A Formalisation of the Myhill-Nerode Theorem based on Regular Expressions

Christian Urban

joint work with Chunhan Wu and Xingyuan Zhang from the PLA University of Science and Technology in Nanjing

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or, Regular Languages Done Right

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In Textbooks...

- A regular language is one where there is DFA that recognises it.

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- A regular language is one where there is DFA that recognises it.
- Pumping lemma, closure properties of regular languages (closed under "negation") etc are all described and proved in terms of DFAs.
- Similarly the Myhill-Nerode theorem, which gives necessary and sufficient conditions for a language being regular (also describes a minimal DFA for a language).

Really Bad News!

This is bad news for formalisations in theorem provers. DFAs might be represented as

- graphs
- matrices
- partial functions

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Constable et al needed (on and off) 18 months for a 3-person team to formalise automata theory in Nuprl including Myhill-Nerode. There is only very little other formalised work on regular languages I know of in Coq, Isabelle and HOL.

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- graphs
- matrices
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All constructions are difficult to reason about.

typical textbook reasoning goes like: "...if M and N are any two automata with no inaccessible states ..."

Regular Expressions

...are a simple datatype:

```
rexp ::= NULL
       |  EMPTY
       |  CHR c
       |  ALT rexp rexp
       |  SEQ rexp rexp
       |  STAR rexp
```

Regular Expressions

...are a simple datatype:

$$
\begin{array}{rcl}
r & ::= & 0 \\
  & \mid & [] \\
  & \mid & c \\
  & \mid & r_1 + r_2 \\
  & \mid & r_1 \cdot r_2 \\
  & \mid & r^\star
\end{array}
$$

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\end{array}
$$

Induction and recursion principles come for free.

Semantics of Rexps

$$\begin{array}{rcl}
\mathbb{L}(\mathbf{0}) & = & \varnothing \\
\mathbb{L}([]) & = & \{[]\} \\
\mathbb{L}(c) & = & \{[c]\} \\
\mathbb{L}(r_1 + r_2) & = & \mathbb{L}(r_1) \cup \mathbb{L}(r_2) \\
\mathbb{L}(r_1 \cdot r_2) & = & \mathbb{L}(r_1) \ ; \ \mathbb{L}(r_2) \\
\mathbb{L}(r^\star) & = & \mathbb{L}(r)^\star
\end{array}$$

$$L_1; L_2 \stackrel{\text{def}}{=} \{s_1@s_2 \mid s_1 \in L_1 \wedge s_2 \in L_2\}$$

$$\frac{}{[] \in L^\star} \qquad \frac{s_1 \in L \quad s_2 \in L^\star}{s_1@s_2 \in L^\star}$$

Regular Expression Matching

- Harper in JFP'99: "Functional Pearl: Proof-Directed Debugging"
- Yi in JFP'06: "Educational Pearl: 'Proof-Directed Debugging' revisited for a first-order version"
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> "Unfortunately, regular expression derivatives have been lost in the sands of time, and few computer scientists are aware of them."

Demo

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- will help with closure properties of regular languages and with the pumping lemma.
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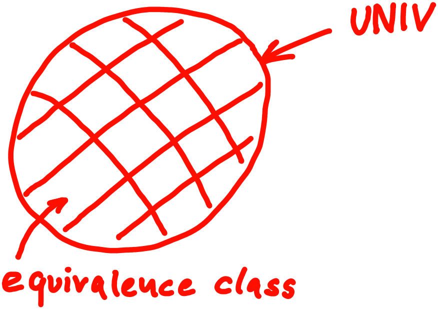
$$x \approx_L y \stackrel{\text{def}}{=} \forall z.\ x@z \in L \Leftrightarrow y@z \in L$$

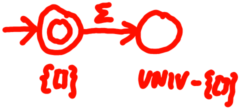
The Myhill-Nerode Theorem



- finite $(UNIV // \approx_L)$ $\Leftrightarrow$ L is regular

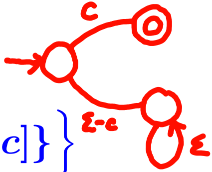
Equivalence Classes

- $L = []$



$$\Big\{\{[]\},\ UNIV - \{[]\}\Big\}$$

- $L = [c]$



$$\Big\{\{[]\},\ \{[c]\},\ UNIV - \{[], [c]\}\Big\}$$

- $L = \varnothing$

$$\Big\{UNIV\Big\}$$

Regular Languages

- L is regular $\stackrel{\text{def}}{=}$ if there is an ~~automaton M~~ regular expression r such that $\mathbb{L}(\not{M}\,r) = L$

- Myhill-Nerode:

  finite $\Rightarrow$ regular
  finite $(UNIV // \approx_L) \Rightarrow \exists r. L = \mathbb{L}(r)$

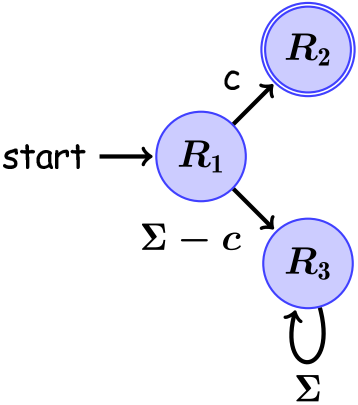
  regular $\Rightarrow$ finite
  finite $(UNIV // \approx_{\mathbb{L}(r)})$

Final States

- $\text{final}_L\ X \stackrel{\text{def}}{=}$
  $X \in (UNIV // \approx_L)\ \wedge\ \forall s \in X.\ s \in L$
- we can prove: $L = \bigcup\{X.\ \text{final}_L\ X\}$

