also needed:

[1] As one of the reviewers remarked, from a technical point of view, every conjunctive grammar and every boolean grammar is still a context-free grammar (the left side of every rule is a single variable, so replacement is independent of context). Therefore, a more accurate naming of these three types of grammars would be "*classical context-free grammars*", "*conjunctive context-free grammars*", and "*boolean context-free grammars*". However, we will retain the usual naming "context-free", "conjunctive" and "boolean" since it is widely used in the literature.

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Definition 5. Let Σ be a finite non-empty set of symbols and let $L_1, \ldots, L_n$ be (three-valued) languages over Σ. We define the *three-valued concatenation* of the languages $L_1, \ldots, L_n$ to be the language L such that:

$$L(w) = \max_{\substack{(w_1,\ldots,w_n):\\ w=w_1\cdots w_n}} \left(\min_{1\le i\le n} L_i(w_i) \right).$$

The concatenation of $L_1, \ldots, L_n$ will be denoted by $L_1 \circ \cdots \circ L_n$.

The above definition can be explained as follows:

- A string w belongs to $L_1 \circ \cdots \circ L_n$ (truth value 1) if it can be partitioned into n parts so that for every i, the i-th part belongs to L_i.
- A string w is excluded from the concatenation (truth value 0) if in every partition of w, there exists some i such that the i-th part is excluded from the language L_i.
- The membership of a string w is undefined in the concatenation (truth value $\frac{1}{2}$) if there exists a partition of w such that no part is excluded from the corresponding language, and there does not exist a partition of w such that every part belongs to the corresponding language.

We can now define the notion of *interpretation* of a given Boolean grammar:

Definition 6. An *interpretation* I of a Boolean grammar $G = (\Sigma, N, P, S)$ is a function $I : N \to \left(\Sigma^* \to \left\{0, \frac{1}{2}, 1\right\}\right)$.

An interpretation I can be recursively extended to apply to expressions that appear as the right-hand sides of Boolean grammar rules:

Definition 7. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and I be an interpretation of G. Then I can be extended to apply to expressions that appear as the right-hand sides of Boolean grammar rules as follows:

- For the empty sequence ϵ and for all $w \in \Sigma^*$, it is $I(\epsilon)(w) = 1$ if $w = \epsilon$ and $I(\epsilon)(w) = 0$ otherwise.
- Let $a \in \Sigma$ be a terminal symbol. Then, for every $w \in \Sigma^*$, $I(a)(w) = 1$ if $w = a$ and $I(a)(w) = 0$ otherwise.
- Let $\alpha = \alpha_1 \cdots \alpha_n$, $n \ge 2$, be a sequence in $(\Sigma \cup N)^*$. Then, for every $w \in \Sigma^*$, it is $I(\alpha)(w) = (I(\alpha_1) \circ \cdots \circ I(\alpha_n))(w)$.
- Let $\alpha \in (\Sigma \cup N)^*$. Then, for every $w \in \Sigma^*$, $I(\neg\alpha)(w) = 1 - I(\alpha)(w)$.
- Let $l_1, \ldots, l_n$ be conjuncts. Then, for every $w \in \Sigma^*$, $I(l_1 \& \cdots \& l_n)(w) = \min\{I(l_1)(w), \ldots, I(l_n)(w)\}$.

We are particularly interested in interpretations that *satisfy* all the rules of a given grammar:

Definition 8. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and I be an interpretation of G. Then, I is a *model* of G if for every rule $A \to l_1 \& \cdots \& l_n$ in P, it is $I(A) \supseteq I(l_1 \& \cdots \& l_n)$.

In the definition of the well-founded model, two orderings on interpretations play a crucial role. Given two interpretations, the first ordering (usually called the *standard ordering*) compares their *degree of truth*:

Definition 9. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and I, J be two interpretations of G. Then, we say that $I \preceq J$ if for all $A \in N$, $I(A) \subseteq J(A)$.

Among the interpretations of a given Boolean grammar, there is one which is the least with respect to the $\preceq$ ordering, denoted by $\perp$, and which assigns the empty language to all nonterminals of the grammar.

The second ordering (usually called the *Fitting ordering*) compares the *degree of information* of two interpretations:

Definition 10. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and I, J be two interpretations of G. Then, we say that $I \preceq_F J$ if for all $A \in N$, $I(A) \subseteq_F J(A)$.

Among the interpretations of a given Boolean grammar, there is one which is the least with respect to the $\preceq_F$ ordering, denoted by $\perp_F$, and which assigns the Fitting-empty language to all nonterminals of the grammar.

Given a set U of interpretations, we will write $\mathrm{lub}_{\preceq} U$ (respectively $\mathrm{lub}_{\preceq_F} U$) for the least upper bound of the members of U under the standard ordering (respectively, the Fitting ordering).

Consider a Boolean grammar G. Then, for any interpretation J of G we define the operator $\Theta_J : \mathcal{I} \to \mathcal{I}$ on the set $\mathcal{I}$ of all 3-valued interpretations of G. Intuitively, J represents information that we have already derived and is considered stable (and therefore it can be safely used to compute the value of negative conjuncts). More specifically, given $I \in \mathcal{I}$, $A \in N$ and $w \in \Sigma^*$, $\Theta_J(I)(A)(w)$ is the value that w gets (using the rules of the grammar) *in one step* when using J in order to evaluate the negative conjuncts in rules defining A in G and I to evaluate the positive ones. More formally:

Definition 11. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, let $\mathcal{I}$ be the set of all three-valued interpretations of G and let $J \in \mathcal{I}$. The operator $\Theta_J : \mathcal{I} \to \mathcal{I}$ is defined as follows. For every $I \in \mathcal{I}$, for all $A \in N$ and for all $w \in \Sigma^*$:

1. $\Theta_J(I)(A)(w) = 1$ if there is a rule $A \to l_1 \& \cdots \& l_n$ in P such that, for every positive l_i it is $I(l_i)(w) = 1$ and for every negative l_i it is $J(l_i)(w) = 1$;

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2. $\Theta_J(I)(A)(w) = 0$ if for every rule $A \to l_1 \& \cdots \& l_n$ in P, either there exists a positive l_i such that $I(l_i)(w) = 0$ or there exists a negative l_i such that $J(l_i)(w) = 0$;
3. $\Theta_J(I)(A)(w) = \frac{1}{2}$, otherwise.

An important fact regarding the operator Θ_J is that it is monotonic with respect to the $\preceq$ ordering of interpretations:

Theorem 12. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and let J be an interpretation of G. Then, the operator Θ_J is monotonic with respect to the $\preceq$ ordering of interpretations. Moreover, Θ_J has a unique least (with respect to $\preceq$) fixed point $\Theta_J^{\uparrow\omega}$ which can be constructed as follows:*

$$\Theta_J^{\uparrow 0} = \bot$$
$$\Theta_J^{\uparrow n+1} = \Theta_J(\Theta_J^{\uparrow n})$$
$$\Theta_J^{\uparrow \omega} = \mathsf{lub}_{\preceq}\{\Theta_J^{\uparrow n} \mid n < \omega\}.$$

We will denote by $\Omega(J)$ the least fixed point $\Theta_J^{\uparrow\omega}$ of Θ_J. Given a grammar G, we can use the Ω operator to construct a sequence of interpretations whose least upper bound M_G (with respect to the Fitting ordering) is a distinguished model of G:

Theorem 13. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar. Then, the operator Ω, where $\Omega(J) = \Theta_J^{\uparrow\omega}$, is monotonic with respect to the $\preceq_F$ ordering of interpretations. Moreover, Ω has a unique least (with respect to $\preceq_F$) fixed point M_G which can be constructed as follows:*

$$M_0 = \bot_F$$
$$M_{n+1} = \Omega(M_n)$$
$$M_G = \mathsf{lub}_{\preceq_F}\{M_n \mid n < \omega\}.$$

Theorem 14. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar. Then, M_G is a model of G (which will be called the* well-founded model *of G).*

The significance of the above result lies in the fact that it specifies for *every* Boolean grammar G a three-valued formal language M_G that can be taken as the meaning of G. It can be seen that the well-founded semantics of Boolean grammars generalizes both the semantics of context-free as well as the semantics of conjunctive grammars.

At this point, it is useful to give some further explanations concerning the construction of M_G. This information will help in understanding the functions that will be introduced in Definition 15, and which will be heavily used in establishing the equivalence of the proposed game to the well-founded semantics.

Consider $A \in N$ and $w \in \Sigma^*$. The monotonicity of the operator Ω with respect to the $\preceq_F$ ordering of interpretations, has different consequences depending on the value of $M_G(A)(w)$. More specifically:

- $M_G(A)(w) = 1$. Then, there exists some $i > 0$, such that[2] for every $n < i$, $M_n(A)(w) = \frac{1}{2}$ and for every $n \geq i$, $M_n(A)(w) = 1$. The former implies (by the definition of Ω and the monotonicity of the Θ operator with respect to $\preceq$) that for every n, $1 \leq n < i$, there exists a $j_n \geq 1$ such that for every $k < j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = 0$ and for every $k \geq j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = \frac{1}{2}$. The latter implies that for every $n \geq i$ there exists a $j_n \geq 1$, such that[3] for every $k < j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) \leq \frac{1}{2}$ and for every $k \geq j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = 1$.
- $M_G(A)(w) = 0$. Then, there exists some $i > 0$, such that for every $n < i$, $M_n(A)(w) = \frac{1}{2}$ and for every $n \geq i$, $M_n(A)(w) = 0$. The former implies that for every n, $1 \leq n < i$, there exists a $j_n \geq 1$, such that for every $k < j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = 0$ and for every $k \geq j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = \frac{1}{2}$. The latter implies that for every $n \geq i$ and for every $k \geq 0$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = 0$.
- $M_G(A)(w) = \frac{1}{2}$. Then, for every $n \geq 0$, $M_n(A)(w) = \frac{1}{2}$. This implies that for every $n \geq 1$ there exists a $j_n \geq 1$, such that for every $k < j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = 0$ and for every $k \geq j_n$, $\Theta_{M_{n-1}}^{\uparrow k}(A)(w) = \frac{1}{2}$. Moreover, since M_G is a fixed point of Ω, there exists a $j \geq 1$, such that for every $k < j$, $\Theta_{M_G}^{\uparrow k}(A)(w) = 0$ and for every $k \geq j$, $\Theta_{M_G}^{\uparrow k}(A)(w) = \frac{1}{2}$.

