Nominal Techniques Course

Thursday-Lecture

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Permutations on Fun's

$$\pi \bullet fn \stackrel{\text{def}}{=} \lambda x. \pi \bullet (fn(\pi^{-1} \bullet x))$$

Example $\lambda x.\mathsf{pr}(a, x)$:

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$$\begin{array}{rcl} a & \mapsto & \mathsf{pr}(a, a) \\ b & \mapsto & \mathsf{pr}(a, b) \\ c & \mapsto & \mathsf{pr}(a, c) \\ d & \mapsto & \mathsf{pr}(a, d) \\ & \vdots & \end{array}$$

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which is the function $\lambda x.\mathsf{pr}(b, x)$.

Permutations on Fun's (ct.)

$$\pi \bullet fn \stackrel{\text{def}}{=} \lambda x. \pi \bullet (fn(\pi^{-1} \bullet x))$$

So $(a\,b) \bullet \lambda x.\mathsf{pr}(a, x) = \lambda x.\mathsf{pr}(b, x)$!

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Equality on Functions

The question arose whether ($a \neq b$):

$$[a].[a].\mathsf{am}(a) = [b].[a].\mathsf{am}(a)?$$

Well, if we knew

- $\pi \bullet ([a].t) = [\pi \bullet a].(\pi \bullet t)$
- $t_1 = t_2 \Leftrightarrow [a].t_1 = [a].t_2$
- $a \neq b \Rightarrow (t_1 = (a\,b) \bullet t_2 \;\wedge\; a \# t_2 \Leftrightarrow [a].t_1 = [b].t_2)$

we could easily decide this question, namely:

Equality on Functions (ct.)

$[a].[a].\mathsf{am}(a) = [b].[a].\mathsf{am}(a)$ where $a \neq b$

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iff $[a].\mathsf{am}(a) = (a\,b)\bullet[a].\mathsf{am}(a)$

and $a \# [a].\mathsf{am}(a)$

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Equality on Functions (ct.)

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$$
\begin{array}{ll}
 & [a].[a].\mathsf{am}(a) = [b].[a].\mathsf{am}(a) \\
\text{iff} & [a].\mathsf{am}(a) = [b].\mathsf{am}(b) \\
 & \text{and } a \# [a].\mathsf{am}(a) \quad \checkmark \\
\text{iff} & \mathsf{am}(a) = \mathsf{am}(a) \quad \checkmark \\
 & \text{and } a \# \mathsf{am}(b) \quad \checkmark
\end{array}
$$

Proofs for the Equalities 1

$$\pi \bullet ([a].t) = [\pi \bullet a].(\pi \bullet t)$$

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$$\lambda x.\pi \bullet \text{if } \pi^{-1} \bullet x = a \text{ then } t$$
$$\text{else if } \pi^{-1} \bullet x \# t \text{ then } (a \ \pi^{-1} \bullet x) \bullet t \text{ else er}$$

$$= \lambda x.\text{if } x = \pi \bullet a \text{ then } \pi \bullet t$$
$$\text{else if } x \# \pi \bullet t \text{ then } (\pi \bullet a \ \ x) \bullet \pi \bullet t \text{ else er}$$

$$[a].t \stackrel{\text{def}}{=} \lambda x.\text{if } x = a \text{ then } t$$
$$\text{else if } x \# t \text{ then } (x\, a) \bullet t \text{ else er}$$

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$$\pi \bullet \mathsf{if} \ldots \mathsf{then} \ldots \mathsf{else} \ldots =$$
$$\mathsf{if} \ldots \mathsf{then}\ \pi \bullet \ldots \mathsf{else}\ \pi \bullet \ldots$$

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= &\lambda x.\text{if } x = \pi\bullet a \text{ then } \pi\bullet t\\
&\qquad \text{else if } x \# \pi\bullet t \text{ then } (\pi\bullet a\ \ x)\bullet\pi\bullet t \text{ else er}
\end{aligned}$$

$$\pi^{-1}\bullet x \# t \ \text{ iff } \ x \# \pi\bullet t$$

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Done.

Proofs for the Equalities 2

$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

which means we can assume that:

$$\begin{aligned} & \lambda x.\text{if } x = a \text{ then } t_1 \\ & \qquad \text{else if } x \# t_1 \text{ then } (a\,x) \bullet t_1 \text{ else er} \\ = \; & \lambda x.\text{if } x = a \text{ then } t_2 \\ & \qquad \text{else if } x \# t_2 \text{ then } (a\,x) \bullet t_2 \text{ else er} \end{aligned}$$

Proofs for the Equalities 2

$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

which means we can assume that:

$$\forall x.\ \text{if } x = a \text{ then } t_1 \text{ else if } x \# t_1 \text{ then } (a\,x)\bullet t_1 \text{ else er}$$

$$=$$

$$\text{if } x = a \text{ then } t_2 \text{ else if } x \# t_2 \text{ then } (a\,x)\bullet t_2 \text{ else er}$$

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which means we can assume that:

if $a = a$ then t_1
else if $a \# t_1$ then $(a\,a){\bullet}t_1$ else er

$=$

if $a = a$ then t_2
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Proofs for the Equalities 2

$$[a].t_1 = [a].t_2 \Rightarrow t_1 = t_2$$

which means we can assume that:

$$t_1$$
$$=$$
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Done.

