Nominal Techniques
or, “The Real Thing”

Christian Urban (TU Munich)

http://isabelle.in.tum.de/nominal/

A Formalisation of a CK Machine:

$\_ \Downarrow \_$ CK

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A Formalisation of a CK Machine:

$$\_ \Downarrow \_ \longrightarrow \text{CK}$$

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or, “The Real Thing”

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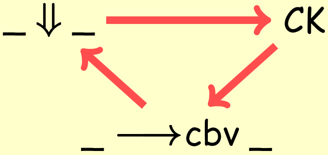
A Formalisation of a CK Machine:



Lambda-Terms

- We build on the theory Nominal (which in turn builds on HOL). Nominal provides an infra-structure to reason with binders.

```
atom_decl name

nominal_datatype lam =
  Var "name"
| App "lam" "lam"
| Lam "«name»lam" ("Lam [_]._")
```

Lambda-Terms

- We build on the theory Nominal (which in turn builds on HOL). Nominal provides an infra-structure to reason with binders.

```
atom_decl name

nominal_datatype lam =
   Var "name"
 | App "lam" "lam"
 | Lam "«name»lam" ("Lam [_]._")
```

- We allow more than one kind of atoms.
- At the moment we only support single, but nested binders (future: arbitrary binding structures).

Contexts

```
datatype ctx =
    Hole ("□")
  | CAppL "ctx" "lam"
  | CAppR "lam" "ctx"
  | CLam  "name" "ctx" ("CLam [_]._")

fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
| "(CLam [x].E)⟦t⟧ = Lam [x].(E⟦t⟧)"

lemma alpha_test:
  shows "x≠y ⟹ (CLam [x].□) ≠ (CLam [y].□)"
  and   "(CLam [x].□)⟦Var x⟧ = (CLam [y].□)⟦Var y⟧"
by (simp_all add: ctx.inject lam.inject alpha swap_simps fresh_atm)
```

Backtrack One Step

- For our CK machines we actually do not need contexts for lambdas.

```
datatype ctx =
    Hole ("□")
  | CAppL "ctx" "lam"
  | CAppR "lam" "ctx"

fun
  filling :: "ctx ⇒ lam ⇒ lam" ("_⟦_⟧")
where
  "□⟦t⟧ = t"
| "(CAppL E t')⟦t⟧ = App (E⟦t⟧) t'"
| "(CAppR t' E)⟦t⟧ = App t' (E⟦t⟧)"
```

Context Composition

```
fun ctx_compose :: "ctx ⇒ ctx ⇒ ctx" ("_ ∘ _")
where
  "□ ∘ E' = E'"
| "(CAppL E t') ∘ E' = CAppL (E ∘ E') t'"
| "(CAppR t' E) ∘ E' = CAppR t' (E ∘ E')"

lemma ctx_compose:
  shows "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧"
by (induct E₁ rule: ctx.induct) (simp_all)

types ctxs = "ctx list"

fun ctx_composes :: "ctxs ⇒ ctx" ("_↓")
where
    "[]↓ = □"
  | "(E#Es)↓ = (Es↓) ∘ E"
```

Context Composition

```
fun ctx_compose :: "ctx ⇒ ctx ⇒ ctx" ("_ ∘ _")
where
  "□ ∘ E' = E'"
| "(CAppL E t') ∘ E' = CAppL (E ∘ E') t'"
| "(CAppR t' E) ∘ E' = CAppR t' (E ∘ E')"

lemma ctx_compose:
  shows "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧"
by (induct E₁ rule: ctx.induct) (simp_all)
```

Subgoals

1. $\Box \circ E_2[\![t]\!] = \Box[\![E_2[\![t]\!]]\!]$
2. $\bigwedge ctx\ lam.\ ctx \circ E_2[\![t]\!] = ctx[\![E_2[\![t]\!]]\!] \Longrightarrow CAppL\ ctx\ lam \circ E_2[\![t]\!] = CAppL\ ctx\ lam[\![E_2[\![t]\!]]\!]$
3. $\bigwedge lam\ ctx.\ ctx \circ E_2[\![t]\!] = ctx[\![E_2[\![t]\!]]\!] \Longrightarrow CAppR\ lam\ ctx \circ E_2[\![t]\!] = CAppR\ lam\ ctx[\![E_2[\![t]\!]]\!]$

Context Composition

```
fun ctx_compose :: "ctx ⇒ ctx ⇒ ctx" ("_ ∘ _")
where
  "□ ∘ E' = E'"
| "(CAppL E t') ∘ E' = CAppL (E ∘ E') t'"
| "(CAppR t' E) ∘ E' = CAppR t' (E ∘ E')"

lemma ctx_compose:
  shows "(E₁ ∘ E₂)⟦t⟧ = E₁⟦E₂⟦t⟧⟧"
by (induct E₁ rule: ctx.induct) (simp_all)

types ctxs = "ctx list"

fun ctx_composes :: "ctxs ⇒ ctx" ("_↓")
where
    "[]↓ = □"
  | "(E#Es)↓ = (Es↓) ∘ E"
```

Definition of Types

```
nominal_datatype ty =
  tVar "string"
| tArr "ty" "ty" ("_ → _")

types ty_ctx = "(name × ty) list"

abbreviation
  "sub_ty_ctx" :: "ty_ctx ⇒ ty_ctx ⇒ bool" ("_ ⊆ _")
where
  "Γ₁ ⊆ Γ₂ ≡ ∀x. x ∈ set Γ₁ ⟶ x ∈ set Γ₂"
```

Definition of Types

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nominal_datatype ty =
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```

- We can overload $\subseteq$, but this might mean we have to give explicit type-annotations so that Isabelle can figure out what is meant.

