Nominal Techniques

or, Something Crazy about Free Variables

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http://isabelle.in.tum.de/nominal/

Free Variables of Lambda-Terms:

$$\begin{aligned} \mathsf{fv}(x) &= \{x\} \\ \mathsf{fv}(t_1\, t_2) &= \mathsf{fv}(t_1) \cup \mathsf{fv}(t_2) \\ \mathsf{fv}(\lambda x.t) &= \mathsf{fv}(t) - \{x\} \end{aligned}$$

What are the free variables of pairs, sets, functions...?

Informal Reasoning

Fluet: “Expressions differing only in names of bound variables are equivalent.”

Harper and Pfenning about contexts: “...when we write Γ,x:A we assume that x is not already declared in Γ. If necessary, we tacitly rename x before adding it to the context Γ.”

Pfenning in Logical Frameworks - A Brief Introduction: “We allow tacit α-conversion (renaming of bound variables) ...”

Plan

- How do we get a type for lambda-terms where we have the equation

$$\lambda x.x = \lambda y.y?$$

- For this we will have a closer look at the notion of free variables and describe abstractly what abstractions are. (Lots of fun!)

A Non-Starter

- If we define

```
datatype lam =
   Var "name"
  | App "lam" "lam"
  | Lam "name" "lam"
```

  then we do not have $\lambda x.x = \lambda y.y$.

- In this case we have to make sure (manually) that everything we do is invariant modulo alpha-equivalence. Curry & Feys need in "Combinatory Logic" 10 pages just for showing that

$$M \approx_\alpha M', N \approx_\alpha N' \Rightarrow M[x:=N] \approx_\alpha M'[x:=N']$$

Types in HOL

HOL includes a mechanism for introducing new types:

- If you can identify a non-empty subset in an existing type, then you can turn this set into a new type.

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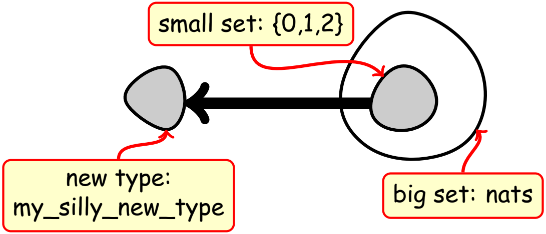

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- As a result, we will be able to introduce the **type** of **named** α-equivalence classes.

```
nominal_datatype lam =
    Var "name"
  | App "lam" "lam"
  | Lam "«name»lam"
```