Similar situations occur if, more generally, a sequence of symbols $\alpha \in (\Sigma \cup N)^*$ instead of a single nonterminal $A \in N$ is considered.

In the following definition, we denote by E the set $(\Sigma \cup N)^* - (\Sigma^* \cup N)$ (that is, E consists of all sequences of terminal and nonterminal symbols of length at least 2, that contain at least one nonterminal symbol). Thus, Σ^*, N, and E form a partition of $(\Sigma \cup N)^*$.

[2] Notice that in Definition 15, i will be denoted by $\mathsf{odp}(A, w)$ (intuitively, the *outer determination point* of the value of $M_G(A)(w)$).

[3] In Definition 15, j_i will be denoted by $\mathsf{idp}(A, w)$ (intuitively, the *inner determination point* of the value of $M_G(A)(w)$).

Definition 15. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$. We define the functions odp and idp (standing for *outer determination point* and *inner determination point* respectively) as follows:

$$\mathsf{odp}(\alpha, w) = \begin{cases} \min\{i \mid M_i(\alpha)(w) \in \{0,1\}\}, & \text{if } M_G(\alpha)(w) \in \{0,1\} \\ \text{undefined}, & \text{if } M_G(\alpha)(w) = \frac{1}{2} \end{cases}$$

$$\mathsf{odp}(\neg\alpha, w) = \mathsf{odp}(\alpha, w)$$

$$\mathsf{idp}(\alpha, w) = \begin{cases} 0, & \text{if } M_G(\alpha)(w) = 1 \\ & \text{and } \mathsf{odp}(\alpha, w) = 0 \\ \min\{j \mid \Theta^{\uparrow j}_{M_{\mathsf{odp}(\alpha,w)-1}}(\alpha)(w) = 1\}, & \text{if } M_G(\alpha)(w) = 1 \\ & \text{and } \mathsf{odp}(\alpha, w) > 0 \\ \min\{j \mid \Theta^{\uparrow j}_{M_G}(\alpha)(w) = \frac{1}{2}\}, & \text{if } M_G(\alpha)(w) = \frac{1}{2} \\ \text{undefined}, & \text{if } M_G(\alpha)(w) = 0. \end{cases}$$

Notice that the definitions of the functions odp and idp can be justified based on the discussion given just before Definition 15. The following lemma will prove useful in a later section of the paper:

Lemma 16. *If* $M_G(\alpha)(w) = 1$ *and* $\mathsf{odp}(\alpha, w) > 0$ *then* $\mathsf{idp}(\alpha, w) > 0$.

Proof. Suppose, for the sake of contradiction, that $\mathsf{odp}(\alpha, w) = i > 0$ and $\mathsf{idp}(\alpha, w) = 0$. Then from the definition of idp we have $\Theta^{\uparrow 0}_{M_{i-1}}(\alpha)(w) = 1$, which implies that $\perp(\alpha)(w) = 1$. Therefore, it must be $\alpha = w$, from which we obtain that also $\perp_F(\alpha)(w) = 1$. Thus, $M_0(\alpha)(w) = 1$. From Definition 15 it follows that $\mathsf{odp}(\alpha, w) = 0$, which is a contradiction. □

3. The game for Boolean grammars

Consider a Boolean grammar $G = (\Sigma, N, P, S)$ and let $A \in N$ and $w \in \Sigma^*$. We describe at an intuitive level a two-player game $\Gamma_G(A, w)$ which has the property that $M_G(A)(w) = 0$ if and only if Player I has a winning strategy in $\Gamma_G(A, w)$; similarly, $M_G(A)(w) = 1$ if and only if Player II has a winning strategy in $\Gamma_G(A, w)$. Finally, $M_G(A)(w) = \frac{1}{2}$ if and only if both Players have strategies that ensure for them at least a tie in $\Gamma_G(A, w)$.

The following definition will be needed:

Definition 17. Let $u \in \Sigma^*$. Then, a *partition* π of u of length n, is a tuple $\langle u_1, \ldots, u_n\rangle \in (\Sigma^*)^n$ such that $u_1 \cdots u_n = u$.

We will refer to the i-th element of a partition π as $\pi(i)$. Similarly, given $\alpha \in (\Sigma \cup N)^+$, we will write $\alpha(i)$ for the i'th symbol of α.

When a play of the game $\Gamma_G(A, w)$ starts, Player I initially has the role of the *Doubter* and Player II the role of the *Believer*. It is possible that during a play the two players swap roles (in the extreme case this may happen infinitely many times). If a move is played by the Believer (respectively, Doubter), then this is indicated by a superscript "+" (respectively, a superscript "−").

In the beginning of a play of the game $\Gamma_G(A, w)$ Player I does not believe that the string w can be produced by the nonterminal A of the Boolean grammar G. For this reason, he plays the move $(A, w)^-$. The intuitive explanation for this move is "*I doubt that w can be produced from A*". On the other hand, Player II believes that the string w can be produced by the nonterminal A of the Boolean grammar G. For this reason, he replies to the move of Player I with a pair $(A \to l_1\& \cdots \&l_m, w)^+$, where $A \to l_1\& \cdots \&l_m$ is a rule in G. The intuitive explanation for this move is "*I believe that w can be produced from A and I can prove this by using this specific rule of the grammar*". Now the reply of Player I to the move of Player II is a pair of the form $(l_i, w)^-$, where l_i is one of the conjuncts in the body of the rule that Player I has just played. The intuition in this case is "*I doubt that w can be produced from the rule you have just played and more specifically I doubt that w can be produced from l_i*". We now have to specify the reaction of a player to a move of the form $(l, u)^-$, for some conjunct l and $u \in \Sigma^*$. We distinguish the following cases:

Case 1: l is positive.

Subcase 1.1: l contains nonterminals. The next move depends on the length of l:

- $|l| = 1$, i.e., $l = B$ where B is a nonterminal. Then, the Believer plays a pair $(B \to l_1\& \cdots \&l_m, u)^+$, where $B \to l_1\& \cdots \&l_m$ is a rule in G. The explanation for this move as well as the reaction of the Doubter, are identical to those specified in the beginning of the previous paragraph for the first move in the game.
- $|l| > 1$, i.e., $l = \alpha$ where α contains at least one nonterminal and $|\alpha| \geq 2$. Then, the Believer partitions u into $|\alpha|$ parts (possibly equal to ϵ and not necessarily of the same size), and plays $(\alpha, \pi)^+$ where π is the partition just mentioned. The intuition behind this rule is as follows: "*I believe that u can be produced from α and I can demonstrate this to you by partitioning u into $|\alpha|$ substrings such that each symbol from α can produce the corresponding substring from u*". The Doubter will then have to choose one of the symbols of the sequence α, say $\alpha(i)$, together with the corresponding string from the partition π, namely $\pi(i)$, and play the move $(\alpha(i), \pi(i))^-$. The intuition now is: "*I doubt that $\alpha(i)$ can produce $\pi(i)$, and therefore I was right to believe that α cannot produce u*".

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Subcase 1.2: l does not contain any nonterminals. Then, the next move depends on whether l is equal to u or not:

- $l = u$, i.e., the last move was of the form $(u, u)^-$. Then, the Believer plays the move (I've won). The intuition here is "*You have just doubted that u can be produced from u, which is completely unreasonable and I have just won*".
- $l = v$, where $v \neq u$. Then, the Believer plays the move (I've lost). The intuition is: "*You have just doubted that u can be produced from v, where $v \neq u$; you are right, I have just lost*".

Case 2: l is negative, i.e., $l = \neg\alpha$, where $\alpha \in (\Sigma \cup N)^*$. Then, the player who was previously the Believer must now become the Doubter and play the pair $(\alpha, u)^-$ as the next move. The intuitive reading here is: "*I doubt that u can be produced from α (and therefore I was right in my previous belief that u can be produced by the rule that contains the conjunct $\neg\alpha$ in its body)*". Therefore, a consequence of this rule of the game is that when negation is encountered, the players swap roles: the Believer now becomes a Doubter and vice-versa.

The above concludes the description of the reaction of a player to a move of the form $(l, u)^-$. It now only remains to provide responses regarding the (I've lost) and the (I've won) moves in order to ensure that the game is infinite in all cases. So, if a move of a player is (I've won) then the reply of the other player is (I've lost), and vice-versa.

By following the above rules, we can form a play of the game (i.e., an infinite legal sequence of moves). Assume that we are given such a play of the game. We can now state the criteria for winning, losing and obtaining a tie. First of all, *any player who first plays the* (I've won) *move, wins*. Furthermore, *if the game play does not contain any* (I've won) *moves, and after a certain point one of the players remains a Doubter, this player wins* (the Doubter is considered a winner in this case since the Believer fails to establish his belief in a finite number of steps). Finally, *if during a play the two players swap roles infinitely often, the result is a tie*: intuitively, none of the players has managed to win the game in a finite number of moves; moreover, none of the two players has managed to remain a Doubter indefinitely (since the roles of the two players are interchanged infinitely many times).

We can now illustrate the game with a few simple examples.

Example 18. Consider the Boolean grammar G with the following rules:

$$S \to bbS \,\&\, \neg bSb$$
$$S \to a.$$

Moreover, consider the string $w = bba$. The following is a possible play of the game $\Gamma_G(S, bba)$:

Player I	*Player II*
$(S,\ bba)^-$	$(S \to bbS \,\&\, \neg bSb,\ bba)^+$
$(\neg bSb,\ bba)^-$	$(bSb,\ bba)^-$
$(bSb,\ \langle b, b, a\rangle)^+$	$(b, a)^-$
(I've lost)	(I've won)
…	…

Obviously, Player I loses since he is the first who plays the (I've lost) move. Actually, one can easily see that Player II has a strategy in the game $\Gamma_G(S, bba)$ that always "corners" Player I. As we are going to see shortly, this means that the string *bba* belongs to the language of the grammar.