Typing Judgements

```
inductive
  valid :: "ty_ctx ⇒ bool"
where
  v1: "valid []"
| v2: "⟦valid Γ; x#Γ⟧⟹ valid ((x,T)#Γ)"

inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t1 : T1→T2; Γ ⊢ t2 : T1⟧ ⟹ Γ ⊢ App t1 t2 : T2"
| t_Lam: "⟦x#Γ; (x,T1)#Γ ⊢ t : T2⟧ ⟹ Γ ⊢ Lam [x].t : T1 → T2"
```

Typing Judgements

$$\frac{\text{valid}\ \Gamma \qquad (x, T) \in \text{set}\ \Gamma}{\Gamma \vdash \text{Var}\ x : T} \qquad \frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash \text{App}\ t_1\ t_2 : T_2}$$

$$\frac{x \# \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \text{Lam}\ [x].t : T_1 \rightarrow T_2}$$

induct
valid
where
v_1: "v
| v_2: "⟦valid Γ; x#Γ⟧ ⟹ valid ((x,T)#Γ)"

inductive
typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t_1 : T_1→T_2; Γ ⊢ t_2 : T_1⟧ ⟹ Γ ⊢ App t_1 t_2 : T_2"
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Typing Judgements

```
inductive
  valid :: "ty_ctx ⇒ bool"
where
  v₁: "valid []"
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  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
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| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"

declare typing.intros[intro] valid.intros[intro]
```

Typing Judgements

```
inductive
  valid :: "ty_ctx ⇒
where
  v1: "valid []"
| v2: "⟦valid Γ; x#Γ

inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t1 : T1→T2; Γ ⊢ t2 : T1⟧ ⟹ Γ ⊢ App t1 t2 : T2"
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declare typing.intros[intro] valid.intros[intro]
```

We want to have the strong induction principle for the typing judgement.

1.) The relation needs to be equivariant.

Typing Judgements

```
inductive
  valid :: "ty_ctx ⇒ bool"
where
  v₁: "valid []"
| v₂: "⟦valid Γ; x#Γ⟧⟹ valid ((x,T)#Γ)"

inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
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declare typing.intros[intro] valid.intros[intro]

equivariance valid
equivariance typing
```

Typing Judgements

```
inductive
  valid :: "ty_ctx ⇒ [
where
  v1: "valid []"
| v2: "⟦valid Γ; x#Γ⟧⟹ valid ((x,T)#Γ)"
```

This proves for us:

$$\text{valid } \Gamma \Longrightarrow \text{valid } (\pi \bullet \Gamma)$$

$$\Gamma \vdash t : T \Longrightarrow \pi \bullet \Gamma \vdash \pi \bullet t : \pi \bullet T$$

```
inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t1 : T1→T2; Γ ⊢ t2 : T1⟧ ⟹ Γ ⊢ App t1 t2 : T2"
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declare typing.intros[intro] valid.intros[intro]

equivariance valid
equivariance typing
```

Typing Judgements (2)

```
inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
| t_App: "⟦Γ ⊢ t₁ : T₁→T₂; Γ ⊢ t₂ : T₁⟧ ⟹ Γ ⊢ App t₁ t₂ : T₂"
| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"
```

```
nominal_inductive typing
```

Typing Judgements (2)

```
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  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
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| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"
```

Subgoals

1. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1)::\Gamma \vdash t : T_2]\!] \Longrightarrow x \# \Gamma$
2. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1)::\Gamma \vdash t : T_2]\!] \Longrightarrow x \# \mathrm{Lam}\ [x].t$
3. $\bigwedge x\ \Gamma\ T_1\ t\ T_2.\ [\![x \# \Gamma; (x, T_1)::\Gamma \vdash t : T_2]\!] \Longrightarrow x \# T_1 \to T_2$

```
nominal_inductive typing
```

Typing Judgements (2)

```
inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
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| t_App: "⟦Γ ⊢ t₁ : T₁→T₂; Γ ⊢ t₂ : T₁⟧ ⟹ Γ ⊢ App t₁ t₂ : T₂"
| t_Lam: "⟦x#Γ; (x,T₁)#Γ ⊢ t : T₂⟧ ⟹ Γ ⊢ Lam [x].t : T₁ → T₂"

lemma ty_fresh:
  fixes x::"name"
  and   T::"ty"
  shows "x#T"
by (induct T rule: ty.induct)
   (simp_all add: fresh_string)

nominal_inductive typing
```

Typing Judgements (2)

```
inductive
  typing :: "ty_ctx ⇒ lam ⇒ ty ⇒ bool" ("_ ⊢ _ : _")
where
  t_Var: "⟦valid Γ; (x,T) ∈ set Γ⟧ ⟹ Γ ⊢ Var x : T"
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lemma ty_fresh:
  fixes x::"name"
  and   T::"ty"
  shows "x#T"
by (induct T rule: ty.induct)
   (simp_all add: fresh_string)

nominal_inductive typing
  by (simp_all add: abs_fresh ty_fresh)
```

Weakening

```
lemma weakening:
  fixes Γ1 Γ2::"ty_ctx"
  assumes a: "Γ1 ⊢ t : T"
  and     b: "valid Γ2"
  and     c: "Γ1 ⊆ Γ2"
  shows "Γ2 ⊢ t : T"
using a b c
by (nominal_induct Γ1 t T avoiding: Γ2 rule: typing.strong_induct)
   (auto simp add: atomize_all atomize_imp)
```

Weakening

```
lemma weakening:
  fixes Γ1 Γ2::"ty_ctx"
  assumes a: "Γ1 ⊢ t : T"
  and     b: "valid Γ2"
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  shows "Γ2 ⊢ t : T"
using a b c
by (nominal_induct Γ1 t T avoiding: Γ2 rule: typing.strong_induct)
  (auto simp add: atomize_all atomize_imp)
```

- This proof is can be found automatically, but that tells us not much. . .

Lemma / Theorem / Corollary

- Lemmas / Theorems / Corollary are of the form:

```
theorem theorem_name:
 fixes      x::"type"
 ...
 assumes    "assm1"
 and        "assm2"
 ...
 shows   "statement"
 ...
```

- Grey parts are optional.
- Assumptions and the (goal)statement must be of type bool.

Lemma / Theorem / Corollary

- Lemmas / Theorems / Corollary are of the form:

```
theorem theorem_name:
  fixes    x::"type"
  ...
  assumes  "assm1"
  and      "assm2"
  ...
  shows   "s
  ...
```

- Grey parts are optional.
- Assumptions and the (g… type bool.

lemma weakening:
  fixes Γ_1 Γ_2::"ty_ctx"
  assumes a: "$\Gamma_1 \vdash t : T$"
  and b: "valid Γ_2"
  and c: "$\Gamma_1 \subseteq \Gamma_2$"
  shows "$\Gamma_2 \vdash t : T$"

Struct. of an Ind. Proof

```
lemma weakening:
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof(nominal_induct Γ₁ t T avoiding: Γ₂ rule: typing.strong_induct)
  case (t_Var Γ₁ x T)
  ...
  show "Γ₂ ⊢ Var x : T"
  ...
next
  case (t_App Γ₁ t₁ T₁ T₂ t₂)
  ...
  show "Γ₂ ⊢ App t₁ t₂ : T₂"
  ...
next
  case (t_Lam x Γ₁ T₁ t T₂)
  ...
  show "Γ₂ ⊢ Lam [x].t : T₁ → T₂"
  ...
qed
```

Cases

- Each case is of the form:

```
case (Name x...)
 have n1: "statment1" by justification
 have n2: "statment2" by justification
 ...
 show "statment" by justification
```

- Grey parts are optional.
- Justifications can also be: **using** ... **by** ...