First Naive Attempt

- We can define 'raw' lambda-terms (i.e. trees) as

```
datatype raw_lam =
    Var "name"
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- and then quotient them modulo α.

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typedef lam = "(UNIV::raw_lam set) // alpha"
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- Problem: This is not an inductive definition and we have to provide an induction principle for lam (recall Barendregt's substitution lemma). This is painful.

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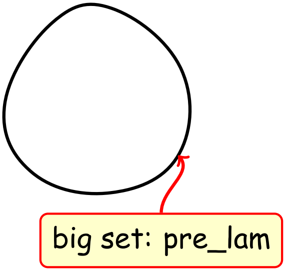

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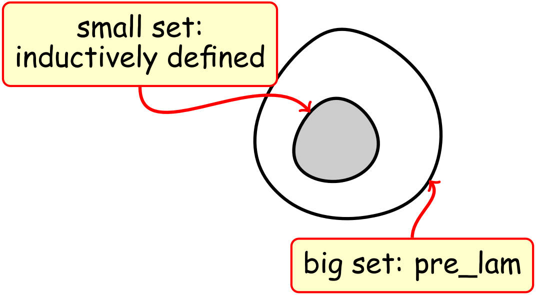

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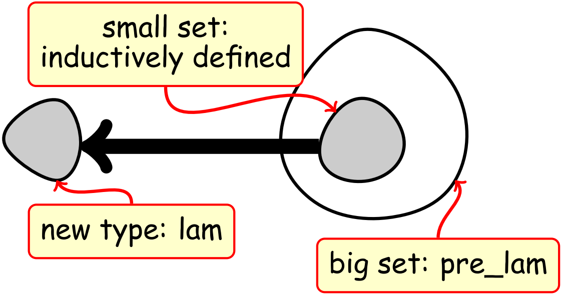

Second Naive Attempt (2)

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- But a Cantor argument will tell us that pre_lam set will always be bigger than pre_lam.

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- In the following we will make this **idea** to work by finding an alternative representation for α-equivalence classes.

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- What are the free variables of a set?

$$\mathsf{fv}(S) \stackrel{\mathsf{def}}{=} \bigcup_{t \in S} \mathsf{fv}(t)$$

Free Variables (2)

- What are the free variables of a function, for example the identity function?

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  But you just told me what the free variables of pairs and sets are. The identity function can be seen as the set of pairs (inputs and outputs):

  $$\{(x, x), (y, y), (z, z), \ldots, (t_1\, t_2, t_1, t_2), \ldots\}$$

  This would imply that the free variables of $\lambda x.x$ is the set of **all** variables?!

Free Variables (3)

- We like to have an (overloaded) definition recursing over the type hierarchy.
  - Starting with definitions for the base types (such as natural numbers, strings and the object languages we want to study).
  - Then for type-formers where the definition should depend on earlier defined notions:

$$\mathsf{fv}(t_1, t_2) \stackrel{\mathsf{def}}{=} \mathsf{fv}(t_1) \cup \mathsf{fv}(t_2)$$

$$\mathsf{fv}([]) \stackrel{\mathsf{def}}{=} \varnothing$$

$$\mathsf{fv}(t :: ts) \stackrel{\mathsf{def}}{=} \mathsf{fv}(t) \cup \mathsf{fv}(ts)$$

  - But what shall we do about functions, $\tau \Rightarrow \sigma$?

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- We start with a countably infinite set of **atoms**.
  - They will be used for object language variables.
  - They are the 'things' that can be bound.
- We restrict ourselves here to just one kind of atoms.
- **Permutations** are lists of pairs of atoms:

$$(a_1, b_1) \ldots (a_n, b_n)$$

Permutations

A permutation **acts** on atoms as follows:

$$[]\bullet a \stackrel{\text{def}}{=} a$$

$$((a_1\, a_2) :: \pi)\bullet a \stackrel{\text{def}}{=} \begin{cases} a_1 & \text{if } \pi\bullet a = a_2 \\ a_2 & \text{if } \pi\bullet a = a_1 \\ \pi\bullet a & \text{otherwise} \end{cases}$$

- $[]$ stands for the empty list (the identity permutation), and
- $(a_1\, a_2) :: \pi$ stands for the permutation π followed by the swapping $(a_1\, a_2)$. (We usually drop the $::$.)

Permutations (2)

- the **composition** of two permutations is given by list-concatenation, written as $\pi'@\pi$,
- the **inverse** of a permutation is given by list reversal, written as π^{-1}, and
- **permutation equality**, two permutations π and π' are equal iff

$$\pi \sim \pi' \stackrel{\text{def}}{=} \forall a.\ \pi \bullet a = \pi' \bullet a$$

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- Example calculations:

$$(b\,d)(b\,c)(a\,c) \bullet a = d$$