Consider on the other hand the string $w = abb$ and the corresponding game $\Gamma_G(S, abb)$. The following is a possible play:

Player I	*Player II*
$(S,\ abb)^-$	$(S \to bbS \,\&\, \neg bSb,\ abb)^+$
$(bbS,\ abb)^-$	$(bbS,\ \langle a, \epsilon, bb\rangle)^+$
$(b,\ a)^-$	(I've lost)
(I've won)	(I've lost)
…	…

In this case, Player I is the winner of the play. Actually, it is easy to see that Player I has a strategy that always "corners" Player II. As we are going to see shortly, this means that the string *abb* does not belong to the language of the grammar. □

Example 19. Consider the context-free grammar G with only the rule $S \to SS$ (which does not produce any string). The following is a possible play of the game $\Gamma_G(S, \epsilon)$:

Player I	*Player II*
$(S,\ \epsilon)^-$	$(S \to SS,\ \epsilon)^+$
$(SS,\ \epsilon)^-$	$(SS,\ \langle\epsilon, \epsilon\rangle)^+$
$(S,\ \epsilon)^-$	$(S \to SS,\ \epsilon)^+$
…	…

The above play goes on for ever in the same manner. Observe that this is actually the only possible play of this game, as both players always have only one legal response. Therefore, Player I does not have a strategy to enforce Player II to play the move (I've lost). However, even in this case, the winner of the play is Player I: if one of the players manages to remain a Doubter for ever, then this player wins. □

Example 20. Consider the Boolean grammar G with only the rule $S \to \neg S$. The following is a possible play of the game $\Gamma_G(S, aa)$:

Player I	*Player II*
$(S,\ aa)^-$	$(S \to \neg S,\ aa)^+$
$(\neg S,\ aa)^-$	$(S,\ aa)^-$
$(S \to \neg S,\ aa)^+$	$(\neg S,\ aa)^-$
$(S,\ aa)^-$	$(S \to \neg S,\ aa)^+$
...	...

In this case the play goes on for ever with none of the players being in a position to announce a victory. Moreover, in this play the two players swap roles (the Believer becomes a Doubter and vice versa) infinitely many times. The result of this play is a tie. □

4. A formalization of the game

In this section we formalize the game we have just described. At first, we present some basic background on infinite games of perfect information, which we then use in order to define the proposed game for Boolean grammars in a formal way.

4.1. Infinite games of perfect information

Infinite games of perfect information [2] are games that take place between two players that we will call *Player I* and *Player II*. In such games there does not exist any "hidden information": both players know all the moves that have been played so far, and there are no simultaneous moves. The games are infinite in the sense that they do not terminate at a finite stage and therefore in order to derive the outcome of a play it may be necessary to examine an infinite sequence of moves.

Before defining perfect information games in a formal way, we need to introduce some notation. Sequences (finite or infinite in length) will usually be denoted by s or x. A finite sequence of length k will be denoted by $\langle s_0, s_1, \ldots, s_{k-1}\rangle$ and the empty sequence by $\langle\rangle$. Given a set X, an *infinite tree* on X is a set $R \subseteq X^\omega$ of infinite sequences[4] of members of X.

During a perfect information game, the two players exchange moves from a non-empty set X, called the *set of moves*. Initially, Player I chooses some $x_0 \in X$, then Player II chooses $x_1 \in X$, and so on. There also exists a set of *rules* specifying the possible moves of the two players. The rules will usually be defined by putting down restrictions on the choice of x_n that depend on the preceding moves $x_0, \ldots, x_{n-1}$. The rules (see for example [6]) implicitly define an infinite tree R on X:

$$\langle x_0, x_1, \ldots\rangle \in R \Leftrightarrow \text{for each } i \geq 0, x_i \text{ is allowed by the restrictions.}$$

Additionally, we assume the existence of a set D, called the *set of payoffs*, which consists of all possible outcomes of the game. Finally, we consider a function Φ, called the *payoff function*, which calculates the outcome of a play of the game. The above notions are formalized as follows:

Definition 21. An infinite game of perfect information is a quadruple $\Gamma = (X, R, D, \Phi)$, where:

- X is a nonempty set, called the set of moves for Players I and II.
- R is an infinite tree on X (i.e., $R \subseteq X^\omega$), usually implicitly specified by a set of rules.
- D is a linearly ordered set called the set of *rewards*, with the property that for all $S \subseteq D$, $\mathsf{lub}(S)$ and $\mathsf{glb}(S)$ belong to D.
- $\Phi : R \to D$, is the *payoff function* of the game.

Based on the set of moves X of a game, we define two sets $\mathsf{Strat}^I(\Gamma)$ and $\mathsf{Strat}^{II}(\Gamma)$ which correspond to the set of strategies for Player I and Player II respectively. A strategy $\sigma \in \mathsf{Strat}^I(\Gamma)$ assigns a move to each even length legal sequence of moves; similarly for $\tau \in \mathsf{Strat}^{II}(\Gamma)$ and odd length legal sequences of moves.

Definition 22. Let $\Gamma = (X, R, D, \Phi)$ be a game. Let R_n be the set of initial segments of elements of R that have length n. Then, a strategy for Player I is a function $\sigma : (\bigcup_{n<\omega} R_{2n}) \to X$ such that for every $n < \omega$ and for every $\langle x_0, \ldots, x_{2n-1}\rangle \in R_{2n}$,

[4] The definition of an infinite tree as a set of infinite sequences can be intuitively justified as follows: the nodes of the tree are all the initial segments of the infinite sequences and the root of the tree is the empty sequence $\langle\rangle$. A consequence of this definition is that an infinite tree is not allowed to contain terminal nodes (leaves), i.e., it is purely infinite.

$\langle x_0, \ldots, x_{2n-1}, \sigma(\langle x_0, \ldots, x_{2n-1}\rangle)\rangle \in R_{2n+1}$. Similarly, a strategy for Player II is a function $\tau : (\bigcup_{n<\omega} R_{2n+1}) \to X$ such that for every $n < \omega$ and for every $\langle x_0, \ldots, x_{2n}\rangle \in R_{2n+1}$, $\langle x_0, \ldots, x_{2n}, \tau(\langle x_0, \ldots, x_{2n}\rangle)\rangle \in R_{2n+2}$. We denote by $\mathsf{Strat}^{I}(\Gamma)$ and by $\mathsf{Strat}^{II}(\Gamma)$ the sets of strategies of Players I and II respectively.

Two strategies, when played one against the other, define a play of the game:

Definition 23. Let Γ be a game and let $\sigma \in \mathsf{Strat}^{I}(\Gamma)$ and $\tau \in \mathsf{Strat}^{II}(\Gamma)$. We define the following sequence:

$$s_0 = \sigma(\langle\rangle)$$
$$s_{2i} = \sigma(\langle s_0, s_1, \ldots, s_{2i-1}\rangle), \quad \text{for all } i \geq 1$$
$$s_{2i+1} = \tau(\langle s_0, s_1, \ldots, s_{2i}\rangle), \quad \text{for all } i \geq 0.$$

The *play of the game determined by the strategies* σ *and* τ, which is denoted by $\sigma \star \tau$, is the infinite sequence $\langle s_0, s_1, s_2, \ldots\rangle$. The s_i's will be called the *moves of the play*.

Until now we have focused on particular plays of a game. We would like to have a notion that gives us the outcome of the whole game provided that Player I tries his best to minimize the result while Player II tries his best to maximize it. Moreover, we would like that during this process, each player can decide for his best strategy, independently of the corresponding choice of the other player. This idea is captured by *determinacy*:

Definition 24 (*Determinacy*). Let $\Gamma = (X, R, D, \Phi)$ be a game and let $\mathcal{S} = \mathsf{Strat}^{I}(\Gamma)$ and $\mathcal{T} = \mathsf{Strat}^{II}(\Gamma)$. Then Γ is *determined* with *value* v if:

$$\mathsf{lub}_{\tau\in\mathcal{T}}\, \mathsf{glb}_{\sigma\in\mathcal{S}}\, \Phi(\sigma \star \tau) = \mathsf{glb}_{\sigma\in\mathcal{S}}\, \mathsf{lub}_{\tau\in\mathcal{T}}\, \Phi(\sigma \star \tau) = v.$$

The following lemma can be established (see for example [7]):

Lemma 25. *Let* $\Gamma = (X, R, D, \Phi)$ *be a game and let* $\mathcal{S} = \mathsf{Strat}^{I}(\Gamma)$ *and* $\mathcal{T} = \mathsf{Strat}^{II}(\Gamma)$. *Then:*

$$\mathsf{lub}_{\tau\in\mathcal{T}}\, \mathsf{glb}_{\sigma\in\mathcal{S}}\, \Phi(\sigma \star \tau) \leq \mathsf{glb}_{\sigma\in\mathcal{S}}\, \mathsf{lub}_{\tau\in\mathcal{T}}\, \Phi(\sigma \star \tau).$$

Determinacy is a very important notion which is in general not straightforward to establish. In fact, one can define infinite games that are not determined (see for example [2]). For the game we are considering here, we demonstrate its determinacy in Section 5.

Actually, for our game, one can intuitively understand what determinacy means and be convinced that it indeed holds. Given a string $w \in M_G(S)$, Player II can design a strategy for proving this fact which succeeds against any strategy of Player I (i.e., Player II does not have to worry about how Player I is going to try to reject the membership of w in $M_G(S)$). Symmetrically, given a string $w \notin M_G(S)$, Player I can design a strategy for proving this fact which succeeds against any strategy of Player II.

4.2. A formal definition of the game

Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$. We will define the perfect information game $\Gamma_G(\alpha, w) = (X, R_{(\alpha,w)}, D, \Phi_{(\alpha,w)})$. This game generalizes the one introduced informally in the previous section, as α is now allowed to be any sequence of terminal and nonterminal symbols rather than a single nonterminal.

Without loss of generality, we may assume that for every nonterminal there exists at least one rule that defines it: if there is a nonterminal B that is not defined in G (i.e., there does not exist in P any rule with left hand side equal to B), we can always construct an equivalent grammar by adding the rule $B \to \epsilon \& \neg\epsilon$. Recall that we denote by E the set $(\Sigma \cup N)^* - (\Sigma^* \cup N)$.