Cases

- Each case is of the form:

```
case (Name x. . . )
  have n1: "statment1" by justification
  have n2: "statment2" by justification
  . . .
  show "statment" by justification
```

- Grey parts are optional.
- Justifications can also be: using . . . by . . .

```
using ih by . . .
using n1 n2 n3 by . . .
using lemma_name. . . by . . .
```

Cases

- Each case is of the form:

```
case (Name x. . . )
 have n1: "statment1" by justification
 have n2: "statment2" by justification
 . . .
 show "statment" by justification
```

- Grey parts are optional.
- Justifications can also be: using . . . by . . .

```
using ih by . . .
using n1 n2 n3 by . . .
using lemma_name. . . by . . .
```

Justifications

- Omitting proofs

  sorry

- Assumptions

  by fact

- Automated proofs

by simp	simplification (equations, definitions)
by auto	simplification & proof search (many goals)
by force	simplification & proof search (first goal)
by blast	proof search

  . . .

$$\frac{\text{valid } \Gamma \qquad (x, T) \in \text{set } \Gamma}{\Gamma \vdash \text{Var } x : T}$$

```
lemma weakening:
  fixes Γ₁ Γ₂::"ty_ctx"
  assumes a: "Γ₁ ⊢ t : T"
  and     b: "valid Γ₂"
  and     c: "Γ₁ ⊆ Γ₂"
  shows "Γ₂ ⊢ t : T"
using a b c
proof(nominal_induct Γ₁ t T avoiding: Γ₂ rule: typing.strong_induct)
  case (t_Var Γ₁ x T)
  have a1: "valid Γ₂" by fact
  have a2: "Γ₁ ⊆ Γ₂" by fact
  have a3: "(x,T) ∈ (set Γ₁)" by fact
  have a4: "(x,T) ∈ (set Γ₂)" using a2 a3 by simp
  show "Γ₂ ⊢ Var x : T" using a1 a4 by auto
next ...
```

$$\frac{x \# \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \text{Lam } [x].t : T_1 \to T_2}$$

```
next
  case (t_Lam x Γ1 T1 t T2)
  have vc: "x#Γ2" by fact
  have ih: "⟦valid ((x,T1)#Γ2); (x,T1)#Γ1 ⊆ (x,T1)#Γ2⟧
                                   ⟹ (x,T1)#Γ2 ⊢ t:T2" by fact
  have a1: "Γ1 ⊆ Γ2" by fact
  have a2: "(x,T1)#Γ1 ⊆ (x,T1)#Γ2" using a1 by simp
  have b1: "valid Γ2" by fact
  have b2: "valid ((x,T1)#Γ2)" using vc b1 by auto
  have b3: "(x,T1)#Γ2 ⊢ t : T2" using ih b2 a2 by simp
  show "Γ2 ⊢ Lam [x].t : T1→T2" using b3 vc by auto
next ...
```

$$\frac{x \# \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \text{Lam}\,[x].t : T_1 \rightarrow T_2}$$

```
next
  case (t_Lam x Γ₁ T₁ t T₂)
  have vc: "x#Γ₂" by fact
  have ih: "⟦valid ((x,T₁)#Γ₂); (x,T₁)#Γ₁ ⊆ (x,T₁)#Γ₂⟧
                                    ⟹ (x,T₁)#Γ₂ ⊢ t:T₂" by fact
  have "Γ₁ ⊆ Γ₂" by fact
  then have a2: "(x,T₁)#Γ₁ ⊆ (x,T₁)#Γ₂" by simp
  have "valid Γ₂" by fact
  then have b2: "valid ((x,T₁)#Γ₂)" using vc by auto
  have "(x,T₁)#Γ₂ ⊢ t : T₂" using ih b2 a2 by simp
  then show "Γ₂ ⊢ Lam [x].t : T₁→T₂" using vc by auto
next ...
```

A Sequence of Facts

```
have n1: "..."
have n2: "..."

...

have nn: "..."
have "..." using n1 n2...nn
```

```
have "..."
moreover have "..."

...

moreover have "..."
ultimately have "..."
```

$$\frac{x \# \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathrm{Lam}\ [x].t : T_1 \rightarrow T_2}$$

```
next
  case (t_Lam x Γ₁ T₁ t T₂)
  have vc: "x#Γ₂" by fact
  have ih: "⟦valid ((x,T₁)#Γ₂); (x,T₁)#Γ₁ ⊆ (x,T₁)#Γ₂⟧
                                   ⟹ (x,T₁)#Γ₂ ⊢ t:T₂" by fact
  have "Γ₁ ⊆ Γ₂" by fact
  then have "(x,T₁)#Γ₁ ⊆ (x,T₁)#Γ₂" by simp
  moreover
  have "valid Γ₂" by fact
  then have "valid ((x,T₁)#Γ₂)" using vc by auto
  ultimately have "(x,T₁)#Γ₂ ⊢ t : T₂" using ih by simp
  then show "Γ₂ ⊢ Lam [x].t : T₁→T₂" using vc by auto
next ...
```

$$\frac{x \# \Gamma \qquad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathrm{Lam}\ [x].t : T_1 \rightarrow T_2}$$

```
next
  case (t_Lam x Γ1 T1 t T2)
  have vc: "x#Γ2" by fact
  have ih: "⟦valid ((x,T1)#Γ2); (x,T1)#Γ1 ⊆ (x,T1)#Γ2⟧
                                  ⟹ (x,T1)#Γ2 ⊢ t:T2" by fact
  have "Γ1 ⊆ Γ2" by fact
  then have "(x,T1)#Γ1 ⊆ (x,T1)#Γ2" by simp
  moreover
  have "valid Γ2" by fact
  then have "valid ((x,T1)#Γ2)" using vc by auto
  ultimately have "(x,T1)#Γ2 ⊢ t : T2" using ih by simp
  then show "Γ2 ⊢ Lam [x].t : T1→T2" using vc by auto
qed (auto)
```

Capture-Avoiding Subst.