Transitions between Equivalence Classes

$L = \{[c]\}$



$UNIV // \approx_L$ produces

R_1: $\{[]\}$
R_2: $\{[c]\}$
R_3: $UNIV - \{[], [c]\}$

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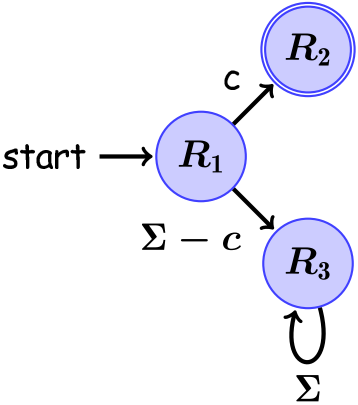


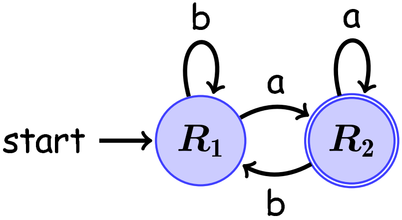
$UNIV // \approx_L$ produces

R_1: $\{[]\}$
R_2: $\{[c]\}$
R_3: $UNIV - \{[], [c]\}$

$$X \xrightarrow{c} Y \stackrel{\text{def}}{=} X ; [c] \subseteq Y$$

Systems of Equations

Inspired by a method of Brzozowski '64, we can build an equational system characterising the equivalence classes:



$$R_1 \equiv R_1; b + R_2; b$$
$$R_2 \equiv R_1; a + R_2; a$$

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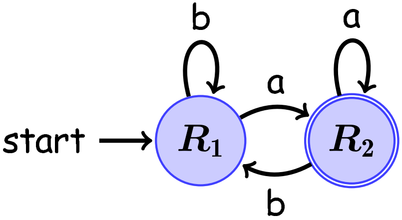

$$R_1 \equiv R_1; b + R_2; b + \lambda; []$$
$$R_2 \equiv R_1; a + R_2; a$$

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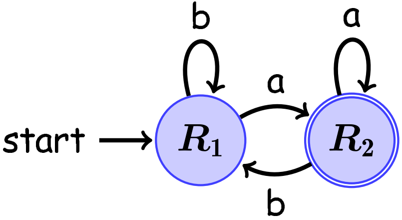

$$R_1 \equiv R_1; b + R_2; b + \lambda; []$$
$$R_2 \equiv R_1; a + R_2; a$$

we can prove
$$R_1 = R_1; \mathbb{L}(b) \cup R_2; \mathbb{L}(b) \cup \{[]\}; \{[]\}$$
$$R_2 = R_1; \mathbb{L}(a) \cup R_2; \mathbb{L}(a)$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

A Variant of Arden's Lemma

Arden's Lemma:
If $[] \not\in A$ then

$$X = X; A + \text{something}$$

has the (unique) solution

$$X = \text{something}; A^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = \underbrace{R_1; a}_{\text{something}} + R_2; \underset{\uparrow \atop A}{a}$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a \cdot a^\star$$

by Arden

$$R_1 = R_2; b \cdot b^\star + \lambda; b^\star$$
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$$R_1 = R_1; b + R_2; b + \lambda; []$$
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by Arden

$$R_1 = R_2; b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

by substitution

$$R_1 = R_1; a \cdot a^\star \cdot b \cdot b^\star + \lambda; b^\star$$
$$R_2 = R_1; a \cdot a^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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$$R_1 = R_1; b + R_2; b + \lambda; []$$
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by Arden

$$R_1 = \lambda; b^\star \cdot (a \cdot a^\star \cdot b \cdot b^\star)^\star$$
$$R_2 = R_1; a \cdot a^\star$$

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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solved form !!

$$R_1 = R_1; b + R_2; b + \lambda; []$$
$$R_2 = R_1; a + R_2; a$$

by Arden

$$R_1 = R_1; b + R_2; b + \lambda; []$$
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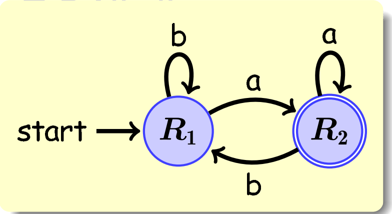
by Arden

R

R



by substitution

R

$$R_2 = R_1; a \cdot a^\star$$

by Arden

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solved form

The Equ's Solving Algorithm

- The algorithm must terminate: Arden makes one equation smaller; substitution deletes one variable from the right-hand sides.
- This is still a bit hairy to formalise because of our set-representation for equations:

$$\{(X, \{(Y_1, r_1), (Y_2, r_2), \ldots\}), \\ \ldots \\ \}$$

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$$\ldots$$
$$\}$$

They are generated from $UNIV // \approx_L$

Other Direction

One has to prove

$$\text{finite}(UNIV // \approx_{\mathbb{L}(r)})$$

by induction on r. Not trivial, but after a bit of thinking (by Chunhan), one can prove that if

$$\text{finite}(UNIV // \approx_{\mathbb{L}(r_1)}) \qquad \text{finite}(UNIV // \approx_{\mathbb{L}(r_2)})$$

then

$$\text{finite}(UNIV // \approx_{\mathbb{L}(r_1) \cup \mathbb{L}(r_2)})$$

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- finite $(UNIV // \approx_L)$ $\Leftrightarrow$ L is regular
- regular languages are closed under 'inversion'
$$UNIV // \approx_L \;=\; UNIV // \approx_{-L}$$
- regular expressions are not good if you look for a minimal one of a language (DFA have this notion)
- if you want to do regular expression matching (see Scott's paper)

Conclusion

- on balance regular expression are superior to DFAs
- I cannot think of a reason to not teach regular languages to students this way
- I have never ever seen a proof of Myhill-Nerode based on regular expressions
- no application, but a lot of fun
- great source of examples