Nominal Datatypes in Isabelle/HOL

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joint work with Stefan Berghofer and the Isabelle-team in Munich

The Nominal Logic Work

- Implemented a version of the nominal logic work as a definitional extension of Isabelle/HOL—that is as "syntactic sugar" (early nominal logic work had problems with the axiom of choice).
- Added an infrastructure for defining nominal datatypes—some of the more advanced reasoning infrastructure is however not yet in place. :o(
- My goal with this work is to make formalisations of informal "pencil-and-paper" proofs easy.

α-Equivalence Classes

Implemented Barendregt's:

Convention: Terms that are α-congruent are identified. So now we write $\lambda x.x \equiv \lambda y.y$ etcetera.

- You could define the type of lists containing only 1s, 2s and 3s from the existing type of lists.
- The polymorphic type $\alpha \times \beta$ is defined as a subset of the type $\alpha \Rightarrow \beta \Rightarrow$ bool. The pair (a, b) is represented as the function:

$$\lambda x\, y.\ (x = a) \wedge (y = b)$$

α-Equivalence Classes

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$=$ is equality in HOL

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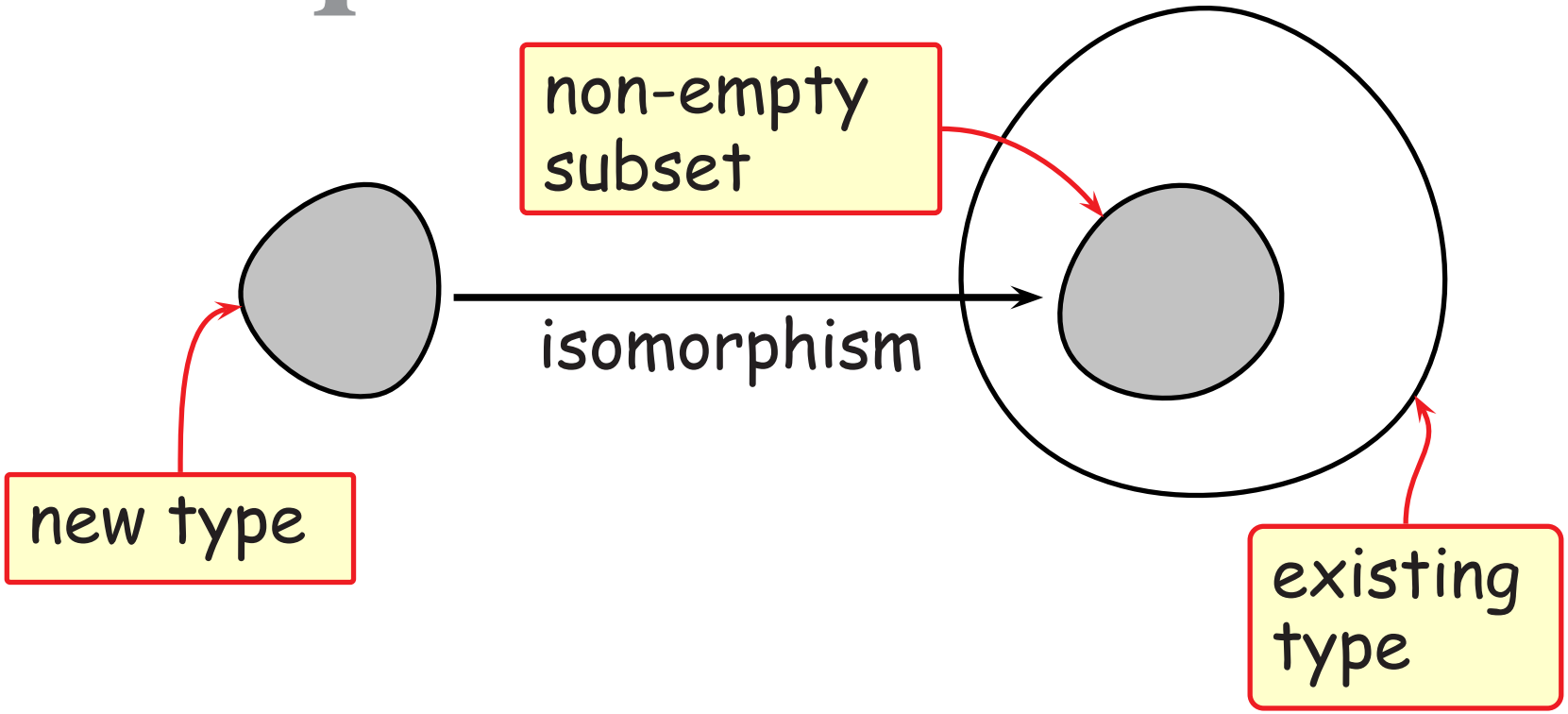
α-Equivalence Classes