Freshness 2

Lemma: $a \neq b \wedge b \# [a].t \Rightarrow b \# t$

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(4) $(b\,c)\bullet c \# (b\,c)\bullet t$ from $c \# t$

Freshness 2

Lemma: $a \neq b \wedge b \# [a].t \Rightarrow b \# t$

Proof:

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(4) $b \# t$ from $c \# t$ and (3)

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Lemma: $a \neq b \wedge b \;\#\; [a].t \Rightarrow b \;\#\; t$

Proof:

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(3) $(b\,c) \bullet t = t$ by "same abstraction"

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Done.

Freshness 3

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Lemma: $a \# [a].t$

Proof:

(1) $(\exists c) c \# (a, t)$ "finitely supported"

(2) $c \# [a].t$ by (Freshness 1)

(3) $(a\, c) \bullet c \# (a\, c) \bullet [a].t$ from (2)

Freshness 3

Lemma: $a \# [a].t$

Proof:

(1) $(\exists c) c \# (a, t)$ "finitely supported"

(2) $c \# [a].t$ by (Freshness 1)

(3) $a \# [c].((a\, c) \bullet t)$ from (2)

Freshness 3

Lemma: $a \# [a].t$

Proof:

(1) $(\exists c) c \# (a, t)$ — "finitely supported"

(2) $c \# [a].t$ — by (Freshness 1)

(3) $a \# [c].((a\,c) \bullet t)$ — from (2)

(4) $[c].((a\,c) \bullet t) = [a].t$ — provided $c \# t$ and $(a\,c) \bullet t = (a\,c) \bullet t$

Freshness 3

Lemma: $a \# [a].t$

Proof:

(1) $(\exists c) c \# (a, t)$ "finitely supported"

(2) $c \# [a].t$ by (Freshness 1)

(3) $a \# [c].((a\,c) \bullet t)$ from (2)

(4) $[c].((a\,c) \bullet t) = [a].t$ provided $c \# t$ and $(a\,c) \bullet t = (a\,c) \bullet t$

Both hold, therefore $a \# [a].t$

Done.

Equivariance

$eqvt(P) \stackrel{\text{def}}{=} (\forall t : \Lambda_\alpha)(\forall x : FSType)(\forall pi)$

We have an induction principle for Λ_α (by way of how it is constructed), but this induction principle is not as weak (in the good sense) as that from Nominal Logic. If we prove an equivariant property, we can do better.

slightly unusual definition for equivariance.

Equivariance

$$\mathsf{eqvt}(P) \stackrel{\mathsf{def}}{=} (\forall t : \Lambda_a)(\forall x : FSType)(\forall pi)$$
$$P\,t\,x \Rightarrow P(\pi \bullet t)(\pi \bullet x)$$

Later we shall often consider predicates having an Λ_α-term as first argument and an $FSType$ as second argument. Therefore, this slightly unusual definition for equivariance.

Some /Any-Property

Assuming $\mathsf{eqvt}(P)$ then

$$(\exists x)\ a \# x \wedge (\forall t)\ P\,([a].t)\ x$$

if and only if

$$(\forall x)\ a \# x \Rightarrow (\forall t)\ P\,([a].t)\ x$$

Some /Any-Property

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$$(\forall x)\ a \# x \Rightarrow (\forall t)\ P\,([a].t)\ x$$

Proof: Same as on Tuesday.

Induction

$$\frac{\begin{array}{l}(\forall a)\ P\,(\mathsf{am}(a))\,x \\ (\forall t_1, t_2)\ P\,t_1\,x \wedge P\,t_2\,x \Rightarrow P\,(\mathsf{pr}(t_1, t_2))\,x \\ (\exists a)\ a \# x \wedge (\forall t)\ P\,t\,x \Rightarrow P\,([a].t)\,x\end{array}}{(\forall t)\ P\,t\,x}$$

Proof: By induction on n (the "stage" when constructing Λ_α).

Induction

$$\frac{\begin{array}{l}\mathsf{eqvt}(P)\\ (\forall a)\ P\,(\mathsf{am}(a))\,x\\ (\forall t_1, t_2)\ P\,t_1\,x \wedge P\,t_2\,x \Rightarrow P\,(\mathsf{pr}(t_1, t_2))\,x\\ (\forall a)\ a \,\#\, x \Rightarrow (\forall t)\ P\,t\,x \Rightarrow P\,([a].t)\,x\end{array}}{(\forall t)\ P\,t\,x}$$

Proof: By induction on n (the "stage" when constructing Λ_α).

What Has Been Achieved

- we gave an inductive definition of a set (Λ_α) that is bijective with the α-equated lambda-terms
- Λ_α has very much the feel of (named) lambda-terms (equated up to α-equivalence)
- if we can prove equivariance for the IH, then we only need to prove the abstraction case for one fresh atom
- **and** we can put the money where our mouth is. . . ;o)