The game $\Gamma_G(\alpha, w)$ can now be formally defined. We first define the set of moves, which is independent of α and w and depends only on G:

$$X = \{(\beta, u)^- \mid \beta \in (\Sigma \cup N)^*, u \in \Sigma^*\} \cup \{(\neg\beta, u)^- \mid \beta \in (\Sigma \cup N)^*, u \in \Sigma^*\} \cup \{(Z, u)^+ \mid Z \in P, u \in \Sigma^*\}$$
$$\cup \{(\beta, \pi)^+ \mid \beta \in (\Sigma \cup N)^*, \pi \in (\Sigma^*)^{|\beta|}\} \cup \{(\text{I've won}), (\text{I've lost})\}.$$

We next define the infinite tree $R_{(\alpha,w)}$ of the game $\Gamma_G(\alpha, w)$: $R_{(\alpha,w)}$ consists of all sequences $\langle x_0, x_1, \ldots, x_k, \ldots\rangle$, which satisfy the following restrictions for each $k \geq 0$:

R0. $x_0 = (\alpha, w)^-$.
R1. If $x_k = (B, u)^-$, where $B \in N$, then $x_{k+1} = (B \to l_1 \& \cdots \& l_m, u)^+$, where $B \to l_1 \& \cdots \& l_m$ is a rule of G.
R2. If $x_k = (B \to l_1 \& \cdots \& l_m, u)^+$, then $x_{k+1} = (l_i, u)^-$, for some i, where $1 \leq i \leq m$.
R3. If $x_k = (\beta, u)^-$, where $\beta \in E$, then $x_{k+1} = (\beta, \pi)^+$, where π is a partition of u of length $|\beta|$.
R4. If $x_k = (\beta, \pi)^+$, where $\beta \in E$ and $\pi \in (\Sigma^*)^{|\beta|}$, then $x_{k+1} = (\beta(i), \pi(i))^-$, for some i, where $1 \leq i \leq |\beta|$.
R5. If $x_k = (\neg\beta, u)^-$, then $x_{k+1} = (\beta, u)^-$. A transition of this form from x_k to x_{k+1} will be called a *role-switch*.
R6. If $x_k = (u, u)^-$, where $u \in \Sigma^*$, or $x_k = (\text{I've lost})$, then $x_{k+1} = (\text{I've won})$.
R7. If $x_k = (v, u)^-$, where $v, u \in \Sigma^*$ and $v \neq u$, or $x_k = (\text{I've won})$, then $x_{k+1} = (\text{I've lost})$.

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We should repeat at this point that since we are dealing with *infinite games*, a play continues even if at some point the play of the game has essentially ended in favor of one of the two players; this is achieved using the two moves (I've won) and (I've lost). The player who has won the play keeps on playing the move (I've won), while the other player keeps on playing the move (I've lost). This way every play is infinite. A play that does not contain (I've won) and (I've lost) moves will be called a *genuinely infinite play*.

Consider now the set of rewards. We define $D = \{0, \frac{1}{2}, 1\}$. In other words, a play of the game can be assigned the value 0 (this means that Player I has won the play), the value 1 (Player II has won), or the value $\frac{1}{2}$ (the result is a tie). It remains to formally define the payoff function. The following definitions are needed:

Definition 26 (*True-Play, False-Play*). Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, $w \in \Sigma^*$ and $\alpha \in (\Sigma \cup N)^*$, and let s be a play of the corresponding game $\Gamma_G(\alpha, w)$. Then, s is called a *true-play* if either Player II plays the (I've won) move in s or s is a genuinely infinite play that contains an odd number of role-switches. Similarly, s is called a *false-play* if either Player I plays the (I've won) move in s or s is a genuinely infinite play that contains an even number of role-switches.

The payoff function is defined as follows:

$$\Phi_{(\alpha,w)}(s) = \begin{cases} 1, & \text{if } s \text{ is a true-play} \\ 0, & \text{if } s \text{ is a false-play} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

This completes the formal presentation of the game. It should be noted at this point that since conjunctive and context-free grammars are subcases of Boolean grammars, the game (actually in a simpler form) is also applicable to them. More specifically, in the case of conjunctive grammars, rule R5 is not needed; in the case of context-free grammars, rule R5 is also not needed; moreover, in rules R1 and R2, the form of the grammar rule is much simpler (i.e., just one conjunct). Notice also that since rule R5 is not used in both of these cases, Player I remains always the Doubter and Player II is always the Believer. Finally, also notice that in the simplified games for conjunctive and context-free grammars, the set of rewards is equal to $\{0, 1\}$ and the payoff function can be defined in a simpler way.

5. Equivalence to the well-founded semantics

There still remain two crucial issues that need to be clarified in order for the game to be "well-defined" and appropriate for capturing the meaning of Boolean grammars. First, we still have not argued regarding the *determinacy* of the game, and second, we have not investigated the relationship of the game to the well-founded semantics of Boolean grammars [3,4]. For infinite games that are win–lose (i.e., no ties), there exists a well-known result, namely *Martin's theorem* [5], which can be used to establish determinacy in most practical cases. In [1], based on Martin's theorem, a criterion is defined that ensures that certain three-valued games are determined. This criterion presupposes the use of the theory of Borel sets (see [1] for details).

In the following, we circumvent the use of Martin's theorem by demonstrating *at the same time* both the determinacy of the game *and* its equivalence to the well-founded semantics. Our new proof can also be adapted to work for the case of logic programs.

In the next subsections, we are going to establish the equivalence of the game to the well-founded semantics (see Theorem 32 at the end of the current section). The proof of this theorem is based on defining optimal strategies for the two players of the game. The strategies are defined using the well-founded model as a guide. The detailed proof is presented in the following subsections.

5.1. Defining two optimal strategies

In this subsection we define a strategy $\hat{\sigma}_{(\alpha,w)}$ for Player I and a strategy $\hat{\tau}_{(\alpha,w)}$ for Player II for the game $\Gamma_G(\alpha, w)$, which will help us establish the equivalence of the game to the well-founded semantics. As it will become clear later on, these two strategies are optimal for the two players.[5] The strategies are defined using an auxiliary mapping $\mathsf{next} : X \to X$, which specifies a legal reply to each move (i.e., x and $\mathsf{next}(x)$ are allowed to be consecutive moves in some play of the game).

The definition of $\mathsf{next}(x)$ consists of four cases depending on the form of the move x. In order to make this definition clear, we first give an intuitive explanation of the various cases.

We first seek for the optimal response of the Believer to a move of the form $(B, u)^-$, $B \in N$, $u \in \Sigma^*$ (*Case 1* in the definition of next). If $M_G(B)(u) = 1$, then the Believer, in order to win the play, follows the steps in the construction of the well-founded model of G in "reverse", starting from the point in which the value 1 for the membership of w in the language produced by B is obtained for the first time during this construction. This point is indicated by the pair of values

5 In the following, we will use $\hat{\sigma}_{(\alpha,w)}$ and $\hat{\tau}_{(\alpha,w)}$ to denote these two fixed optimal strategies while $\sigma_{(\alpha,w)}$ and $\tau_{(\alpha,w)}$ will be used to refer to arbitrary strategies.

$i = \mathsf{odp}(B, u)$, $r = \mathsf{idp}(B, u)$, i.e., it is $\Theta^{\uparrow r}_{M_{i-1}}(B)(u) = 1$. Then, by the definition of $\Theta_{M_{i-1}}$ there exists at least one rule that causes $\Theta^{\uparrow r}_{M_{i-1}}(B)(u)$ to take the value 1, and the Believer selects a rule with this property.

On the other hand, if $M_G(B)(u) = 0$, no strategy of the Believer can guarantee a win or even a tie: if the Doubter follows an appropriate strategy, he can always win against any strategy of the Believer, and the Believer knows that. Therefore, his choice can be any rule with head B.

Finally, if $M_G(B)(u) = \frac{1}{2}$, then the Believer knows that he cannot definitely win the game and therefore he tries to lead the play to a tie. The first step towards achieving this goal is to lead the play to a role-switch. For this reason, the Believer considers the computation of $\Omega(M_G)$. Since M_G is a fixed point of Ω, it holds that $\Omega(M_G)(B)(u) = M_G(B)(u)$. Therefore, there exists a rule that causes the membership value of u in the language $M_G(B)$ to become $\frac{1}{2}$. The selected rule has the property that for every positive conjunct α_i, the membership of u in the language produced by α_i has taken a value of at least $\frac{1}{2}$ *earlier* in the construction of $\Omega(M_G)$. This guarantees that in the worst case after finitely many moves, a rule with only negative conjuncts will be selected, resulting to a role-switch after two additional moves.

The optimal response of the Believer to a move of the form $x = (\beta, u)^-$, where $\beta \in E$ and $u \in \Sigma^*$ can be chosen in an analogous way as above (*Case 2* in the definition of next).

We now seek for the optimal response of the Doubter to a move of the form $(B \to l_1\& \cdots \& l_m, u)^+$ (*Case 3* in the definition of next). If $M_G(l_1\& \cdots \& l_m)(u) = 0$, then the Doubter, in order to win the play, should obviously play a move of the form $(l_j, u)^-$ with $M_G(l_j)(u) = 0$. However, if he chooses an arbitrary l_j with this property it is possible that the Believer will achieve to lead the play to an infinite number of role switches. Thus, the Doubter has to choose l_j using additional conditions, to guarantee that a finite number of role switches occur in the worst case. The intuition behind the additional conditions can be better understood in the case that $M_G(B)(u) = 0$. Suppose that $\mathsf{odp}(B, u) = i$; then, there exists some l_j in the body of rule $B \to l_1\& \cdots \& l_m$ that causes the membership of u to the language produced by the body of the rule to be 0 under M_i, i.e., $M_i(l_j)(u) = 0$, and thus $\mathsf{odp}(l_j, u) \le i$. Then, the Doubter selects a conjunct that has the above property.

On the other hand, if $M_G(l_1\& \cdots \& l_m)(u) = 1$, then also $M_G(B)(u) = 1$ and the Doubter knows that he has no winning strategy. Thus, he simply chooses the first conjunct in the body of the rule.

Similarly, if $M_G(l_1\& \cdots \& l_m)(u) = \frac{1}{2}$, the Doubter knows that he can achieve at least a tie, by selecting a conjunct l with $M_G(l)(u) = \frac{1}{2}$.

Finally, the optimal response of the Doubter to a move of the form $x = (\beta, \pi)^+$, where $\beta \in E$, and $\pi \in (\Sigma^*)^{|\beta|}$ can be chosen similarly (*Case 4* in the definition of next).

