Recap from Last Time

- For "free" came the following structural induction principle:

$$\frac{\begin{array}{l}(\forall a)\ P\,(\mathsf{Var}(a)) \\ (\forall t_1, t_2)\ P\,t_1 \wedge P\,t_2 \Rightarrow P\,(\mathsf{App}(t_1, t_2)) \\ (\forall a, t)\ P\,t \Rightarrow P\,([a].t)\end{array}}{(\forall t)\ P\,t}$$

Recap from Last Time

- We then derived the following structural induction principle:

$$\frac{\begin{array}{l}(\forall a)\; P\,(\mathsf{Var}(a))\,x \\ (\forall t_1, t_2)\; P\,t_1\,x \wedge P\,t_2\,x \Rightarrow P\,(\mathsf{App}(t_1, t_2))\,x \\ (\forall a, t)\; a \mathrel{\#} x \;\wedge\; P\,t\,x \Rightarrow P\,([a].t)\,x\end{array}}{(\forall t)\; P\,t\,x}$$

Recap from Last Time

- I think it should be strengthened as follows:

$$\frac{\begin{array}{l}(\forall x\, a)\ P\,(\mathsf{Var}(a))\,x \\ (\forall x\, t_1, t_2)\ P\,t_1\,x \wedge P\,t_2\,x \Rightarrow P\,(\mathsf{App}(t_1, t_2))\,x \\ (\forall x\, a\, t)\ a \# x \wedge\ ((\forall z)\ P\,t\,z) \Rightarrow P\,([a].t)\,x\end{array}}{(\forall x\, t)\ P\,t\,x}$$

The proof of this induction principle stays the same.
The strengthening is needed in some cases.

Weakening Property

$$\frac{\mathsf{valid}(\Gamma) \quad a : \tau \in \Gamma}{\Gamma \vdash a : \tau}$$

$$\frac{\Gamma \vdash t_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\, t_2 : \tau_2}$$

$$\frac{a : \tau_1, \Gamma \vdash t : \tau_2 \quad a \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash [a].t : \tau_1 \rightarrow \tau_2}$$

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If $\Gamma \vdash t : \tau$, $\Gamma \subseteq \Gamma'$ and $\mathsf{valid}(\Gamma')$, then also $\Gamma' \vdash t : \tau$.

Attempting the Proof

We proceed by rule induction and try to show that the predicate $\varphi(\Gamma; [t]_\alpha; \tau)$ given by

$$\Gamma \subseteq \Gamma' \Rightarrow \mathsf{valid}(\Gamma') \Rightarrow \Gamma' \vdash M : \tau$$

is closed under the axiom and the two inference rules. Interesting case:

$$\frac{a : \tau_1, \Gamma \vdash t : \tau_2 \quad a \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash [a].t : \tau_1 \rightarrow \tau_2}$$

Attempti

We proceed by rule
that the predicate φ

$$\Gamma \subseteq \Gamma' \Rightarrow \mathsf{val}$$

is closed under the c
rules. Interesting ca

We know (for the premise):

1. $\varphi(\Gamma'; M; \tau_2)$
2. $a \notin \mathsf{dom}(\Gamma)$

We have to prove:

$\varphi(\Gamma'; [a].M; \tau_2)$

for **all** τ' and a.

$$\frac{a : \tau_1, \Gamma \vdash t : \tau_2 \quad a \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash [a].t : \tau_1 \rightarrow \tau_2}$$

Attempti...

We proceed by rule ...
that the predicate φ ...

$$\Gamma \subseteq \Gamma' \Rightarrow \mathsf{val}...$$

is closed under the ...
rules. Interesting ca...

$$\frac{a : \tau_1, \Gamma \vdash t : \tau_2 \quad a \not\in \mathsf{dom}(\Gamma)}{\Gamma \vdash [a].t : \tau_1 \longrightarrow \tau_2}$$

We know (for the premise):

1. $\varphi(\Gamma'; M; \tau_2)$
2. $a \not\in \mathsf{dom}(\Gamma)$

We have to prove:

$\varphi(\Gamma'; [a].M; \tau_2)$

for **all** τ' and a.

But this fails for all $\mathsf{valid}(\Gamma')$ with $a \in \mathsf{dom}(\Gamma')$!

Moral of this Example

- Does this mean the weakening property does **not** hold for the simply-typed lambda-calculus?

  Clearly, **NO**!

  Just our simple-minded reasoning did not work. We have to take into account some facts about α-equivalent classes and do renamings.