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- Example calculations:

$$(a\,a) \sim []$$

Three Properties

We require of all permutation operations that:

- $[] \bullet x = x$
- $(\pi_1 @ \pi_2) \bullet x = \pi_1 \bullet (\pi_2 \bullet x)$
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From this we have:

- $\pi^{-1}\bullet(\pi\bullet x) = x$
- $\pi\bullet x_1 = x_2$ if and only if $x_1 = \pi^{-1}\bullet x_2$
- $x_1 = x_2$ if and only if $\pi\bullet x_1 = \pi\bullet x_2$

Permutations on λ-Terms

$$\begin{array}{rcl} \pi\bullet(a) & & \text{given by the action on atoms} \\ \pi\bullet(t_1\,t_2) & \stackrel{\text{def}}{=} & (\pi\bullet t_1)(\pi\bullet t_2) \\ \pi\bullet(\lambda a.t) & \stackrel{\text{def}}{=} & \lambda(\pi\bullet a).(\pi\bullet t) \end{array}$$

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we treat lambdas as if there were no binders

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An aside: This definition leads also to a simple definition of α-equivalence:

$$\frac{t_1 \approx t_2}{\lambda a.t_1 \approx \lambda a.t_2}$$

$$\frac{a \neq b \quad t_1 \approx (a\, b) \bullet t_2 \quad a \# t_2}{\lambda a.t_1 \approx \lambda b.t_2}$$

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$$\frac{a \neq b \quad t_1 = (a\,b)\bullet t_2 \quad a \# t_2}{\lambda a.t_1 = \lambda b.t_2}$$

Perm's for Other Types

- $\pi \bullet (x_1, x_2) \stackrel{\text{def}}{=} (\pi \bullet x_1, \pi \bullet x_2)$ pairs
- $\pi \bullet [] \stackrel{\text{def}}{=} []$ lists

  $\pi \bullet (x :: xs) \stackrel{\text{def}}{=} (\pi \bullet x) :: (\pi \bullet xs)$
- $\pi \bullet X \stackrel{\text{def}}{=} \{\pi \bullet x \mid x \in X\}$ sets

  $\pi \bullet [\lambda x.N]_\alpha = [\lambda(\pi \bullet x).(\pi \bullet N)]_\alpha$
- $\pi \bullet f \stackrel{\text{def}}{=} \lambda x.\pi \bullet (f\ (\pi^{-1} \bullet x))$ functions

  $\pi \bullet (f\ x) = (\pi \bullet f)\ (\pi \bullet x)$
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$$
\begin{aligned}
(\pi \bullet f)\ (\pi \bullet x) &\stackrel{\text{def}}{=} (\lambda x.\pi \bullet (f\ (\pi^{-1} \bullet x)))\ (\pi \bullet x) \\
&= \pi \bullet (f\ (\pi^{-1} \bullet (\pi \bullet x))) \\
&= \pi \bullet (f\ x)
\end{aligned}
$$

…s

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Support and Freshness

The support of an object x is a set of atoms:

$$\mathsf{supp}(x) \stackrel{\mathsf{def}}{=} \{a \mid \mathsf{infinite}\,\{b \mid (a\,b)\bullet x \neq x\}\}$$

$$a \# x \stackrel{\mathsf{def}}{=} a \not\in \mathsf{supp}(x)$$

In words: all atoms a where the set

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is infinite (each swapping $(a\,b)$ needs to change something in x).

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OK, this definition is a tiny bit complicated, so let's go slowly. . .

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Support of an Atom

What is the support of the atom c?

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Let's check the (infinitely many) atoms one by one:

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We know

$$(t_1, t_2) = (s_1, s_2) \ \text{ iff } \ t_1 = s_1 \wedge t_2 = s_2$$

hence

$$(t_1, t_2) \neq (s_1, s_2) \ \text{ iff } \ t_1 \neq s_1 \vee t_2 \neq s_2$$

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Support of a Pair

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$$\mathsf{supp}(t_1) \qquad \cup \qquad \mathsf{supp}(t_2)$$

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$$\mathsf{supp}(t_1) \qquad \cup \qquad \mathsf{supp}(t_2)$$

So $\mathsf{supp}(t_1, t_2) = \mathsf{supp}(t_1) \cup \mathsf{supp}(t_2)$.

However, such things are proved for you:
the user does **not** have to bother with them.