- We next want to introduce an evaluation relation and a CK machine.
- For this we need the notion of capture-avoiding substitution.

```
consts
 subst :: "lam ⇒ name ⇒ lam ⇒ lam"  ("_[_::=_]")

nominal_primrec
 "(Var x)[y::=s] = (if x=y then s else (Var x))"
 "(App t₁ t₂)[y::=s] = App (t₁[y::=s]) (t₂[y::=s])"
 "x#(y,s) ⟹ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"
```

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 "x#(y,s) ⟹ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"
```

- Despite its looks, this is a total function!

Bound Names Function

- However there is a problem with the bound names function:

```
consts
  bnds :: "lam ⇒ name set"

nominal_primrec
  "bnds (Var x) = {}"
  "bnds (App t₁ t₂) = bnds (t₁) ∪ bnds (t₂)"
  "bnds (Lam [x].t) = bnds (t) ∪ {x}"

lemma
  shows "bnds (Lam [x].Var x) = {x}"
  and   "bnds (Lam [y].Var y) = {y}"
by (simp_all)
```

Bound Names Function

Assume $x \neq y$.

- Howe
  funct

consts
  bnds :: "lo

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Assume $x \neq y$.

Lam [x].Var x = Lam [y].Var y

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- Howe
  funct

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bnds :: "lc

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"bnds (App t_1 t_2) = bnds (t_1) ∪ bnds (t_2)"
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by (simp_all)

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Lam [x].Var x = Lam [y].Var y

bnds (Lam [x].Var x) = bnds (Lam [y].Var y)

Bound Names Function

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  and "bnds (Lam [y].Var y) = {y}"
by (simp_all)

Assume $x \neq y$.

Lam [x].Var x = Lam [y].Var y

bnds (Lam [x].Var x) = bnds (Lam [y].Var y)

{x} = {y}

Bound Names Function

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Freshness Condition for Binders (FCB)

$$\forall a\, ts.\ a \# f \Rightarrow a \# f\, a\, ts$$

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  "x#(y,s) ⟹ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"
```

Freshness Condition for Binders (FCB)

$$\forall a\, ts.\ a \# f \Rightarrow a \# f\ a\ ts$$

$$\bigwedge x1\ y1.\ \ldots\ \ldots \Longrightarrow x1 \# \mathrm{Lam}\ [x1].y1$$

Capture-Avoiding Subst.

```
consts
  subst :: "lam ⇒ name ⇒ lam ⇒ lam"  ("_[_::=_]")

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  "(Var x)[y::=s] = (if x=y then s else (Var x))"
  "(App t₁ t₂)[y::=s] = App (t₁[y::=s]) (t₂[y::=s])"
  "x#(y,s) ⟹ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"
apply(finite_guess)+
apply(rule TrueI)+
apply(simp add: abs_fresh)+
apply(fresh_guess)+
done
```

Freshness Condition for Binders (FCB)

$$\forall a\, ts.\ a \# f \Rightarrow a \# f\ a\ ts$$

⋀x1 y1. ... ... ⟹ x1 # Lam [x1].y1

Capture-Avoiding Subst.

FCB for Bound Variable Function:

$\bigwedge$x1 y1. ... ... ⟹ x1 # (y1 ∪ {x1})

consts
subst :: "lam ⇒ name ⇒ lam ⇒ lam" ("_[_::=_]")

nominal_primrec
"(Var x)[y::=s] = (if x=y then s else (Var x))"
"(App t_1 t_2)[y::=s] = App (t_1[y::=s]) (t_2[y::=s])"
"x#(y,s) ⟹ (Lam [x].t)[y::=s] = Lam [x].(t[y::=s])"
apply(finite_guess)+
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apply(fresh_guess)+
done

Freshness Condition for Binders (FCB)

$$\forall a\, ts.\ a \# f \Rightarrow a \# f\ a\ ts$$

$\bigwedge$x1 y1. ... ... ⟹ x1 # Lam [x1].y1

Evaluation Relation

```
inductive
  eval :: "lam ⇒ lam ⇒ bool" ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
| e_App: "⟦t₁⇓ Lam [x].t; t₂⇓ v'; t[x::=v']⇓ v⟧ ⟹ App t₁ t₂ ⇓ v"

declare eval.intros[intro]
```

Evaluation Relation

```
inductive
  eval :: "lam ⇒ lam ⇒ bool" ("_ ⇓ _")
where
  e_Lam: "Lam [x].t ⇓ Lam [x].t"
| e_App: "⟦t₁⇓ Lam [x].t; t₂⇓ v'; t[x::=v']⇓ v⟧ ⟹ App t₁ t₂ ⇓ v"

declare eval.intros[intro]
```

$$\frac{}{\mathsf{Lam}\ [x].t \Downarrow \mathsf{Lam}\ [x].t}$$

$$\frac{t_1 \Downarrow \mathsf{Lam}\ [x].t \qquad t_2 \Downarrow v' \qquad t[x::=v'] \Downarrow v}{\mathsf{App}\ t_1\ t_2 \Downarrow v}$$

Values

```
inductive
  val :: "lam ⇒ bool"
where
  v_Lam[intro]:   "val (Lam [x].e)"

lemma eval_to_val:
  assumes a: "t ⇓ t'"
  shows "val t'"
using a by (induct) (auto)
```

Values

```
inductive
  val :: "lam ⇒ bool"
where
  v_Lam[intro]:   "val (Lam [x].e)"

lemma eval_to_val:
  assumes a: "t ⇓ t'"
  shows "val t'"
using a by (induct) (auto)
```

- If our language contained natural numbers, booleans, etc., we would expand on this definition.

CK Machine

- A CK machine works on configurations ⟨_,_⟩ consisting of a lambda-term and a list of contexts.

```
inductive
  machine :: "lam⇒ctxs⇒lam⇒ctxs⇒bool"   ("⟨_,_⟩ ↦ ⟨_,_⟩")
where
  m1: "⟨App e1 e2,Es⟩ ↦ ⟨e1,(CAppL □ e2)#Es⟩"
| m2: "val v ⟹ ⟨v,(CAppL □ e2)#Es⟩ ↦ ⟨e2,(CAppR v □)#Es⟩"
| m3: "val v ⟹ ⟨v,(CAppR (Lam [x].e) □)#Es⟩ ↦ ⟨e[x::=v],Es⟩"
```

CK Machine

- A CK machine works on configurations ⟨_,_⟩ consisting of a lambda-term and a list of contexts.

```
inductive
  machine :: "lam⇒ctxs⇒lam⇒ctxs⇒bool"   ("⟨_,_⟩ ↦ ⟨_,_⟩")
where
  m1: "⟨App e1 e2,Es⟩ ↦ ⟨e1,(CAppL □ e2)#Es⟩"
| m2: "val v ⟹ ⟨v,(CAppL □ e2)#Es⟩ ↦ ⟨e2,(CAppR v □)#Es⟩"
| m3: "val v ⟹ ⟨v,(CAppR (Lam [x].e) □)#Es⟩ ↦ ⟨e[x::=v],Es⟩"
```