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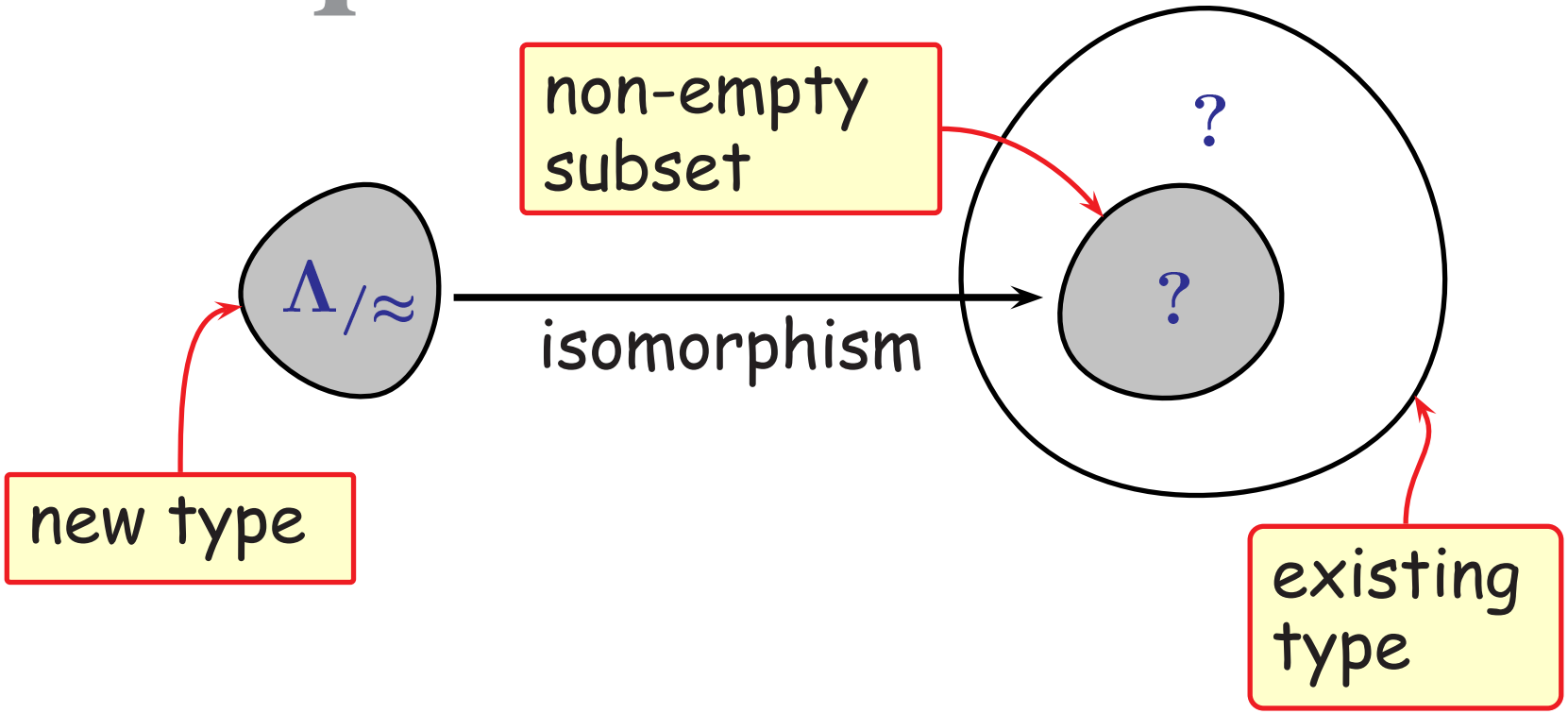
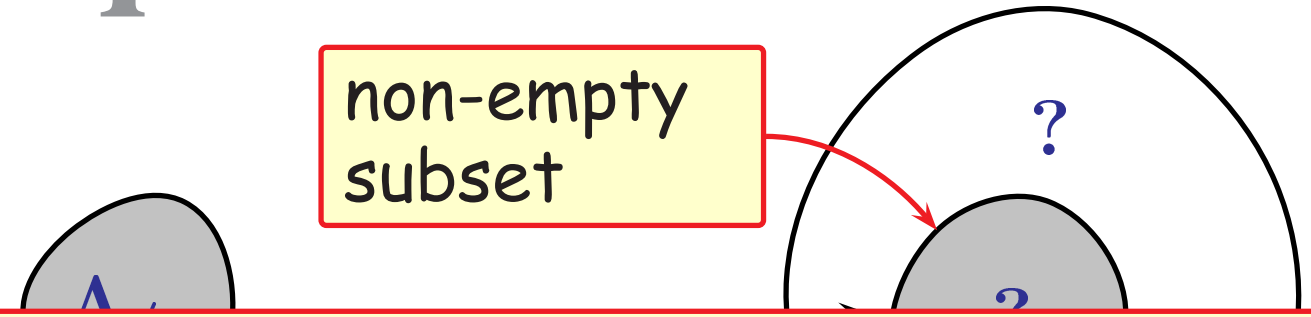
α-Equivalence Classes