```
lemma
  shows "supp (t₁,t₂) = supp t₁ ∪ ((supp t₂)::atom set)"
proof -
 have "supp (t₁,t₂) = {a. inf {b. [(a,b)]•(t₁,t₂) ≠ (t₁,t₂)}}"
   by (simp add: supp_def)
 also have "... = {a. inf {b. ([(a,b)]•t₁,[(a,b)]•t₂) ≠ (t₁,t₂)}}" by simp
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It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

Proof: case $a = b$ clear.

It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

Proof: case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\mathsf{fin}\{c \mid (b\,c)\bullet x \neq x\}$

$$a \# x \stackrel{\text{def}}{=} a \notin \mathsf{supp}(x)$$

$$\mathsf{supp}(x) \stackrel{\text{def}}{=} \{a \mid \mathsf{inf}\{c \mid (a\,c)\bullet x \neq x\}\}$$

It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

Proof: case $a \neq b$:

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(2) $\text{fin}(\{c \mid (a\,c)\bullet x \neq x\} \cup \{c \mid (b\,c)\bullet x \neq x\})$ f'rm (1)

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Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b){\bullet}x = x$

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$\quad\text{fin}\{c \mid (b\,c){\bullet}x \neq x\}$

(2) $\text{fin}\{c \mid (a\,c){\bullet}x \neq x \vee (b\,c){\bullet}x \neq x\}$ f'rm (1)

(3) $\text{inf}\{c \mid \neg((a\,c){\bullet}x \neq x \vee (b\,c){\bullet}x \neq x)\}$ f'rm (2)

Given a finite set of atoms, its 'co-set' must be infinite.

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Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

Proof: case $a \neq b$:

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(2) $\text{fin}\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f'rm (1)

(3) $\text{inf}\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x\}$ f'rm (2)

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(3) $\text{inf}\{c \mid (a\,c){\bullet}x = x \wedge (b\,c){\bullet}x = x\}$ f'rm (2)

(4) (i) $(a\,c){\bullet}x = x$ (ii) $(b\,c){\bullet}x = x$ for a $c \in$ (3)

If a set is infinite, it must contain a few elements. Let's pick c.

It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b){\bullet}x = x$

Proof: case $a \neq b$:

(1) $\text{fin}\{c \mid (a\,c){\bullet}x \neq x\}$ from Ass. +Def. of $\#$
$\text{fin}\{c \mid (b\,c){\bullet}x \neq x\}$

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(5) $(a\,c){\bullet}x = x$ by (4i)

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Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

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(5) $(a\,c)\bullet x = x$ by (4i)

(6) $(b\,c)\bullet(a\,c)\bullet x = (b\,c)\bullet x$ by bij.

bij.: $x = y$ iff $\pi\bullet x = \pi\bullet y$

It's as Simple as This

Lemma: $a \mathrel{\#} x \wedge b \mathrel{\#} x \Rightarrow (a\,b)\bullet x = x$

Proof: case $a \neq b$:

(1) fin$\{c \mid (a\,c)\bullet x \neq x\}$ — from Ass. +Def. of $\#$

fin$\{c \mid (b\,c)\bullet x \neq x\}$

(2) fin$\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ — f'rm (1)

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(4) (i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$ — for a $c \in$ (3)

(5) $(a\,c)\bullet x = x$ — by (4i)

(6) $(b\,c)\bullet(a\,c)\bullet x = x$ — by bij.,(4ii)

It's as Simple as This

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Proof: case $a \neq b$:

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It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\, b) \bullet x = x$

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$\text{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2) $\text{fin}\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f'rm (1)

(3) $\text{inf}\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x\}$ f'rm (2)

(4) (i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$ for a $c \in$ (3)

(5) $(a\,c)\bullet x = x$ by (4i)

(6) $(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4ii)

(7) $(a\,c)\bullet(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4i)

$$(a\,c)(b\,c)(a\,c)\bullet a = b$$
$$(a\,c)(b\,c)(a\,c)\bullet b = a$$
$$(a\,c)(b\,c)(a\,c)\bullet c = c$$

It's as Simple as This

Lemma: $a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$

Proof: case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x\}$ — from Ass. +Def. of $\#$

$\mathsf{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2) fin … $\neq x\}$ — f'rm (1)

(3) inf … $= x\}$ — f'rm (2)

> property of permutation:
>
> $\pi_1 \sim \pi_2 \Rightarrow \pi_1\bullet x = \pi_2\bullet x$

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(8) $(a\,b)\bullet x = x$ by prop. of perms

Done.