Induction for Free

This is the induction principle that comes with the definition of typing:

$$(\forall \Gamma\, a\, \tau)\ \mathsf{valid}(\Gamma) \wedge a : \tau \in \Gamma \Rightarrow P\ \Gamma\ \mathsf{Var}(a)\ \tau$$

$$\begin{array}{ll} (\forall \Gamma\, M\, \tau\, \sigma) & \Gamma \vdash M : \tau \to \sigma \wedge P\ \Gamma\ M\ (\tau \to \sigma) \wedge \\ & \Gamma \vdash N : \tau \wedge P\ \Gamma\ N\ \tau \Rightarrow \\ & P\ \Gamma\ \mathsf{App}(M, N)\ \sigma \end{array}$$

$$\begin{array}{ll} (\forall \Gamma\, a\, M\, \tau\, \sigma) & a \# \Gamma \wedge (a : \tau, \Gamma) \vdash M : \sigma \wedge \\ & P\ (a : \tau, \Gamma)\ M\ \sigma \Rightarrow \\ & P\ \Gamma\ [a].M\ (\tau \to \sigma) \end{array}$$

$$(\forall\, \Gamma\, M\, \tau)\ \Gamma \vdash M : \tau \Rightarrow P\ \Gamma\ M\ \tau$$

Is it useful for something?

Yes, we can prove:

$$(\forall\pi)\ \Gamma \vdash M : \tau \Rightarrow \pi\bullet\Gamma \vdash \pi\bullet M : \tau$$

using valid$(\pi\bullet\Gamma)$ provided valid(Γ). With this we can make the simple proof to work, but we have to do renamings by hand. BUT we want to get the simple proof to work.
We can use the induction principle for “free”to derive a new one, namely...

New Induction Principle

$$(\forall x\,\Gamma\,a\,\tau)\ \mathsf{valid}(\Gamma) \wedge a : \tau \in \Gamma \Rightarrow P\ \Gamma\ \mathsf{Var}(a)\ \tau\ x$$

$$\begin{array}{l}(\forall x\,\Gamma\,M\,\tau\,\sigma)\\ \qquad\qquad \Gamma \vdash M : \tau \rightarrow \sigma \wedge P\ \Gamma\ M\ (\tau \rightarrow \sigma)\ x\ \wedge\\ \qquad\qquad \Gamma \vdash N : \tau \wedge P\ \Gamma\ N\ \tau\ x \Rightarrow\\ \qquad\qquad P\ \Gamma\ \mathsf{App}(M, N)\ \sigma\ x\end{array}$$

$$\begin{array}{l}(\forall x\,\Gamma\,a\,M\,\tau\,\sigma)\\ \qquad\qquad a \# x \wedge a \# \Gamma \wedge (a : \tau, \Gamma) \vdash M : \sigma\ \wedge\\ \qquad\qquad ((\forall z)\ P\ (a : \tau, \Gamma)\ M\ \sigma\ z) \Rightarrow\\ \qquad\qquad P\ \Gamma\ [a].M\ (\tau \rightarrow \sigma)\ x\end{array}$$

$$(\forall x\,\Gamma\,M\,\tau)\ \Gamma \vdash M : \tau \Rightarrow P\ \Gamma\ M\ \tau\ x$$

Proof is a bit Hairy

- We strengthen the induction hypotheisis to

$$(\forall \pi\, x\, \Gamma\, M\, \tau)\ \Gamma \vdash M : \tau \Rightarrow P\ \pi \bullet \Gamma\ \pi \bullet M\ \tau\ x$$

- in the lambda case we have to prove:

$$P\ (\pi \bullet \Gamma)\ [\pi \bullet a].(\pi \bullet M)\ \tau\ x$$

- we know

$$a \# \Gamma, \quad a : \tau, \Gamma \vdash M : \sigma$$
$$\forall \pi\ \Gamma\ M\ \tau\ x.\ P\ (\pi \bullet (a : \tau, \Gamma))\ (\pi \bullet M)\ \sigma\ x$$
$$\begin{aligned}&(\forall x\, \Gamma\, a\, M\, \tau\, \sigma)\\ &\qquad a \# x \wedge a \# \Gamma \wedge (a : \tau, \Gamma) \vdash M : \sigma \wedge\\ &\qquad ((\forall z)\ P\ (a : \tau, \Gamma)\ M\ \sigma\ z) \Rightarrow\\ &\qquad P\ \Gamma\ [a].M\ (\tau \rightarrow \sigma)\ x\end{aligned}$$

Weakening Lemma

Subgoals to be proved:

$$\mathsf{valid}(\Gamma_1') \wedge (a:\tau) \in \Gamma_1' \wedge \Gamma_1' \subseteq \Gamma_2' \wedge \mathsf{valid}(\Gamma_2')$$
$$\Rightarrow \Gamma_2' \vdash \mathsf{Var}(a) : \tau$$