We can now proceed to a precise definition of next. In order to make this definition functional, we assume that the set of rules P of the given grammar $G = (\Sigma, N, P, S)$, has a predetermined ordering. Similarly, we assume that the conjuncts that appear in the body of every rule in P also have a predetermined ordering. Therefore, we will freely use expressions like "the first rule in P that has the property...", or "the first conjunct in the body of the rule that has the property...". Finally, given a string w, we will assume that the set of partitions of w into n parts, is also ordered. Again, we will use expressions like "the first partition of w that satisfies...".

We will define $\mathsf{next}(x)$, for every move $x \in X$, for which there exist more than one legal choices; in the remaining cases $\mathsf{next}(x)$ is the unique legal move that may follow x. The definition distinguishes four cases, as already mentioned in the intuitive explanation:

Case 1: $x = (B, u)^-$, with $B \in N$ and $u \in \Sigma^*$. Then, $\mathsf{next}(x) = (Z, u)^+$, where Z is a rule in P selected as follows:

- Suppose that $M_G(B)(u) = 1$. Let $i = \mathsf{odp}(B, u)$ and $r = \mathsf{idp}(B, u)$. Then, Z is the first rule of G of the form $B \to l_1\& \cdots \& l_m$ which has the property that for every positive l_j, $\Theta^{\uparrow r-1}_{M_{i-1}}(l_j)(u) = 1$, and for every negative l_j, $M_{i-1}(l_j)(u) = 1$.
- Suppose that $M_G(B)(u) = 0$. Then, Z is the first rule of G with head B.
- Suppose that $M_G(B)(u) = \frac{1}{2}$. Let $r = \mathsf{idp}(B, u)$. Then, Z is the first rule of G of the form $B \to l_1\& \cdots \& l_m$ which has the property that for every positive l_j, $\Theta^{\uparrow r-1}_{M_G}(l_j)(u) \ge \frac{1}{2}$, and for every negative l_j, $M_G(l_j)(u) \ge \frac{1}{2}$.

Case 2: $x = (\beta, u)^-$, where $\beta \in E$ and $u \in \Sigma^*$. Then, $\mathsf{next}(x) = (\beta, \pi)^+$, where π a partition of u of length $|\beta|$, selected as follows:

- Suppose that $M_G(\beta)(u) = 1$. Let $i = \mathsf{odp}(\beta, u)$ and let $r = \mathsf{idp}(\beta, u)$. Then, π is the first partition of u of length $|\beta|$ which has the property that for every j, $1 \le j \le |\beta|$, $\Theta^{\uparrow r}_{M_{i-1}}(\beta(j))(\pi(j)) = 1$.
- Suppose that $M_G(\beta)(u) = 0$. Then, π is the first partition of u of length $|\beta|$.
- Suppose that $M_G(\beta)(u) = \frac{1}{2}$. Let $r = \mathsf{idp}(\beta, u)$. Then, π is the first partition of u of length $|\beta|$ which has the property that for every j, $1 \le j \le |\beta|$, $\Theta^{\uparrow r}_{M_G}(\beta(j))(\pi(j)) \ge \frac{1}{2}$.

Case 3: $x = (B \to l_1\& \cdots \& l_m, u)^+$. Then, $\mathsf{next}(x) = (l_j, u)^-$, where j is selected as follows:

- Suppose that $M_G(l_1\& \cdots \& l_m)(u) = 0$. Let $i = \min\{\mathsf{odp}(l_k, u) \mid 1 \le k \le m, M_G(l_k)(u) = 0\}$. Then, j is the minimum index such that l_j is positive, $M_G(l_j)(u) = 0$ and $\mathsf{odp}(l_j, u) = i$, if such an index exists; otherwise j is the minimum index such that l_j is negative, $M_G(l_j)(u) = 0$ and $\mathsf{odp}(l_j, u) = i$.

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- Suppose that $M_G(l_1\& \cdots \&l_m)(u) = 1$. Then, $j = 1$.
- Suppose that $M_G(l_1\& \cdots \&l_m)(u) = \frac{1}{2}$. Then, j is the minimum index such that $M_G(l_j)(u) = \frac{1}{2}$.

Case 4: $x = (\beta, \pi)^+$, where $\beta \in E$, and $\pi \in (\Sigma^*)^{|\beta|}$. Then, $\mathsf{next}(x) = (\beta(j), \pi(j))^-$, where j is selected as follows:

- Suppose that $\min_{k=1}^{|\beta|} M_G(\beta(k))(\pi(k)) = 0$. Let $i = \min\{\mathsf{odp}(\beta(k), \pi(k)) \mid 1 \le k \le |\beta|, M_G(\beta(k))(\pi(k)) = 0\}$. Then, j is the minimum index such that $M_G(\beta(j))(\pi(j)) = 0$ and $\mathsf{odp}(\beta(j), \pi(j)) = i$.
- Suppose that $\min_{k=1}^{|\beta|} M_G(\beta(k))(\pi(k)) = 1$. Then, $j = 1$.
- Suppose that $\min_{k=1}^{|\beta|} M_G(\beta(k))(\pi(k)) = \frac{1}{2}$. Then, j is the minimum index such that $M_G(\beta(j))(\pi(j)) = \frac{1}{2}$.

The fact that the above functions are well-defined, follows easily from the definition of odp and idp. We can now define the strategies $\hat{\sigma}_{(\alpha,w)}$ and $\hat{\tau}_{(\alpha,w)}$:

Definition 27. Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$. Consider the game $\Gamma_G(\alpha, w)$. Then, the strategies $\hat{\sigma}_{(\alpha,w)}$ of Player I and $\hat{\tau}_{(\alpha,w)}$ of Player II are defined as follows:

$$\hat{\sigma}_{(\alpha,w)}(\langle\rangle) = (\alpha, w)^-$$
$$\hat{\sigma}_{(\alpha,w)}(\langle x_0, x_1, \ldots, x_{2i-1}\rangle) = \mathsf{next}(x_{2i-1}), \quad \text{for all } i \ge 1$$
$$\hat{\tau}_{(\alpha,w)}(\langle x_0, x_1, \ldots, x_{2i}\rangle) = \mathsf{next}(x_{2i}), \quad \text{for all } i \ge 0.$$

The properties of the above strategies will be proved in the remainder of this section (Lemmata 28–31). Notice that, although we have defined an infinite family of strategies (indexed by (α, w)), all of them are very similar in nature: they are all based on the same response to the previous move, specified by the function next. This property will allow us to relate plays of different games. To demonstrate this, suppose that Player II follows the strategy $\hat{\tau}_{(\alpha,w)}$ in some play of the game $\Gamma_G(\alpha, w)$ and that a move $(\beta, u)^-$ is played by Player I (or Player II) during this play. Then, from the Player II's point of view, the sub-play starting with this move is equivalent to a whole play of the game $\Gamma_G(\beta, u)$, in which he is initially the Believer (resp. Doubter) and follows the strategy $\hat{\tau}_{(\beta,u)}$ (resp. $\hat{\sigma}_{(\beta,u)}$). Similar considerations can be made for Player I.

The consequences of the above facts are formalized in the following two lemmata, which will be very useful in the proof of the main result of this section.

Lemma 28. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$. Let $\sigma_{(\alpha,w)}$ be a strategy of Player I for the game $\Gamma_G(\alpha, w)$ and let $\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)} = \langle x_0, x_1, x_2, \ldots\rangle$. Assume there exists $i > 0$ such that $x_i = (\beta, u)^-$, where $\beta \in (\Sigma \cup N)^*$ and $u \in \Sigma^*$. Then the following statements hold for all $v \in \{0, \frac{1}{2}, 1\}$:*

(a) *If i is an even number and for every strategy σ of Player I for the game $\Gamma_G(\beta, u)$ it is $\Phi_{(\beta,u)}(\sigma \star \hat{\tau}_{(\beta,u)}) \ge v$, then $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \ge v$.*
(b) *If i is an odd number and for every strategy τ of Player II for the game $\Gamma_G(\beta, u)$ it is $\Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau) \le v$, then $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \ge 1 - v$.*

Proof. We give the proof of statement (b). Statement (a) can be proved along the same lines.

Define a strategy $\tau_{(\beta,u)}$ of Player II for the game $\Gamma_G(\beta, u)$ as follows:

$$\tau_{(\beta,u)}(\langle s_0, s_1, \ldots, s_{2j}\rangle) = \sigma_{(\alpha,w)}(\langle x_0, x_1, \ldots, x_{i-1}, s_0, s_1, \ldots, s_{2j}\rangle).$$

It is easy to verify that $\tau_{(\beta,u)}$ is actually a valid strategy (that is, it respects the rules R0–R7). Let $\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)} = \langle y_0, y_1, y_2, \ldots\rangle$.

We will prove by induction on j that $y_j = x_{i+j}$. For the basis case ($j = 0$) we have:

$$y_0 = \hat{\sigma}_{(\beta,u)}(\langle\rangle) = (\beta, u)^- = x_i.$$

Suppose that $y_k = x_{i+k}$ holds for all $k \le j$. We will show that $y_{j+1} = x_{i+j+1}$. If j is an even number then

$$\begin{aligned} y_{j+1} &= \tau_{(\beta,u)}(\langle y_0, y_1, \ldots, y_j\rangle) \\ &= \sigma_{(\alpha,w)}(\langle x_0, x_1, \ldots, x_{i-1}, y_0, y_1, \ldots, y_j\rangle) \\ &= \sigma_{(\alpha,w)}(\langle x_0, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+j}\rangle) \quad \text{(by ind. hyp.)} \\ &= x_{i+j+1}. \end{aligned}$$

If j is an odd number then

$$\begin{aligned} y_{j+1} &= \hat{\sigma}_{(\beta,u)}(\langle y_0, y_1, \ldots, y_j\rangle) \\ &= \mathsf{next}(y_j) \\ &= \mathsf{next}(x_{i+j}) \quad \text{(by ind. hyp.)} \\ &= \hat{\tau}_{(\alpha,w)}(\langle x_0, x_1, \ldots, x_{i+j}\rangle) \\ &= x_{i+j+1}. \end{aligned}$$

We now show that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1 - \Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)})$.