Existence of a Fresh Atom

Q: Why do we assume that there are countably infinitely many atoms?

A: For any finitely supported x:

$$\exists a.\ \ a \# x$$

If something is finitely supported, then we can always choose a fresh atom (also for finitely supported functions).

Exercises about Support

- Given a finite set of atoms. What is the support of this set?
- What is the support of the set of all atoms?
- From the set of all atoms take one atom out. What is the support of the resulting set?
- Are there any sets of atoms that have infinite support?

"Support by Andrew Pitts"

In Daily Use there Is Nothing Scary about Support

- We usually restrict ourselves to finitary structures (lists, lambda-terms, etc). In those structures, the notion of support coincides with the usual notion of what the free variables are.
- We just have to be careful with sets and functions (we treat them on a case-by-case basis and they usually turn out to have empty support).

In Daily Use there Is Nothing Scary about Support

- We usually restrict ourselves to finitary structures (lists, lambda-terms, etc). In those structures, the notion of support coincides with the usual notion of what the free variables are.
- We just have to be careful with sets and functions (we treat them on a case-by-case basis and they usually turn out to have empty support).
- There are two reasons for wanting to find out what the free variables of functions are: when we define functions over the "structure" of α-equivalence classes and because of a trick.

Nominal Abstractions

We are now going to specify what abstraction 'abstractly' means: it is an operation

$$[\_].(\_) : \mathsf{atom} \Rightarrow \mathsf{trm} \Rightarrow \mathsf{trm}$$

and has to satisfy two properties:

- $\pi\bullet([a].x) = [\pi\bullet a].(\pi\bullet x)$
- $[a].x \;=\; [b].y$ iff

  $$(a = b \wedge x = y) \vee$$
  $$(a \neq b \wedge x = (a\,b)\bullet y \wedge a \# y)$$

- These two properties imply for finitely supported x

  $$\mathsf{supp}([a].x) = \mathsf{supp}(x) - \{a\}$$

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- $[a].x$

  $(a =$

  $$\frac{}{a \,\#\, [a].x} \qquad \frac{b \neq a \quad b \,\#\, x}{b \,\#\, [a].x}$$

  $(a \neq b \wedge x = (a\,b) \bullet y \wedge a \,\#\, y)$

- These two properties imply for finitely supported x

$$\text{supp}([a].x) = \text{supp}(x) - \{a\}$$

Function $[a].t$ ‘=’ $[\lambda a.t]_\alpha$

$$[a].t \stackrel{\text{def}}{=} (\lambda b.\text{if } a = b \text{ then Some}(t) \text{ else if } b \# t \text{ then Some}((b\,a)\bullet t) \text{ else None})$$

type: atom $\to$ trm option

Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

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This is supposed to stand for the α-equivalence class of $\lambda a.t$.

Function $[a].t$ '=' $[\lambda a.t]_\alpha$

$[a].(a,c) \stackrel{\text{def}}{=}$

$(\lambda b.$if $a = b$
then Some(a,c)
else if $b \# (a,c)$
then Some$((b\,a)\bullet(a,c))$ else None$)$

Let's check this for $[a].(a,c)$:

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c 'applied to' $[a].(a,c)$ 'gives' None
d 'applied to' $[a].(a,c)$ 'gives' Some(d,c)
⋮

Function $[a].t$ '=' $[\lambda a.t]_\alpha$

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$$(\lambda b.\text{if } a = b$$
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a 'applied to' $[a].(a,c)$ 'gives' Some(a,c)		'$\lambda a.(a\,c)$'
b 'applied to' $[a].(a,c)$ 'gives' Some(b,c)		'$\lambda b.(b\,c)$'
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⋮		⋮

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$\vdots$

$[\lambda a.(a\,c)]_\alpha$

‘$\lambda a.(a\,c)$’

‘$\lambda b.(b\,c)$’

‘$\lambda d.(d\,c)$’

$\vdots$

Function $[a].t$ '=' $[\lambda a.t]_\alpha$

$$[a].t \stackrel{\text{def}}{=} (\lambda b.\text{if } a = b \\ \quad \text{then Some}(t) \\ \quad \text{else if } b \# t \text{ then Some}((b\,a)\bullet t) \text{ else None})$$

This function 'takes' a lambda-abstraction and an atom, and tries to rename the abstraction according to the given atom.