Initial state of the CK machine:

⟨t,[]⟩

CK Machine

- A CK machine works on configurations $\langle\_,\_\rangle$ consisting of a lambda-term and a list of contexts.

inductive
  machine :: "lam⇒ctxs⇒lam⇒ctxs⇒bool"   ("$\langle\_,\_\rangle \mapsto \langle\_,\_\rangle$")
where
  m_1: "$\langle$App e_1 e_2,Es$\rangle \mapsto \langle e_1$,(CAppL □ e_2)#Es$\rangle$"
| m_2: "val v $\Longrightarrow$ $\langle$v,(CAppL □ e_2)#Es$\rangle \mapsto \langle e_2$,(CAppR v □)#Es$\rangle$"
| m_3: "val v $\Longrightarrow$ $\langle$v,(CAppR (Lam [x].e) □)#Es$\rangle \mapsto \langle$e[x::=v],Es$\rangle$"

inductive
  "machines" :: "lam⇒ctxs⇒lam⇒ctxs⇒bool" ("$\langle\_,\_\rangle \mapsto^* \langle\_,\_\rangle$")
where
  ms_1: "$\langle$e,Es$\rangle \mapsto^* \langle$e,Es$\rangle$"
| ms_2: "$[\![\langle e_1,Es_1\rangle \mapsto \langle e_2,Es_2\rangle; \langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle]\!]$
    $\Longrightarrow \langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle$"

Our Goal

- Our goal is to show that the result the machine calculates corresponds to the value the evaluation relation generates and vice versa. That means:

$$t \Downarrow v \Longleftrightarrow \langle t, [] \rangle \mapsto^* \langle v, [] \rangle$$

with v being a value.

Left-to-Right Direction

```
corollary eval_implies_machines:
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a using eval_implies_machines_ctx by simp
```

Left-to-Right Direction

```
lemma ms₃:
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩" "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a by (induct) (auto)
```

```
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  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a by (induct) (auto)

theorem eval_implies_machines_ctx:
  assumes a: "t ⇓ t'"
  shows "⟨t,Es⟩ ↦* ⟨t',Es⟩"
using a
by (induct arbitrary: Es)
    (metis eval_to_val machine.intros ms₁ ms₂ ms₃ v_Lam)+

corollary eval_implies_machines:
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a using eval_implies_machines_ctx by simp
```

Left-to-Right Direction

lemma ms_3:
  assumes a: "$\langle e_1,Es_1\rangle \mapsto^* \langle e_2,Es_2\rangle$" "$\langle e_2,Es_2\rangle \mapsto^* \langle e_3,Es_3\rangle$"
  shows "$\langle e_1,Es_1\rangle \mapsto^* \langle e_3,Es_3\rangle$"

Sledgehammer:

Can be used at any point in the development.

Isabelle

Left-to-Right Direction

lemma ms$_3$:
  assumes a: "$\langle e_1, Es_1\rangle \mapsto^* \langle e_2, Es_2\rangle$" "$\langle e_2, Es_2\rangle \mapsto^* \langle e_3, Es_3\rangle$"
  shows "$\langle e_1, Es_1\rangle \mapsto^* \langle e_3, Es_3\rangle$"

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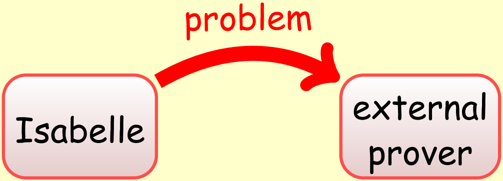

Left-to-Right Direction

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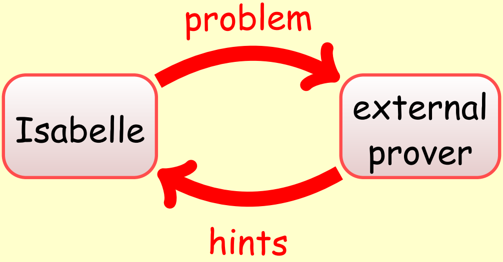

Left-to-Right Direction

```
lemma ms₃:
  assumes a: "⟨e₁,Es₁⟩ ↦* ⟨e₂,Es₂⟩" "⟨e₂,Es₂⟩ ↦* ⟨e₃,Es₃⟩"
  shows "⟨e₁,Es₁⟩ ↦* ⟨e₃,Es₃⟩"
using a by (induct) (auto)

theorem eval_implies_machines_ctx:
  assumes a: "t ⇓ t'"
  shows "⟨t,Es⟩ ↦* ⟨t',Es⟩"
using a
by (induct arbitrary: Es)
    (metis eval_to_val machine.intros ms₁ ms₂ ms₃ v_Lam)+

corollary eval_implies_machines:
  assumes a: "t ⇓ t'"
  shows "⟨t,[]⟩ ↦* ⟨t',[]⟩"
using a using eval_implies_machines_ctx by simp
```

Right-to-Left Direction

- The statement for the other direction is as follows:

```
lemma machines_implies_eval:
  assumes a: "⟨t,[]⟩ ↦* ⟨v,[]⟩"
  and     b: "val v"
  shows "t ⇓ v"
```

Right-to-Left Direction

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```
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  assumes a: "⟨t,[]⟩ ↦* ⟨v,[]⟩"
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  oops
```

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```
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  assumes a: "⟨t,[]⟩ ↦* ⟨v,[]⟩"
  and       b: "val v"
  shows "t ⇓ v"
  oops
```

- We can prove this direction by introducing a small-step reduction relation.

CBV Reduction

inductive
  cbv :: "lam⇒lam⇒bool" ("_ ⟶cbv _")
where
  cbv_1: "val v ⟹ App (Lam [x].t) v ⟶cbv t[x::=v]"
| cbv_2: "t ⟶cbv t' ⟹ App t t_2 ⟶cbv App t' t_2"
| cbv_3: "t ⟶cbv t' ⟹ App t_2 t ⟶cbv App t_2 t'"

- Later on we like to use the strong induction principle for this relation.

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where
cbv_1: "val v ⟹ App (Lam [x].t) v ⟶cbv t[x::=v]"
| cbv_2: "t ⟶cbv t' ⟹ App t t_2 ⟶cbv App t' t_2"
| cbv_3: "t ⟶cbv t' ⟹ App t_2 t ⟶cbv App t_2 t'"

- Later on we like to use the strong induction principle for this relation.