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Suppose first that $\Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)}) = 0$. We distinguish two cases:

- If Player I plays the move (I've won) in $\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)}$, then it is $y_k =$ (I've won) for some even number k. Then it is also $x_{i+k} =$ (I've won), where $i + k$ is an odd number (since by assumption i is odd). This implies that Player II plays an (I've won) move in $\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}$. Therefore, $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1$.
- Otherwise, $\langle y_0, y_1, y_2, \ldots\rangle$ contains an even number of role switches. Now since move x_i is of the form $(\beta, u)^-$, where $\beta \in (\Sigma \cup N)^*$ and $u \in \Sigma^*$, and i is an odd number, Player II is the Doubter when x_i is played, which implies that the number of role switches in $\langle x_0, x_1, \ldots, x_i\rangle$ is odd. Since $\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)} = \langle x_0, x_1, \ldots, x_{i-1}, y_0, y_1, y_2, \ldots\rangle$, it follows that this play contains an odd number of role switches (recall that $y_0 = x_i$). Therefore, $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1$.

Suppose now that $\Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)}) = 1$. In a similar way we obtain that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 0$.

Finally suppose that $\Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)}) = \frac{1}{2}$. Then $\langle y_0, y_1, y_2, \ldots\rangle$ contains an infinite number of role switches. Therefore, the same holds for the sequence $\langle x_0, x_1, \ldots, x_i, y_0, y_1, y_2, \ldots\rangle$, which implies $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = \frac{1}{2}$.

We have proved that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1 - \Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau_{(\beta,u)})$, from which statement (b) follows immediately. □

The following Lemma is dual to the previous one and can be proved in the same way.

Lemma 29. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar, $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$. Let $\tau_{(\alpha,w)}$ be a strategy of Player II for the game $\Gamma_G(\alpha, w)$ and let $\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)} = \langle x_0, x_1, x_2, \ldots\rangle$. Assume there exists $i > 0$ such that $x_i = (\beta, u)^-$, where $\beta \in (\Sigma \cup N)^*$ and $u \in \Sigma^*$. Then the following statements hold for all $v \in \{0, \frac{1}{2}, 1\}$:*

(a) *If i is an even number and for every strategy τ of Player II for the game $\Gamma_G(\beta, u)$ it is $\Phi_{(\beta,u)}(\hat{\sigma}_{(\beta,u)} \star \tau) \leq v$, then $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) \leq v$.*
(b) *If i is an odd number and for every strategy σ of Player I for the game $\Gamma_G(\beta, u)$ it is $\Phi_{(\beta,u)}(\sigma \star \hat{\tau}_{(\beta,u)}) \geq v$, then $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) \leq 1 - v$.*

5.2. *The proof of equivalence to the well-founded semantics*

Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and M_G be its well-founded model. Moreover, let $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$. Consider now the game $\Gamma_G(\alpha, w)$. We would like to demonstrate that $M_G(\alpha)(w)$ is always equal to the value of the game $\Gamma_G(\alpha, w)$. In this subsection, we establish this equality in two steps. First, we demonstrate that if $M_G(\alpha)(w) \in \{0, 1\}$, then the value of the game $\Gamma_G(\alpha, w)$ is equal to $M_G(\alpha)(w)$ (Lemma 30). Then, we demonstrate that if $M_G(\alpha)(w) = \frac{1}{2}$ then the value of the game $\Gamma_G(\alpha, w)$ is equal to $\frac{1}{2}$ (Lemma 31). Actually, the proof of Lemma 31 uses Lemma 30.

Lemma 30. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and let M_G be the well-founded model of G. Moreover, let $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$, such that $M_G(\alpha)(w) \in \{0, 1\}$. Then the strategies $\hat{\sigma}_{(\alpha,w)}$ and $\hat{\tau}_{(\alpha,w)}$ for the game $\Gamma_G(\alpha, w) = (X, R_{(\alpha,w)}, D, \Phi_{(\alpha,w)})$ satisfy the following statements:*

(a) *For every strategy $\tau_{(\alpha,w)}$ of Player II for the game $\Gamma_G(\alpha, w)$, it holds that $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) \leq M_G(\alpha)(w)$.*
(b) *For every strategy $\sigma_{(\alpha,w)}$ of Player I for the game $\Gamma_G(\alpha, w)$, it holds that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \geq M_G(\alpha)(w)$.*

Proof. We will prove statements (a) and (b) by induction on $\mathrm{odp}(\alpha, w)$. The basis of the induction is for $\mathrm{odp}(\alpha, w) = 0$. We consider two cases, based on the value of $M_G(\alpha)(w)$.

Case 1: $M_G(\alpha)(w) = 1$. In this case, it is $M_0(\alpha)(w) = 1$, which implies $\alpha = w$. Therefore, Player II's first move is (I've won) (which is his only legal move) and obviously statement (b) holds. Moreover, statement (a) is trivial in this case.

Case 2: $M_G(\alpha)(w) = 0$. In this case, it is $M_0(\alpha)(w) = 0$, which implies that $\alpha \notin N$ (since $M_0(A)(w) = \frac{1}{2}$ for every $A \in N$). If $\alpha \in \Sigma^*$, then it holds $\alpha \neq w$ and Player II's first move is (I've lost). If $\alpha \in E$ then Player II's first move is $(\alpha, \pi)^+$, where π is a partition of w into $|\alpha|$ parts. Since, $M_0(\alpha)(w) = 0$, it must be $\min_{k=1}^{|\alpha|} M_0(\alpha(k))(\pi(k)) = 0$. By the definition of $\hat{\sigma}_{(\alpha,w)}$, the reply of Player I is a move $(\alpha(j), \pi(j))^-$ for some j, $1 \leq j \leq |\alpha|$, such that $M_0(\alpha(j))(\pi(j)) = 0$. This implies that $\alpha(j) \notin N$. Therefore, $\alpha(j) \in \Sigma$ and $\alpha(j) \neq \pi(j)$. Then, the next move of Player II is (I've lost). Since Player II plays (I've lost) in any case, statement (a) holds. Moreover, statement (b) is trivial in this case.

For the induction step, assume that if $\mathrm{odp}(\alpha, w) = i$ then statements (a) and (b) hold. We show that they also hold in the case that $\mathrm{odp}(\alpha, w) = i + 1$. Similarly to the basis of the induction, we distinguish two cases.

Case 1: $M_G(\alpha)(w) = 1$. In this case, statement (a) is trivial. We will show by an inner induction on r that statement (b) holds for all $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$ such that $\mathrm{odp}(\alpha, w) = i + 1$ and $\mathrm{idp}(\alpha, w) = r$.

The basis of the inner induction is for $r = 0$, which holds vacuously from Lemma 16.

Suppose that the statement holds for r and consider $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$ such that $M_G(\alpha)(w) = 1$, $\mathrm{odp}(\alpha, w) = i + 1$ and $\mathrm{idp}(\alpha, w) = r + 1$. Moreover, assume that Player II follows the strategy $\hat{\tau}_{(\alpha,w)}$ and consider an arbitrary strategy $\sigma_{(\alpha,w)}$ of Player I for $\Gamma_G(\alpha, w)$. Let $\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)} = \langle x_0, x_1, x_2, \ldots\rangle$. We distinguish two subcases:

Subcase 1.1: $\alpha = A \in N$. By the definition of $\hat{\tau}_{(\alpha,w)}$, Player II plays a move of the form $x_1 = (A \to l_1 \& \cdots \& l_m, w)^+$, such that for every positive l_j, $\Theta_{M_i}^{\uparrow r}(l_j)(w) = 1$ and for every negative l_j, $M_i(l_j)(w) = 1$. Then, Player I plays a move of the form $x_2 = (l_k, w)^-$.

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If l_k is a positive conjunct, then $\Theta^{\uparrow r}_{M_i}(l_k)(w) = 1$ implies that $M_G(l_k)(w) = 1$ and $\mathsf{odp}(l_k, w) \le i + 1$; moreover, if $\mathsf{odp}(l_k, w) = i + 1$, then $\Theta^{\uparrow r}_{M_{\mathsf{odp}(l_k,w)-1}}(l_k)(w) = 1$, which implies (by the Definition 15) that $\mathsf{idp}(l_k, w) \le r$. Using the outer induction hypothesis (if $\mathsf{odp}(l_k, w) < i+1$) or the inner induction hypothesis (if $\mathsf{odp}(l_k, w) = i+1$) we obtain that for every strategy $\sigma_{(l_k,w)}$ of Player I for $\Gamma_G(l_k, w)$, $\Phi_{(l_k,w)}(\sigma_{(l_k,w)} \star \hat{\tau}_{(l_k,w)}) \ge 1$. Then Lemma 28(a) implies that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \ge 1$.

Otherwise, it is $l_k = \neg\beta$, for some $\beta \in (\Sigma \cup N)^*$. Then, the next move of Player II is $x_3 = (\beta, w)^-$. Since $M_i(l_k)(w) = 1$, it is $M_i(\beta)(w) = 0$, which implies that $M_G(\beta)(w) = 0$ and $\mathsf{odp}(\beta, w) < i + 1$. Using the outer induction hypothesis we obtain that for every strategy $\tau_{(\beta,w)}$ of Player II for $\Gamma_G(\beta, w)$, $\Phi_{(\beta,w)}(\hat{\sigma}_{(\beta,w)} \star \tau_{(\beta,w)}) \le 0$. Then Lemma 28(b) implies that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \ge 1$.

Subcase 1.2: $\alpha \notin N$. Since $\mathsf{odp}(\alpha, w) = i+1 > 0$, it is $\alpha \notin \Sigma^*$. Therefore, $\alpha \in E$, which implies (by the definition of $\hat{\tau}_{(\alpha,w)}$) that Player II will play a move of the form $x_1 = (\alpha, \pi)^+$, where π is a partition of w, such that $\Theta^{\uparrow r+1}_{M_i}(\alpha(j))(\pi(j)) = 1$, for every j, $1 \le j \le |\alpha|$. This implies that $\mathsf{odp}(\alpha(j), \pi(j)) \le i+1$. Moreover, if $\mathsf{odp}(\alpha(j), \pi(j)) = i+1$, it must be $\mathsf{idp}(\alpha(j), \pi(j)) \le r+1$ and $\alpha(j) \in N$ (since it is $\mathsf{odp}(\alpha(j), \pi(j)) > 0$). Now Player I plays a move of the form $x_2 = (\alpha(k), \pi(k))^-$. Using the outer induction hypothesis (if $\mathsf{odp}(\alpha(k), \pi(k)) < i + 1$) or the inner induction hypothesis (if $\mathsf{odp}(\alpha(k), \pi(k)) = i + 1$ and $\mathsf{idp}(\alpha(k), \pi(k)) < r + 1$) or Subcase 1.1 (if $\mathsf{odp}(\alpha(k), \pi(k)) = i + 1$ and $\mathsf{idp}(\alpha(k), \pi(k)) = r + 1$) we obtain that for every strategy $\sigma_{(\alpha(k),\pi(k))}$ of Player I for $\Gamma_G(\alpha(k), \pi(k))$, $\Phi_{(\alpha(k),\pi(k))}(\sigma_{(\alpha(k),\pi(k))} \star \hat{\tau}_{(\alpha(k),\pi(k))}) \ge 1$. Then Lemma 28(a) implies that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \ge 1$.