Question:

- What is a suitable existing type and a suitable subset to represent α-equivalent lambda-terms? (And α-equivalence classes in general?)
- Answer: this talk.

α-Equivalence Classes



There are many ways how to obtain α-equivalence classes. I will present my way (in HOL4 the answer would be different: Norrish uses a package written by Homeier).

In some sense it is a completely uninteresting question, because the user is only interested in an "interface",that is they are α-equivalence classes, but I hope you are interested in what is going on behind the scenes.

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- Answer: this talk.

α-Equivalence

Take one step back and consider "raw" lambda-terms. The following four rules define α-equivalence:

$$\frac{}{a \approx a}\ \approx\text{-var}$$

$$\frac{t_1 \approx s_1 \quad t_2 \approx s_2}{t_1\, t_2 \approx s_1\, s_2}\ \approx\text{-app}$$

$$\frac{t \approx s}{\lambda a.t \approx \lambda a.s}\ \approx\text{-lam}_1$$

$$\frac{t \approx (a\,b) \bullet s \quad a \# s}{\lambda a.t \approx \lambda b.s}\ \approx\text{-lam}_2$$

assuming $a \neq b$

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$\lambda a.t \approx \lambda b.s$ iff t is α-equivalent with s in which all occurrences of b have been renamed to a...oops **permuted** to a.

α-Equivalence

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α-

$\overline{\lambda a}$

λa

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to a. . . oops permuted to a.

Permutations acting on lambda-terms (permutations are lists of swappings):

$$[] \bullet a \stackrel{\text{def}}{=} a$$

$$((a_1\, a_2) :: \pi) \bullet a \stackrel{\text{def}}{=} \begin{cases} a_1 & \text{if } \pi \bullet a = a_2 \\ a_2 & \text{if } \pi \bullet a = a_1 \\ \pi \bullet a & \text{otherwise} \end{cases}$$

$$\pi \bullet (t\, s) \stackrel{\text{def}}{=} (\pi \bullet t)(\pi \bullet s)$$

$$\pi \bullet (\lambda a.t) \stackrel{\text{def}}{=} \lambda(\pi \bullet a)(\pi \bullet t)$$

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$$\lambda a$$

wh

to a... oops **permuted** to a.

But this alone leads to an 'unsound' rule! Consider

$$\lambda a.b \quad \text{and} \quad \lambda b.a$$

which are **not** α-equivalent. However, if we apply the permutation $(a\,b)$ to a we get

$$b \approx b$$

which leads to non-sense.

We need to ensure that there are **no** 'free' occurrences of a in s, i.e. $a \# s$.

b

α-Equivalence

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assuming $a \neq b$

$\lambda a.t \approx \lambda b.s$ iff t is α-equivalent with s in which all occurrences of b have been renamed to a...oops **permuted** to a.

Freshness

$$\frac{}{a \# b}\ \#\text{-var} \qquad \frac{a \# t \quad a \# s}{a \# t\, s}\ \#\text{-app}$$

$$\frac{}{a \# \lambda a.t}\ \#\text{-lam}_1 \qquad \frac{a \# t}{a \# \lambda b.t}\ \#\text{-lam}_2$$

assuming $a \neq b$

Be careful, we have defined two relations over lambda-terms/syntax-trees. We have **not** defined what 'bound' or 'free' means. That is a feature, not a bug.™

$\approx$ is an Equivalence

You might be an agnostic and notice that

$$\frac{a \neq b \quad t \approx (a\,b) \bullet s \quad a \# s}{\lambda a.t \approx \lambda b.s}\ \approx\text{-lam}_2$$

is defined rather asymmetrically. Still we have:

Theorem: $\approx$ is an equivalence relation.

(Reflexivity) $t \approx t$

(Symmetry) if $t_1 \approx t_2$ then $t_2 \approx t_1$

(Transitivity) if $t_1 \approx t_2$ and $t_2 \approx t_3$ then $t_1 \approx t_3$

$\Rightarrow$ is rather tricky to prove

$\approx$ is an Equivalence

You might be an agnostic and notice that

$$\frac{a \neq b \quad t \approx (a\,b) \bullet s \quad a \# s}{\lambda a.\, t \approx \lambda b.\, s}\ \approx\text{-lam}_2$$

And coincides with the usual definition:

least congruence which identifies $\lambda a.t$ with $\lambda b.[a := b]t$ provided b is not free in t

Proof left as an exercise.