Conditions:

1. ⋀v x t. val v ⟹ x # App Lam [x].t v
2. ⋀v x t. val v ⟹ x # t[x::=v]

CBV Reduction

```
inductive
  cbv :: "lam⇒lam⇒bool" ("_ ⟶cbv _")
where
  cbv₁: "⟦val v; x#v⟧ ⟹ App (Lam [x].t) v ⟶cbv t[x::=v]"
| cbv₂[intro]: "t ⟶cbv t' ⟹ App t t₂ ⟶cbv App t' t₂"
| cbv₃[intro]: "t ⟶cbv t' ⟹ App t₂ t ⟶cbv App t₂ t'"
```

- The conditions that give us automatically the strong induction principle require us to add the assumption $x \# v$. This makes this rule less useful.

Strong Induction Principle

```
lemma  subst_eqvt[eqvt]:
  fixes π::"name prm"
  shows "π•(t₁[x::=t₂]) = (π•t₁)[(π•x)::=(π•t₂)]"
by (nominal_induct t₁ avoiding: x t₂ rule: lam.strong_induct)
   (auto simp add: perm_bij fresh_atm fresh_bij)

lemma fresh_fact:
  fixes z::"name"
  shows "⟦z#s; (z=y ∨ z#t)⟧ ⟹ z#t[y::=s]"
by (nominal_induct t avoiding: z y s rule: lam.strong_induct)
   (auto simp add: abs_fresh fresh_prod fresh_atm)

equivariance val
equivariance cbv
nominal_inductive cbv
  by (simp_all add: abs_fresh fresh_fact)
```

```
lemma subst_rename:
  assumes a: "y#t"
  shows "t[x::=s] = ([(y,x)]•t)[y::=s]"
using a
by (nominal_induct t avoiding: x y s rule: lam.strong_induct)
   (auto simp add: calc_atm fresh_atm abs_fresh)

lemma better_cbv₁[intro]:
  assumes a: "val v"
  shows "App (Lam [x].t) v ⟶cbv t[x::=v]"
proof -
  obtain y::"name" where fs: "y#(x,t,v)"
    by (rule exists_fresh) (auto simp add: fs_name1)
  have "App (Lam [x].t) v = App (Lam [y].([(y,x)]•t)) v" using fs
    by (auto simp add: lam.inject alpha' fresh_prod fresh_atm)
  also have "... ⟶cbv ([(y,x)]•t)[y::=v]" using fs a
    by (auto simp add: cbv₁ fresh_prod)
  also have "... = t[x::=v]" using fs
    by (simp add: subst_rename[symmetric] fresh_prod)
  finally show "App (Lam [x].t) v ⟶cbv t[x::=v]" by simp
qed
```

CBV Reduction*

```
inductive
  "cbvs" :: "lam ⇒ lam ⇒ bool" (" _ ⟶cbv* _")
where
  cbvs1[intro]: "e ⟶cbv* e"
| cbvs2[intro]: "⟦e1 ⟶cbv e2; e2 ⟶cbv* e3⟧ ⟹ e1 ⟶cbv* e3"

lemma cbvs3[intro]:
  assumes a: "e1 ⟶cbv* e2" "e2 ⟶cbv* e3"
  shows "e1 ⟶cbv* e3"
using a by (induct) (auto)
```

CBV Reduction*

```
inductive
  "cbvs" :: "lam ⇒ lam ⇒ bool" (" _ ⟶cbv* _")
where
  cbvs₁[intro]: "e ⟶cbv* e"
| cbvs₂[intro]: "⟦e₁⟶cbv e₂; e₂ ⟶cbv* e₃⟧ ⟹ e₁ ⟶cbv* e₃"

lemma cbvs₃[intro]:
  assumes a: "e₁ ⟶cbv* e₂" "e₂ ⟶cbv* e₃"
  shows "e₁ ⟶cbv* e₃"
using a by (induct) (auto)

lemma cbv_in_ctx:
  assumes a: "t ⟶cbv t'"
  shows "E⟦t⟧ ⟶cbv E⟦t'⟧"
using a by (induct E) (auto)
```

CK Machine Implies CBV*

```
lemma machines_implies_cbvs:
  assumes a: "⟨e,[]⟩ ↦* ⟨e',[]⟩"
  shows "e ⟶cbv* e'"
using a by (auto dest: machines_implies_cbvs_ctx)
```

CK Machine Implies CBV*

```
lemma machine_implies_cbvs_ctx:
  assumes a: "⟨e,Es⟩ ↦ ⟨e',Es'⟩"
  shows "(Es↓)⟦e⟧ ⟶cbv* (Es'↓)⟦e'⟧"
using a by (induct) (auto simp add: ctx_compose intro: cbv_in_ctx)
```

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  assumes a: "⟨e,[]⟩ ↦* ⟨e',[]⟩"
  shows "e ⟶cbv* e'"
using a by (auto dest: machines_implies_cbvs_ctx)
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```

If we had not derived the better cbv-rule, then we would have to do an explicit renaming here.

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```

CBV* Implies Evaluation

- We need the following scaffolding lemmas in order to show that cbv-reduction implies evaluation.

```
lemma eval_val:
  assumes a: "val t"
  shows "t ⇓ t"
using a by (induct) (auto)

lemma e_App_elim:
  assumes a: "App t1 t2 ⇓ v"
  shows "∃ x t v'. t1 ⇓ Lam [x].t ∧ t2 ⇓ v' ∧ t[x::=v'] ⇓ v"
using a by (cases) (auto simp add: lam.inject)
```

```
lemma cbv_eval:
  assumes a: "t₁ ⟶cbv t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a
by (induct arbitrary: t₃)
   (auto intro: eval_val dest!: e_App_elim)

lemma cbvs_eval:
  assumes a: "t₁ ⟶cbv* t₂" "t₂ ⇓ t₃"
  shows "t₁ ⇓ t₃"
using a by (induct) (auto simp add: cbv_eval)

lemma cbvs_implies_eval:
  assumes a: "t ⟶cbv* v" "val v"
  shows "t ⇓ v"
using a
by (induct)
   (auto simp add: eval_val cbvs_eval dest: cbvs₂)
```

Right-to-Left Direction

- Via the the cbv-reduction relation we can finally show that the CK machine implies the evaluation relation.

```
theorem machines_implies_eval:
  assumes a: "⟨t₁,[]⟩ ↦* ⟨t₂,[]⟩"
  and      b: "val t₂"
  shows "t₁ ⇓ t₂"
proof -
  from a have "t₁ ⟶cbv* t₂" by (simp add: machines_implies_cbvs)
  then show "t₁ ⇓ t₂" using b by (simp add: cbvs_implies_eval)
qed
```