Case 2: $M_G(\alpha)(w) = 0$. In this case, statement (b) is trivial. We will prove statement (a).

Consider $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$ such that $M_G(\alpha)(w) = 0$ and $\mathsf{odp}(\alpha, w) = i + 1$. Moreover, assume that Player I follows the strategy $\hat{\sigma}_{(\alpha,w)}$ and consider an arbitrary strategy $\tau_{(\alpha,w)}$ of Player II for $\Gamma_G(\alpha, w)$. Let $\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)} = \langle x_0, x_1, x_2, \ldots\rangle$.

Let Q denote the following set of moves:

$$Q = \{(\gamma, u)^- \mid \gamma \in (\Sigma \cup N)^*,\ u \in \Sigma^*,\ M_G(\gamma)(u) = 0,\ \mathsf{odp}(\gamma, u) = i + 1\}.$$

Notice that, if $\gamma \in \Sigma^*$, then $M_0(\gamma)(u) \in \{0, 1\}$, for every $u \in \Sigma^*$, which implies that $\mathsf{odp}(\gamma, u) = 0$. Therefore, if $(\gamma, u)^- \in Q$ then $\gamma \in (N \cup E)$.

We distinguish two subcases:

Subcase 2.1: for every $\delta \ge 0$ it is $x_{2\delta} \in Q$. Then, $\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}$ is a genuinely infinite play without role switches, and therefore $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) = 0$.

Subcase 2.2: there exists $\delta \ge 0$ such that $x_{2\delta} \notin Q$. Consider the minimum index p such that $x_{2p} \notin Q$. Notice that, since $x_0 = (\alpha, w)^- \in Q$, it must be $p \ge 1$. Moreover, from the minimality of p it follows that $x_{2p-2} \in Q$. Therefore, $x_{2p-2} = (\gamma, u)^-$, for some $\gamma \in (N \cup E)$ and $u \in \Sigma^*$. We consider the two possible forms of γ:

- $\gamma = A \in N$. Then, $x_{2p-1} = (A \to l_1 \& \cdots \& l_m, u)^+$. Since $x_{2p-2} = (A, u)^- \in Q$, it is $M_G(A)(u) = 0$ and $\mathsf{odp}(A, u) = i + 1$, which implies $M_{i+1}(A)(u) = 0$. Therefore, either there exists some positive l_j such that $M_{i+1}(l_j)(u) = 0$ (which implies $\mathsf{odp}(l_j, u) \le i + 1$) or there exists some negative l_j such that $M_i(l_j)(u) = 0$ (which implies $\mathsf{odp}(l_j, u) < i + 1$). Moreover, $x_{2p} = (l_k, u)^-$ for some conjunct l_k. By the definition of strategy $\hat{\sigma}_{(\alpha,w)}$, $M_G(l_k)(u) = 0$ and either l_k is a positive conjunct and $\mathsf{odp}(l_k, u) \le i + 1$ or l_k is a negative conjunct and $\mathsf{odp}(l_k, u) < i + 1$.
- $\gamma \in E$. Then $x_{2p-1} = (\gamma, \pi)^+$, where π is a partition of u. Since $x_{2p-2} = (\gamma, u)^- \in Q$, it is $M_{i+1}(\gamma)(u) = 0$. Therefore, there exists some j, $1 \le j \le |\gamma|$, such that $M_{i+1}(\gamma(j))(\pi(j)) = 0$, which implies that $\mathsf{odp}(\gamma(j), \pi(j)) \le i + 1$. By the definition of strategy $\hat{\sigma}_{(\alpha,w)}$ it follows that $x_{2p} = (\gamma(k), \pi(k))^-$, with $M_G(\gamma(k))(\pi(k)) = 0$ and $\mathsf{odp}(\gamma(k), \pi(k)) \le i + 1$.

Therefore, for any possible form of γ, we reach one of the following situations:

(a) $x_{2p} = (\beta, z)^-$ for some $\beta \in (\Sigma \cup N)^*$ and $z \in \Sigma^*$, such that $M_G(\beta)(z) = 0$, and $\mathsf{odp}(\beta, z) \le i + 1$. Now the fact that $x_{2p} \notin Q$ implies that $\mathsf{odp}(\beta, z) < i + 1$. Thus, using the induction hypothesis we obtain that for every strategy $\tau_{(\beta,z)}$ of Player II for $\Gamma_G(\beta, z)$, $\Phi_{(\beta,z)}(\hat{\sigma}_{(\beta,z)} \star \tau_{(\beta,z)}) \le 0$. Then Lemma 29(a) implies that $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) \le 0$.

(b) $x_{2p} = (\neg\beta, u)^-$ for some $\beta \in (\Sigma \cup N)^*$, such that $M_G(\beta)(u) = 1$ and $\mathsf{odp}(\beta, u) = \mathsf{odp}(\neg\beta, u) < i + 1$. Then, $x_{2p+1} = (\beta, u)^-$. Using the induction hypothesis we obtain that for every strategy $\sigma_{(\beta,u)}$ of Player I for $\Gamma_G(\beta, u)$, $\Phi_{(\beta,u)}(\sigma_{(\beta,u)} \star \hat{\tau}_{(\beta,u)}) \ge 1$. Then Lemma 29(b) implies that $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) \le 0$.

This completes the proof of the lemma. □

Lemma 31. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and let M_G be the well-founded model of G. Moreover, let $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$, such that $M_G(\alpha)(w) = \frac{1}{2}$. Then the strategies $\hat{\sigma}_{(\alpha,w)}$ and $\hat{\tau}_{(\alpha,w)}$ for the game $\Gamma_G(\alpha, w) = (X, R_{(\alpha,w)}, D, \Phi_{(\alpha,w)})$ satisfy the following statements:*

(a) *For every strategy $\tau_{(\alpha,w)}$ of Player II for the game $\Gamma_G(\alpha, w)$, it holds that $\Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau_{(\alpha,w)}) \le \frac{1}{2}$.*

(b) *For every strategy $\sigma_{(\alpha,w)}$ of Player I for the game $\Gamma_G(\alpha, w)$, it holds that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) \ge \frac{1}{2}$.*

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Proof. We demonstrate the proof of (b); the proof of (a) is symmetrical.

Define $Q = \{(\gamma, u)^- \mid \gamma \in (\Sigma \cup N)^*,\ u \in \Sigma^*,\ M_G(\gamma)(u) = \frac{1}{2}\}$. Observe that, if $\gamma \in \Sigma^*$, then $M_G(\gamma)(u) \in \{0, 1\}$, for every $u \in \Sigma^*$. Therefore, if $(\gamma, u)^- \in Q$ then $\gamma \in (N \cup E)$.

Assume that Player II follows the strategy $\hat{\tau}_{(\alpha,w)}$ and consider an arbitrary strategy $\sigma_{(\alpha,w)}$ of Player I for $\Gamma_G(\alpha, w)$. Let $\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)} = \langle x_0, x_1, x_2, \ldots\rangle$. We distinguish two cases:

Case 1: all the moves in the play $\langle x_0, x_1, x_2, \ldots\rangle$ of the form $(\gamma, u)^-$, where $\gamma \in (\Sigma \cup N)^*$ and $u \in \Sigma^*$, are in Q. Then, this play cannot contain moves of the form $(\gamma, u)^-$ where $\gamma \in \Sigma^*$, which implies that it does not contain any move in {(I've won),(I've lost)}, i.e., it is a genuinely infinite play. If $\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}$ contains an infinite number of role switches then $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = \frac{1}{2}$. Otherwise there exists some $\delta \geq 0$ such that in all moves after x_δ one of the two players remains the Doubter and plays moves of the form $(\gamma, u)^-$ (since any move of the form $(\neg\gamma, u)^-$ would imply a role switch). By our assumption, these moves of the Doubter must be in Q. Consequently, either $x_i \in Q$ for every odd index $i \geq \delta$, or $x_i \in Q$ for every even index $i \geq \delta$. We claim that only the former of these two conditions can be true.

In order to prove this claim suppose, for the sake of contradiction, that for every even index $i \geq \delta$, x_i is a move of the form $(\beta_i, u_i)^- \in Q$ and let $r_i = \mathsf{idp}(\beta_i, u_i)$. We distinguish two subcases, depending on the form of β_i:

Subcase 1.1: $\beta_i = A \in N$. By the definition of $\hat{\tau}_{(\alpha,w)}$, Player II plays a move of the form $x_{i+1} = (A \to l_1 \& \cdots \& l_m, u_i)^+$, such that for every positive conjunct l_j, $\Theta_{M_G}^{\uparrow r_i - 1}(l_j)(u_i) \geq \frac{1}{2}$, and for every negative conjunct l_j, $M_G(l_j)(u_i) \geq \frac{1}{2}$. Since $i + 2$ is also an even number, the move $x_{i+2} = (\beta_{i+2}, u_{i+2})^-$ is in Q. Therefore, $\beta_{i+2} = l_k$, for some positive conjunct l_k such that $M_G(l_k)(u) = \frac{1}{2}$; moreover, $u_i = u$. This implies that $r_{i+2} = \mathsf{idp}(\beta_{i+2}, u_{i+2}) = \mathsf{idp}(l_k, u) < r_i$.

