# How to Write a Definitional Package for Isabelle

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Isabelle Developers Workshop, 14.8.2009

# Outline

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> *The method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil. Let us leave them to others and proceed with our honest toil.*
>
> — Bertrand Russell, Introduction to Mathematical Philosophy

# Definitional Packages

**inductive** $\mathit{even} :: \mathit{nat} \Rightarrow \mathit{bool}$
**where**
$\quad \mathit{even}\ 0$
$|\ \mathit{even}\ n \Longrightarrow \mathit{even}\ (\mathit{Suc}\ (\mathit{Suc}\ n))$

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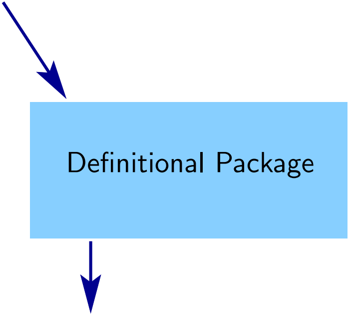


$$even \equiv lfp\ (\lambda p\ x.\ x = 0 \lor (\exists\, n.\ x = Suc\ (Suc\ n) \land p\ n))$$

# Definitional Packages

**inductive** *even* :: *nat* ⇒ *bool*
**where**
  *even 0*
| *even n* ⟹ *even* (*Suc* (*Suc n*))

**Theorem**

*even 0*
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Definitional Package

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**How to tackle the problem?**

1. Try out the construction on some examples
2. Figure out the general construction principle
3. Write code implementing the construction principle

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## *even'* is least predicate closed under above introduction rules

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$\lambda z.\ \forall\, even'.\ even'\ 0 \longrightarrow (\forall\, n.\ even'\ n \longrightarrow even'\ (Suc\ (Suc\ n)))$
$\longrightarrow even'\ z$

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### Intuition

$even'\ x$ holds iff $P\ x$ holds for every predicate $P$ closed under the above rules.

# Deriving the Properties "by hand"

**Demo**

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# Characteristic Rules

## Introduction Rules

$$\bigwedge \vec{x}_i.\ \vec{A}_i \Longrightarrow \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow R_{k_{ij}}\ \vec{p}\ \vec{s}_{ij}\right)_{j=1,\ldots,m_i} \Longrightarrow R_{l_i}\ \vec{p}\ \vec{t}_i$$

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## Definition of Predicates

$$R_i \equiv \lambda \vec{p}\ \vec{z}_i.\ \forall \vec{P}.\ K_1 \longrightarrow \cdots \longrightarrow K_r \longrightarrow P_i\ \vec{z}_i$$

$$K_i \equiv \forall \vec{x}_i.\ \vec{A}_i \longrightarrow \left(\forall \vec{y}_{ij}.\ \vec{B}_{ij} \longrightarrow P_{k_{ij}}\ \vec{s}_{ij}\right)_{j=1,\ldots,m_i} \longrightarrow P_{l_i}\ \vec{t}_i$$

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## Induction rules (weak)

$$R_i\ \vec{p}\ \vec{z}_i \Longrightarrow I_1 \Longrightarrow \cdots \Longrightarrow I_r \Longrightarrow P_i\ \vec{z}_i$$

$$I_i \equiv \bigwedge \vec{x}_i.\ \vec{A}_i \Longrightarrow \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow P_{k_{ij}}\ \vec{s}_{ij}\right)_{j=1,\ldots,m_i} \Longrightarrow P_{l_i}\ \vec{t}_i$$

# Proof of Introduction Rules

$$\bigwedge \vec{x}_i.\ \vec{A}_i \Longrightarrow \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow R_{k_{ij}}\ \vec{p}\ \vec{s}_{ij}\right)_{j=1,\ldots,m_i} \Longrightarrow R_{l_i}\ \vec{p}\ \vec{t}_i$$

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**Unfolding the definition**

$$\bigwedge \vec{x}_i.\ \vec{A}_i \Longrightarrow \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow \forall \vec{P}.\ \vec{K} \longrightarrow P_{k_{ij}}\ \vec{s}_{ij}\right)_{j=1,\dots,m_i} \Longrightarrow \\ \forall \vec{P}.\ \vec{K} \longrightarrow P_{l_i}\ \vec{t}_i$$

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**Applying introduction rules for $\forall$ and $\longrightarrow$**

$$\bigwedge \vec{x}_i\ \vec{P}.\ \vec{A}_i \Longrightarrow \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow \forall \vec{P}.\ \vec{K} \longrightarrow P_{k_{ij}}\ \vec{s}_{ij}\right)_{j=1,\dots,m_i} \Longrightarrow \vec{K} \Longrightarrow P_{l_i}\ \vec{t}_i$$

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**Applying $K_{l_i}$**

$$\begin{array}{l}\bigwedge \vec{x}_i\ \vec{P}.\ \vec{A}_i \Longrightarrow \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow \forall \vec{P}.\ \vec{K} \longrightarrow P_{k_{ij}}\ \vec{s}_{ij}\right)_{j=1,\dots,m_i} \Longrightarrow \\ \qquad \vec{K} \Longrightarrow \left\{ \begin{array}{l} \vec{A}_i \\ \left(\bigwedge \vec{y}_{ij}.\ \vec{B}_{ij} \Longrightarrow P_{k_{ij}}\ \vec{s}_{ij}\right)_{j=1,\dots,m_i} \end{array} \right. \end{array}$$

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4. Store theorems