Preservation and Progress

- Next we like to prove a **type preservation** and an **progress lemma** for the cbv-reduction relation.

```
theorem cbv_type_preservation:
  assumes a: "t ⟶cbv t'"
  and       b: "Γ ⊢ t : T"
  shows "Γ ⊢ t' : T"
```

```
theorem progress:
  assumes a: "[] ⊢ t : T"
  shows "(∃ t'. t ⟶cbv t') ∨ (val t)"
```

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```
theorem progress:
  assumes a: "[] ⊢ t : T"
  shows "(∃ t'. t ⟶cbv t') ∨ (val t)"
```

- We need the property of type-substitutivity.

Some Side-Lemmas

```
lemma valid_elim:
  assumes a: "valid ((x,T)#Γ)"
  shows "x#Γ ∧ valid Γ"
using a by (cases) (auto)

lemma valid_insert:
  assumes a: "valid (Δ@[(x,T)]@Γ)"
  shows "valid (Δ@Γ)"
using a
by (induct Δ)
   (auto simp add: fresh_list_append fresh_list_cons dest!: valid_elim)

lemma fresh_list:
  shows "y#xs = (∀ x ∈ set xs. y#x)"
by (induct xs) (simp_all add: fresh_list_nil fresh_list_cons)

lemma context_unique:
  assumes a1: "valid Γ"
    and   a2: "(x,T) ∈ set Γ"
    and   a3: "(x,U) ∈ set Γ"
  shows "T = U"
using a1 a2 a3
by (induct) (auto simp add: fresh_list fresh_prod fresh_atm)
```

```
lemma type_substitution_aux:
  assum
  and
  shows
using a
proof (r
                                    avoiding: x e' Δ rule: typing.strong_induct)
```

```
corollary type_substitution:
  assumes a: "(x,T')#Γ ⊢ e : T"
  and     b: "Γ ⊢ e' : T'"
  shows "Γ ⊢ e[x::=e'] : T"
```

```
  case (t_Var Γ' y T x e' Δ)
  then have a1: "valid (Δ@[(x,T')]@Γ)"
       and  a2: "(y,T) ∈ set (Δ@[(x,T')]@Γ)"
       and  a3: "Γ ⊢ e' : T'" by simp_all
  from a1 have a4: "valid (Δ@Γ)" by (rule valid_insert)
  { assume eq: "x=y"
    from a1 a2 have "T=T'" using eq by (auto intro: context_unique)
    with a3 have "Δ@Γ ⊢ Var y[x::=e'] : T" using eq a4 by (auto intro: weakening) }
  moreover
  { assume ineq: "x≠y"
    from a2 have "(y,T) ∈ set (Δ@Γ)" using ineq by simp
    then have "Δ@Γ ⊢ Var y[x::=e'] : T" using ineq a4 by auto }
  ultimately show "Δ@Γ ⊢ Var y[x::=e'] : T" by blast
qed (force simp add: fresh_list_append fresh_list_cons)+
```

```
lemma type_substitution_aux:
  assumes a: "Δ@[(x,T')]@Γ ⊢ e : T"
  and     b: "Γ ⊢ e' : T'"
  shows "Δ@Γ ⊢ e[x::=e'] : T"
using a b
proof (nominal_induct Γ'≡"Δ@[(x,T')]@Γ" e T
                                        avoiding: x e' Δ rule: typing.strong_induct)
  case (t_Var Γ' y T x e' Δ)
  then have a1: "valid (Δ@[(x,T')]@Γ)"
       and  a2: "(y,T) ∈ set (Δ@[(x,T')]@Γ)"
       and  a3: "Γ ⊢ e' : T'" by simp_all
  from a1 have a4: "valid (Δ@Γ)" by (rule valid_insert)
  { assume eq: "x=y"
    from a1 a2 have "T=T'" using eq by (auto intro: context_unique)
    with a3 have "Δ@Γ ⊢ Var y[x::=e'] : T" using eq a4 by (auto intro: weakening) }
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qed (force simp add: fresh_list_append fresh_list_cons)+
```

$$\frac{\text{valid } \Gamma \quad (x, T) \in \text{set } \Gamma}{\Gamma \vdash \text{Var } x : T}$$

```
lemma type_substitution_aux:
  assumes a: "Δ@[(x,T')]@Γ ⊢ e : T"
  and     b: "Γ ⊢ e' : T'"
  shows "Δ@Γ ⊢ e[x::=e'] : T"
using a b
proof (nominal_induct Γ'≡"Δ@[(x,T')]@Γ" e T
                                  avoiding: x e' Δ rule: typing.strong_induct)
  case (t_Var Γ' y T x e' Δ)
  then have a1: "valid (Δ@[(x,T')]@Γ)"
        and  a2: "(y,T) ∈ set (Δ@[(x,T')]@Γ)"
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    with a3 have "Δ@Γ ⊢ Var y[x::=e'] : T" using eq a4 by (auto intro: weakening) }
  moreover
  { assume ineq: "x≠y"
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    then have "Δ@Γ ⊢ Var y[x::=e'] : T" using ineq a4 by auto }
  ultimately show "Δ@Γ ⊢ Var y[x::=e'] : T" by blast
qed (force simp add: fresh_list_append fresh_list_cons)+
```

Type Substitutivity

```
lemma type_substitution_aux:
  assumes a: "Δ@[(x,T')]@Γ ⊢ e : T"
  and     b: "Γ ⊢ e' : T'"
  shows "Δ@Γ ⊢ e[x::=e'] : T"

corollary type_substitution:
  assumes a: "(x,T')#Γ ⊢ e : T"
  and       b: "Γ ⊢ e' : T'"
  shows "Γ ⊢ e[x::=e'] : T"
using a b type_substitution_aux[where Δ="[]"]
  by (auto)
```

Inversion Lemmas

```
lemma t_App_elim:
  assumes a: "Γ ⊢ App t1 t2 : T"
  shows "∃ T'. Γ ⊢ t1 : T' → T ∧ Γ ⊢ t2 : T'"
using a by (cases) (auto simp add: lam.inject)

lemma t_Lam_elim:
  assumes ty: "Γ ⊢ Lam [x].t : T"
  and     fc: "x#Γ"
  shows "∃ T₁ T₂. T = T₁ → T₂ ∧ (x,T₁)#Γ ⊢ t : T₂"
using ty fc
by (cases rule: typing.strong_cases)
    (auto simp add: alpha lam.inject abs_fresh ty_fresh)
```

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash \mathsf{App}\ t_1\ t_2 : T_2} \qquad \frac{x \# \Gamma \quad (x, T_1)::\Gamma \vdash t : T_2}{\Gamma \vdash \mathsf{Lam}\ [x].t : T_1 \to T_2}$$

Type Preservation