Subcase 1.2: $\beta_i \in E$. By the definition of $\hat{\tau}_{(\alpha,w)}$, Player II plays a move of the form $x_{i+1} = (\beta_i, \pi)^+$, where π is a partition of u_i, such that $\Theta_{M_G}^{\uparrow r_i}(\beta_i(j))(\pi(j)) \geq \frac{1}{2}$, for every j, $1 \leq j \leq |\beta_i|$. Now Player I plays a move $x_{i+2} = (\beta_{i+2}, u_{i+2})^- \in Q$. This implies that $\beta_{i+2} = \beta_i(k)$ for some k, $1 \leq k \leq |\beta_i|$, such that $\beta_i(k) \in N$ and $M_G(\beta_i(k))(\pi(k)) = \frac{1}{2}$; moreover, $u_{i+2} = \pi(k)$. Thus, $r_{i+2} = \mathsf{idp}(\beta_{i+2}, u_{i+2}) = \mathsf{idp}(\beta_i(k), \pi(k)) \leq r_i$ and $\beta_{i+2} \in N$.

Notice that if Subcase 1.1 applies to the move x_i, then either subcase may apply to x_{i+2}. However, if Subcase 1.2 applies to x_i then only Subcase 1.1 may apply to x_{i+2}. We conclude that for every even index $i \geq \delta$, it is $r_{i+4} < r_i$. This implies that there exists some even index $\ell > \delta$ such that $r_\ell < 0$, which is a contradiction, since idp has non-negative values.

Therefore $x_i \in Q$ for every odd index $i \geq \delta$, which means that Player II remains a Doubter in all moves after x_δ. This implies that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1$.

Case 2: there exists a move x_p in the play $\langle x_0, x_1, x_2, \ldots\rangle$ of the form $(\gamma, u)^-$, where $\gamma \in (\Sigma \cup N)^*$ and $u \in \Sigma^*$, such that $(\gamma, u)^- \notin Q$. Consider the minimum index $p \geq 0$ such that $x_p = (\gamma, u)^-$ and $(\gamma, u)^- \notin Q$. Notice that, since $x_0 = (\alpha, w)^- \in Q$ and x_1 is not of the form $(\gamma, u)^-$, it must be $p \geq 2$.

Subcase 2.1: p is an even number, i.e., the move $x_p = (\gamma, u)^-$ is played by Player I. We consider all the possible forms of x_{p-1}:

- $x_{p-1} = (A \to l_1 \& \cdots \& l_m, u)^+$. Then $\gamma = l_k$ for some positive conjunct l_k and $x_{p-2} = (A, u)^- \in Q$ (by the minimality of p). This implies that $M_G(A)(u) = \frac{1}{2}$. By the definition of $\hat{\tau}_{(\alpha,w)}$ it follows that for all j, $M_G(l_j)(u) \geq \frac{1}{2}$. Thus, $M_G(\gamma)(u) \geq \frac{1}{2}$.
- $x_{p-1} = (\beta, \pi)^+$, where π is a partition of some $z \in \Sigma^*$. Then $\gamma = \beta(k)$ and $u = \pi(k)$, for some k, $1 \leq k \leq |\beta|$, and $x_{p-2} = (\beta, z)^- \in Q$ (by the minimality of p). This implies that $M_G(\beta)(z) = \frac{1}{2}$, and by the definition of $\hat{\tau}_{(\alpha,w)}$ it follows that for all j, $1 \leq j \leq |\beta|$, $M_G(\beta(j))(\pi(j)) \geq \frac{1}{2}$. Thus, $M_G(\gamma)(u) \geq \frac{1}{2}$.
- $x_{p-1} = (\neg\gamma, u)^-$. Then, x_{p-2} exists and must be of the form $(A \to l_1 \& \cdots \& l_m, u)^+$ and $\neg\gamma = l_k$ for some negative conjunct l_k. Moreover, $x_{p-3} = (A, u)^-$ (the form of the move x_{p-2}, implies that $p - 2 > 0$ and thus x_{p-3} exists). Since $(A, u)^- \in Q$ (by the minimality of p) it is $M_G(A)(u) = \frac{1}{2}$, which implies (using the fact that M_G is a model of G) that there exists some j, $1 \leq j \leq m$, such that $M_G(l_j)(u) \leq \frac{1}{2}$. By the definition of $\hat{\tau}_{(\alpha,w)}$ it follows that $M_G(l_k)(u) \leq \frac{1}{2}$, which implies $M_G(\gamma)(u) \geq \frac{1}{2}$.

Thus, for every possible form of x_{p-1} it holds $M_G(\gamma)(u) \geq \frac{1}{2}$. This implies, since $(\gamma, u)^- \notin Q$, that $M_G(\gamma)(u) = 1$. Using Lemmata 30(b) and 28(a) we obtain that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1$.

Subcase 2.2: p is an odd number, i.e., the move $x_p = (\gamma, u)^-$ is played by Player II. Similarly to the previous case, we consider all the possible forms of x_{p-1}:

- $x_{p-1} = (A \to l_1 \& \cdots \& l_m, u)^+$. Then $\gamma = l_k$ for some positive conjunct l_k. Moreover, $x_{p-2} = (A, u)^- \in Q$ (by the minimality of p), which implies $M_G(A)(u) = \frac{1}{2}$. Since M_G is a model of G, there exists some j, such that $M_G(l_j)(u) \leq \frac{1}{2}$. By the definition of $\hat{\tau}_{(\alpha,w)}$ it follows that $M_G(\gamma)(u) \leq \frac{1}{2}$.
- $x_{p-1} = (\beta, \pi)^+$, where π is a partition of some $z \in \Sigma^*$. Then $\gamma = \beta(k)$ and $u = \pi(k)$, for some k, $1 \leq k \leq |\beta|$. Moreover, $x_{p-2} = (\beta, z)^- \in Q$ (by the minimality of p), which implies $M_G(\beta)(z) = \frac{1}{2}$. By the definition of three-valued concatenation, it follows that there exists some j, $1 \leq j \leq |\beta|$, $M_G(\beta(j))(\pi(j)) \leq \frac{1}{2}$. By the definition of $\hat{\tau}_{(\alpha,w)}$ it follows that $M_G(\gamma)(u) \leq \frac{1}{2}$.

- $x_{p-1} = (\neg\gamma, u)^-$. Then, x_{p-2} must be of the form $(A \to l_1\& \cdots \& l_m, u)^+$ and $\neg\gamma = l_k$ for some negative conjunct l_k. Moreover, $x_{p-3} = (A, u)^- \in Q$ (by the minimality of p), which implies that $M_G(A)(u) = \frac{1}{2}$. By the definition of $\hat{\tau}_{(\alpha,w)}$ it follows that for all j, $M_G(l_j)(u) \geq \frac{1}{2}$. In particular, $M_G(l_k)(u) \geq \frac{1}{2}$, which implies $M_G(\gamma)(u) \leq \frac{1}{2}$.

Thus, for every possible form of x_{p-1} it holds $M_G(\gamma)(u) \leq \frac{1}{2}$. This implies, since $(\gamma, u)^- \notin Q$, that $M_G(\gamma)(u) = 0$. Using Lemmata 30(a) and 28(b) we obtain that $\Phi_{(\alpha,w)}(\sigma_{(\alpha,w)} \star \hat{\tau}_{(\alpha,w)}) = 1$.

This completes the proof of statement (b) of the lemma. □

Using the above lemmata, it is easy to prove the following theorem which establishes the equivalence between the game and the well-founded semantics:

Theorem 32. *Let $G = (\Sigma, N, P, S)$ be a Boolean grammar and M_G be its well-founded model. For every $\alpha \in (\Sigma \cup N)^*$ and $w \in \Sigma^*$, the game $\Gamma_G(\alpha, w)$ is determined with value $M_G(\alpha)(w)$.*

Proof. Let $\mathcal{S} = \mathsf{Strat}^I(\Gamma_G(\alpha, w))$ and $\mathcal{T} = \mathsf{Strat}^{II}(\Gamma_G(\alpha, w))$. Then:

$$
\begin{aligned}
M_G(\alpha)(w) &\leq \mathsf{glb}_{\sigma\in\mathcal{S}}\, \Phi_{(\alpha,w)}(\sigma \star \hat{\tau}_{(\alpha,w)}) && \text{(by Lemmata 30(b) and 31(b))}\\
&\leq \mathsf{lub}_{\tau\in\mathcal{T}}\, \mathsf{glb}_{\sigma\in\mathcal{S}}\, \Phi_{(\alpha,w)}(\sigma \star \tau) && \text{(definition of lub)}\\
&\leq \mathsf{glb}_{\sigma\in\mathcal{S}}\, \mathsf{lub}_{\tau\in\mathcal{T}}\, \Phi_{(\alpha,w)}(\sigma \star \tau) && \text{(by Lemma 25)}\\
&\leq \mathsf{lub}_{\tau\in\mathcal{T}}\, \Phi_{(\alpha,w)}(\hat{\sigma}_{(\alpha,w)} \star \tau) && \text{(definition of glb)}\\
&\leq M_G(\alpha)(w) && \text{(by Lemmata 30(a) and 31(a))}
\end{aligned}
$$

Therefore, $\mathsf{lub}_{\tau\in\mathcal{T}}\, \mathsf{glb}_{\sigma\in\mathcal{S}}\, \Phi_{(\alpha,w)}(\sigma\star\tau) = \mathsf{glb}_{\sigma\in\mathcal{S}}\, \mathsf{lub}_{\tau\in\mathcal{T}}\, \Phi_{(\alpha,w)}(\sigma\star\tau) = M_G(\alpha)(w)$, that is, the game $\Gamma_G(\alpha, w)$ is determined with value $M_G(\alpha)(w)$. □

6. Conclusions

We have presented an infinite game semantics for Boolean grammars and have demonstrated that it is equivalent to the well-founded semantics of this type of grammars. The simplicity of the new semantics stems mainly from its anthropomorphic flavor. In this respect, it differs from the well-founded semantics whose construction requires a more heavy mathematical machinery. We believe that these two semantical approaches can be used in a complementary way in the study of Boolean grammars. In our opinion, the game-theoretic approach will prove useful in establishing the correctness of meaning-preserving transformations for Boolean grammars. The reasoning in such a case can proceed as follows. Consider a Boolean grammar G and its transformed version G'. We can verify that the meaning of a nonterminal A in G coincides with the meaning of A in G' if for every string w, Player i has a winning strategy in the game $\Gamma_G(A, w)$ iff Player i has a winning strategy in game $\Gamma_{G'}(A, w)$. On the other hand, the well-founded semantics appears to be more useful in computing the meaning of *specific* grammars. This is due to the iterative-inductive flavor of the well-founded approach (see [8] for an example of an iterative computation of the meaning of a Boolean grammar using a procedure that was inspired by the well-founded construction).

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