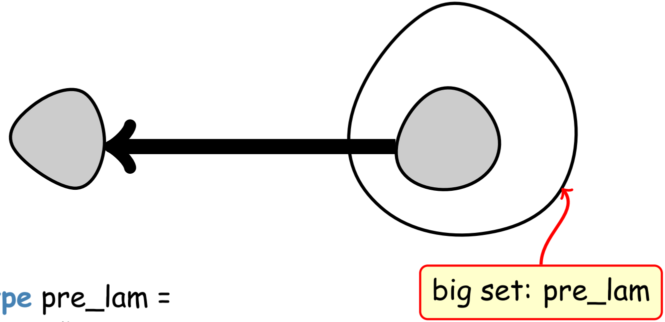
Function $[a].t$ '=' $[\lambda a.t]_\alpha$

$$\begin{aligned}[a].t \stackrel{\text{def}}{=} (\lambda b.\,&\text{if } a = b \\ &\text{then Some}(t) \\ &\text{else if } b\#t \text{ then Some}((b\,a)\bullet t) \text{ else None})\end{aligned}$$

This function 'takes' a lambda-abstraction and an atom, and tries to rename the abstraction according to the given atom.

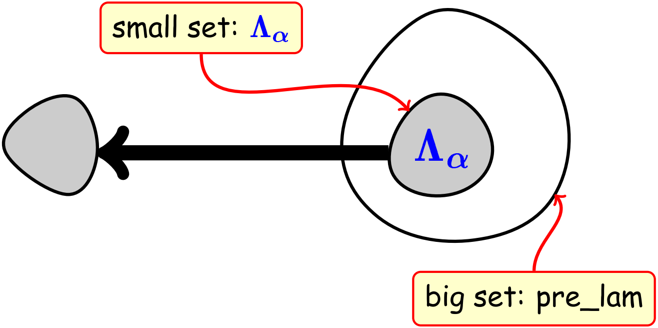
α-Equivalence Classes

We can now define **inductively** **named** α-equivalence classes of lambda-terms:




```
datatype pre_lam =
  Var "atom"
| App "pre_lam" "pre_lam"
| Lam "atom ⇒ pre_lam option"
```

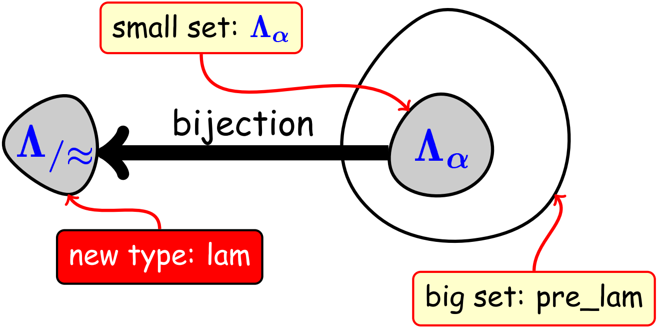
Definition of Small Set



$$\frac{}{\mathsf{Var}\ a \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}\ t_1\ t_2 \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\ [a].t \in \Lambda_\alpha}$$

Definition of Small Set



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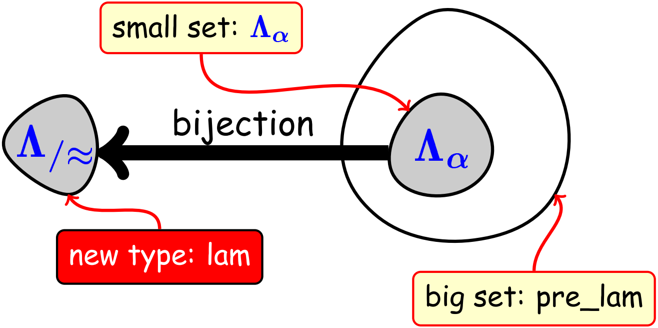

$$\frac{}{\mathsf{Var}\ a \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}\ t_1\ t_2 \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\ [a].t \in \Lambda_\alpha}$$

This means we have the familiar induction principle for Λ_α and so also for $\Lambda_{/\approx}$.