$$\Gamma_1' \vdash M_1 : \tau \rightarrow \sigma \wedge \Gamma_1' \vdash M_2 : \tau \wedge$$
$$(\Gamma_1' \subseteq \Gamma_2' \Rightarrow \mathsf{valid}(\Gamma_2') \Rightarrow \Gamma_2' \vdash M_1 : \tau \rightarrow \sigma) \wedge$$
$$(\Gamma_1' \subseteq \Gamma_2' \Rightarrow \mathsf{valid}(\Gamma_2') \Rightarrow \Gamma_2' \vdash M_2 : \tau) \wedge$$
$$\Gamma_1' \subseteq \Gamma_2' \wedge \mathsf{valid}(\Gamma_2')$$
$$\Rightarrow \Gamma_2' \vdash \mathsf{App}(M_1, M_2) : \sigma$$

$$a \# \Gamma_2' \wedge a \# \Gamma_1' \wedge (a:\tau, \Gamma_1') \vdash M : \sigma \wedge$$
$$(\forall \Gamma_3'.\ (a:\tau, \Gamma_1') \subseteq \Gamma_3' \Rightarrow \mathsf{valid}(\Gamma_3') \Rightarrow \Gamma_3' \vdash M : \sigma) \wedge$$
$$\Gamma_1' \subseteq \Gamma_2' \wedge \mathsf{valid}(\Gamma_2')$$
$$\Rightarrow \Gamma_2' \vdash [a].M : \tau \rightarrow \sigma$$

So What?

So we have done this for one example. Is there a general principle behind. The only property we needed in the induction proof was

$$(\forall \pi)\ \Gamma \vdash M : \tau \Rightarrow \pi \bullet \Gamma \vdash \pi \bullet M : \tau$$

We could define $\pi \bullet \tau = \tau$ for all types (they do not contain any atoms), then

$$(\forall \pi)\ \Gamma \vdash M : \tau \Rightarrow \pi \bullet \Gamma \vdash \pi \bullet M : \pi \bullet \tau$$

This is equivariance—relation is preserved under permutations.

$$R(M_1, \ldots, M_n) \Rightarrow P(M_1, \ldots, M_n)$$

When is a Relation Eqvt?

$$\frac{\mathsf{valid}(\Gamma) \quad a : \tau \in \Gamma}{\Gamma \vdash a : \tau}$$

$$\frac{\Gamma \vdash t_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\, t_2 : \tau_2}$$

$$\frac{a : \tau_1, \Gamma \vdash t : \tau_2 \quad a \not\in \mathsf{dom}(\Gamma)}{\Gamma \vdash [a].t : \tau_1 \rightarrow \tau_2}$$

Can we abstract a bit from the permutations on arguments?

$$(\forall \pi)\ \Gamma \vdash M : \tau \Rightarrow \pi \bullet \Gamma \vdash \pi \bullet M : \pi \bullet \tau$$

Empty Support

Remember that

if $a \# x$ and $b \# x$ implies $(a\,b)\bullet x = x$

and

$$\pi\bullet(f\ x) = (\pi\bullet f)(\pi\bullet x)$$

We have

$$((\forall a\,b)\ (a\,b)\bullet x = x) \Leftrightarrow ((\forall \pi)\ \pi\bullet x = x$$

Therefore for f having empty support:

$$(\forall \pi)\ \pi\bullet(f\ x) = f(\pi\bullet x)$$

Actually we have that with “if-and-only-if”.

When is a Relation Eqvt?

$$\frac{\mathsf{valid}(\Gamma) \quad a : \tau \in \Gamma}{\Gamma \vdash a : \tau}$$

$$\frac{\Gamma \vdash t_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\, t_2 : \tau_2}$$

$$\frac{a : \tau_1, \Gamma \vdash t : \tau_2 \quad a \notin \mathsf{dom}(\Gamma)}{\Gamma \vdash [a].t : \tau_1 \rightarrow \tau_2}$$

Answer: When $\vdash$ has empty support.

Question: When does a definition have empty support?

Permutation on Bools

$$\pi \bullet True \stackrel{\text{def}}{=} True$$
$$\pi \bullet False \stackrel{\text{def}}{=} False$$

Support of the truth values is now given! How about the logic connectives?

- $\pi \bullet (Q = P) \Leftrightarrow (\pi \bullet Q) = (\pi \bullet P)$
- $\pi \bullet (x \in X) \Leftrightarrow (\pi \bullet x) = (\pi \bullet X)$
- $\pi \bullet (P \wedge Q) \Leftrightarrow (\pi \bullet P) \wedge (\pi \bullet Q)$
- $\pi \bullet (\neg Q) \Leftrightarrow \neg(\pi \bullet Q)$
- $\pi \bullet (\forall x.\ Q\ x) \Leftrightarrow \forall x.\ (\pi \bullet Q)\ x$

Support Abstractly

Assuming f is equivariant/has empty support.
When has $f\ x$ empty support? Well, if f has empty support then

$$supp(f\ x) \subseteq (supp\ x)$$

In this proof we need

$$\{b.\ f\ ((b\,a)\bullet x) \neq f\ x\} \subseteq \{b.(b\,a)\bullet x \neq x\}$$

Fazit

If you want to check whether a definition is equivariant, then one needs to check whether all its primitive relations are equivariant.