```
theorem cbv_type_preservation:
  assumes a: "t ⟶cbv t'"
  and       b: "Γ ⊢ t : T"
  shows "Γ ⊢ t' : T"
using a b
by (nominal_induct avoiding: Γ T rule: cbv.strong_induct)
    (auto dest!: t_Lam_elim t_App_elim
       simp add: type_substitution ty.inject)

corollary cbvs_type_preservation:
  assumes a: "t ⟶cbv* t'"
  and     b: "Γ ⊢ t : T"
  shows "Γ ⊢ t' : T"
using a b
by (induct) (auto intro: cbv_type_preservation)
```

Progress Lemma

- Finally we can establish the progress lemma:

```
lemma canonical_tArr:
  assumes a: "[] ⊢ t : T1 → T2"
  and       b: "val t"
  shows "∃ x t'. t = Lam [x].t'"
using b a by (induct) (auto)



theorem progress:
  assumes a: "[] ⊢ t : T"
  shows "(∃ t'. t ⟶cbv t') ∨ (val t)"
using a
by (induct Γ≡"[]::ty_ctx" t T)
    (auto intro!: cbv.intros dest: canonical_tArr)
```

Progress Lemma

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```
lemma canonical_tArr:
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  and      b: "val t"
  shows "∃ x t'. t = Lam [x].t'"
using b a by (induct) (auto)
```

- This lemma is stated with extensions in mind.

```
theorem progress:
  assumes a: "[] ⊢ t : T"
  shows "(∃ t'. t ⟶cbv t') ∨ (val t)"
using a
by (induct Γ≡"[]::ty_ctx" t T)
   (auto intro!: cbv.intros dest: canonical_tArr)
```

Extensions

- With only minimal modifications the proofs can be extended to the language given by:

```
nominal_datatype lam =
  Var "name"
| App "lam" "lam"
| Lam "«name»lam" ("Lam [_]._")
| Num "nat"
| Minus "lam" "lam" ("_ -- _")
| Plus "lam" "lam" ("_ ++ _")
| TRUE
| FALSE
| IF "lam" "lam" "lam"
| Fix "«name»lam" ("Fix [_]._")
| Zet "lam"
| Eqi "lam" "lam"
```

Formalisation of LF

(joint work with Cheney and Berghofer)

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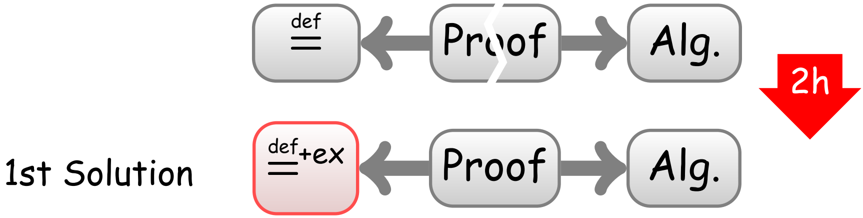

(each time one needs to check ~31pp of informal paper proofs)

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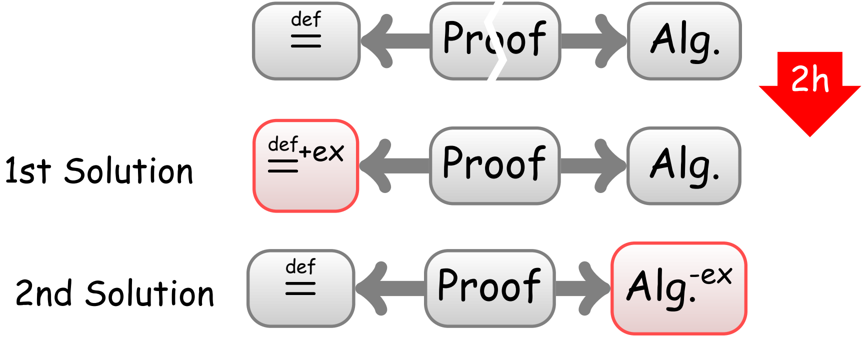

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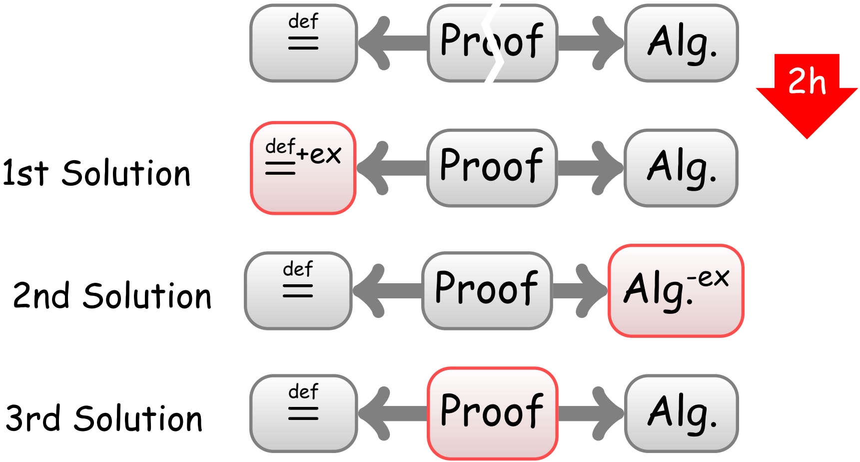

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Two Health Warnings ;o)

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Theorem provers should come with two health warnings:

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- Theorem provers cause you to lose faith in your proofs done by hand!

  (Michael Norrish, Mike Gordon, me, very possibly others)

Answers to Exercises

- Given a finite set of atoms. What is the support of this set?

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- What is the support of the set of all atoms?

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- What is the support of the set of all atoms? Let $A = \{a_0, a_1 \ldots\}$, then $\mathsf{supp}(A) = \varnothing$.

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- From the set of all atoms take one atom out. What is the support of the resulting set? $\text{supp}(A - \{a\}) = \{a\}$.

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- Are there any sets of atoms that have infinite support?

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- From the set of all atoms take one atom out. What is the support of the resulting set? $\mathrm{supp}(A - \{a\}) = \{a\}$.
- Are there any sets of atoms that have infinite support? If both S and $A - S$ are infinite then $\mathrm{supp}(S) = A$.

Thank you very much!