Structural Induction

$$\frac{}{\mathsf{Var}\; a \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}\; t_1\; t_2 \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\,[a].t \in \Lambda_\alpha}$$

...implies the structural induction principle over the **type lam**:

$$\frac{\begin{array}{l}\forall a.\; P\,(\mathsf{Var}\; a) \\ \forall t_1\, t_2.\; P\, t_1 \wedge P\, t_2 \Rightarrow P\,(\mathsf{App}\; t_1\; t_2) \\ \forall a\, t.\; P\, t \Rightarrow P\,(\mathsf{Lam}\,[a].t)\end{array}}{P\, t}$$

Better Structural Induction

$$\frac{\begin{array}{l}\forall a.\ P\,(\mathsf{Var}\,a)\\ \forall t_1\,t_2.\ P\,t_1 \wedge P\,t_2 \Rightarrow P\,(\mathsf{App}\,t_1\,t_2)\\ \forall a\,t.\ P\,t \Rightarrow P\,(\mathsf{Lam}\,[a].t)\end{array}}{P\,t}$$

implies (as seen yesterday)

$$\frac{\begin{array}{l}\forall a\,c.\ P\,c\,(\mathsf{Var}\,a)\\ \forall t_1\,t_2\,c.\ (\forall d.P\,d\,t_1) \wedge (\forall d.P\,d\,t_2) \Rightarrow P\,c\,(\mathsf{App}\,t_1\,t_2)\\ \forall a\,t\,c.\ a\,\#\,c \wedge (\forall d.P\,d\,t) \Rightarrow P\,c\,(\mathsf{Lam}\,[a].t)\end{array}}{P\,c\,t}$$

provided c is finitely supported

“All” for Free

```
nominal_datatype lam =
  Var "name"
| App "lam" "lam"
| Lam "«name»lam" ("Lam [_]._")

lemma alpha_test:
  shows "Lam [x].Var x = Lam [y].Var y"
  by (simp add: lam.inject alpha swap_simps fresh_atm)
```

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```

```
thm lam.inject[no_vars]
(Var x1 = Var y1) = (x1 = y1)
(App x2 x1 = App y2 y1) = (x2 = y2 ∧ x1 = y1)
(Lam [x1].x2 = Lam [y1].y2) = ([x1].x2 = [y1].y2)
```

"All" for Free

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  shows "Lam [x].Var x = Lam [y].Var y"
  by (simp add: lam.inject alpha swap_simps fresh_atm)
```

thm alpha[no_vars]

$([a].x = [b].y) =$
$\quad (a = b \wedge x = y \vee a \neq b \wedge x = [(a, b)] \bullet y \wedge a \# y)$

“All” for Free

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lemma alpha_test:
  shows "Lam [x].Var x = Lam [y].Var y"
  by (simp add: lam.inject alpha swap_simps fresh_atm)
```

```
thm swap_simps[no_vars]
[(a, b)] • a = b
[(a, b)] • b = a
```

"All" for Free

```
nominal_datatype lam =
  Var "name"
| App "lam" "lam"
| Lam "«name»lam" ("Lam [_]._")

lemma alpha_test:
  shows "Lam [x].Var x = Lam [y].Var y"
  by (simp add: lam.inject alpha swap_simps fresh_atm)
```

```
thm fresh_atm[no_vars]
a # b = (a ≠ b)
```

In LF

```
nominal_datatype
    kind =   Type
           | KPi "ty" "«name»kind"
and ty =   TConst "id"
           | TApp "ty" "trm"
           | TPi "ty" "«name»ty"
and trm = Const "id"
           | Var "name"
           | App "trm" "trm"
           | Lam "ty" "«name»trm"

abbreviation KPi_syn :: "name ⇒ ty ⇒ kind ⇒ kind" ("Π[_:_]._")
where "Π[x:A].K ≡ KPi A x K"

abbreviation TPi_syn :: "name ⇒ ty ⇒ ty ⇒ ty" ("Π[_:_]._")
where "Π[x:A1].A2 ≡ TPi A1 x A2"

abbreviation Lam_syn :: "name ⇒ ty ⇒ trm ⇒ trm" ("Lam [_:_]._")
where  "Lam [x:A].M ≡ Lam A x M"
```

In My PhD

```
nominal_datatype trm =
    Ax    "name" "coname"
  | Cut   "«coname»trm" "«name»trm"                      ("Cut ⟨_⟩._ (_)._")
  | NotR  "«name»trm" "coname"                           ("NotR (_)._ _")
  | NotL  "«coname»trm" "name"                           ("NotL ⟨_⟩._ _")
  | AndR  "«coname»trm" "«coname»trm" "coname"           ("AndR ⟨_⟩._ ⟨_⟩._ _")
  | AndL1 "«name»trm" "name"                             ("AndL1 (_)._ _")
  | AndL2 "«name»trm" "name"                             ("AndL2 (_)._ _")
  | OrR1  "«coname»trm" "coname"                         ("OrR1 ⟨_⟩._ _")
  | OrR2  "«coname»trm" "coname"                         ("OrR2 ⟨_⟩._ _")
  | OrL   "«name»trm" "«name»trm" "name"                 ("OrL (_)._ (_)._ _")
  | ImpR  "«name»(«coname»trm)" "coname"                 ("ImpR (_).⟨_⟩._ _")
  | ImpL  "«coname»trm" "«name»trm" "name"               ("ImpL ⟨_⟩._ (_)._ _")
```

- A SN-result for cut-elimination in CL: reviewed by Henk Barendregt and Andy Pitts, and reviewers of conference and journal paper. Still, I found errors in central lemmas; fortunately the main claim was correct :o)

Conclusions

- The support of $\lambda x.x$:

$$
\begin{aligned}
\pi \bullet \lambda x.x &\stackrel{\text{def}}{=} \lambda x.\pi \bullet ((\lambda x.x)\ (\pi^{-1} \bullet x)) \\
&= \lambda x.\pi \bullet \pi^{-1} \bullet x \\
&= \lambda x.x
\end{aligned}
$$

Conclusions

- The support of $\lambda x.x$:

$$\begin{aligned} \pi \bullet \lambda x.x &\stackrel{\text{def}}{=} \lambda x.\pi \bullet ((\lambda x.x)\,(\pi^{-1} \bullet x)) \\ &= \lambda x.\pi \bullet \pi^{-1} \bullet x \\ &= \lambda x.x \end{aligned}$$

- Therefore

$$\begin{aligned} \mathsf{supp}(\lambda x.x) &\stackrel{\text{def}}{=} \{a \,|\, \mathsf{infinite}\{b \,|\, (a\,b) \bullet \lambda x.x \neq \lambda x.x\}\} \\ &= \{a \,|\, \mathsf{infinite}\{b \,|\, \lambda x.x \neq \lambda x.x\}\} \\ &= \varnothing \end{aligned}$$

Conclusions

- To represent α-equivalence classes we used a trick:
  - The same α-equivalence class can be written in many ways ($\lambda x.x$, $\lambda y.y$).
  - Similarly, one and the same function can be written in many ways ($[x]$.Var x, $[y]$.Var y).

Conclusions

- To represent α-equivalence classes we used a trick:
  - The same α-equivalence class can be written in many ways ($\lambda x.x$, $\lambda y.y$).
  - Similarly, one and the same function can be written in many ways ($[x]$.Var x, $[y]$.Var y).

- **Next:** This all might look complicated, but my claim is that nearly all complications can be hidden away. I will show you tomorrow how to formalise a simple CK machine.

Exercises

- Given a finite set of atoms. What is the support of this set?
- What is the support of the set of all atoms?
- From the set of all atoms take one atom out. What is the support of the resulting set?
- Are there any sets of atoms that have infinite support?