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(Transitivity) if $t_1 \approx t_2$ and $t_2 \approx t_3$ then $t_1 \approx t_3$

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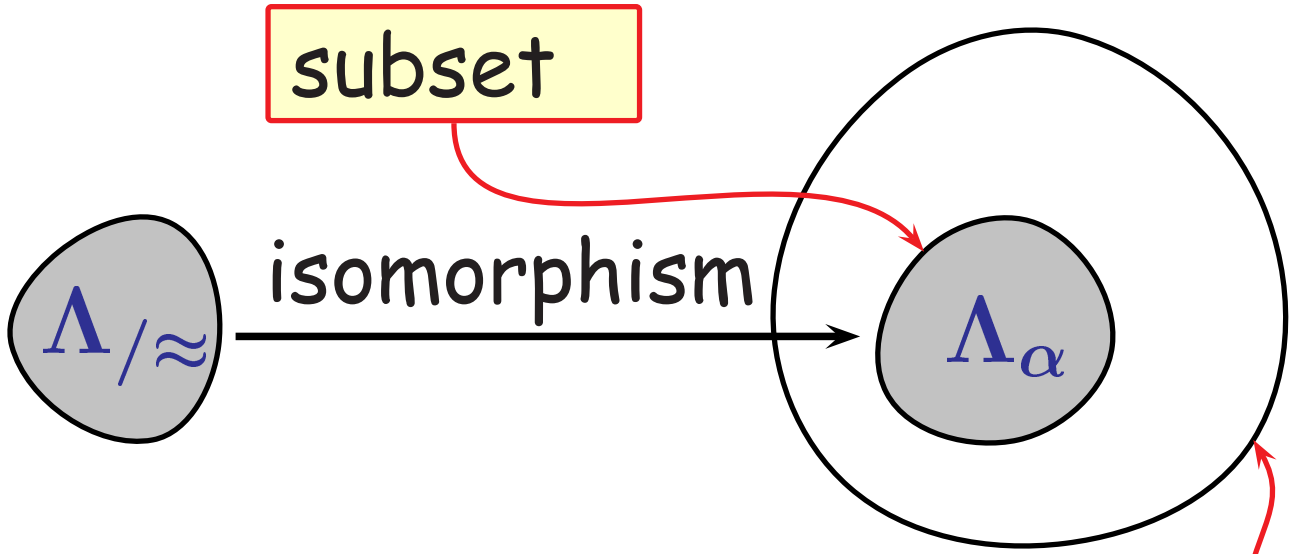
Weak HOAS Encoding

I defined **inductively** the α-equivalence classes of lambda-terms using some carefully chosen "partial" functions.



$$
\begin{array}{rl}
trm := & \mathsf{Var} : \alpha \\
\mid & \mathsf{App} : trm \times trm \\
\mid & \mathsf{Lam} : \alpha \to trm\ \mathsf{option}
\end{array}
$$

Nominal Abstractions

The functions (called nominal abstractions and written $[a].t$) satisfy:

- $\pi \bullet ([a].t) = [\pi \bullet a].(\pi \bullet t)$
- $[a].t \;=\; [b].s$ iff

  $(a = b \wedge t = s) \vee$
  $(a \neq b \wedge t = (a\,b) \bullet s \wedge a \# s)$

- btw., these two properties imply for finitely supported t

  $$\mathrm{supp}([a].t) = \mathrm{supp}(t) - \{a\}$$

Nominal Abstractions

Remember the definition of α-equivalence from the beginning:

$$\frac{t \approx s}{\lambda a.t \approx \lambda a.s} \qquad \frac{a \neq b \;\; t \approx (a\,b)\bullet s \quad a \# s}{\lambda a.t \approx \lambda b.s}$$

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Nominal Abstractions

Definition of support and freshness:

$$\text{supp}\ x \stackrel{\text{def}}{=} \{a \mid \text{infinite}\{b \mid (a\ b) \bullet x \neq x\}\}$$

$$a \# x \stackrel{\text{def}}{=} a \notin \text{supp}(x)$$

$$\text{supp}\ a = \{a\}$$

$$\text{supp}\ (t, s) = \text{supp}(t) \cup \text{supp}(s)$$

finitely supported t

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Function $[a].t$ '=' $[\lambda a.t]_\alpha$

$$[a].t \stackrel{\text{def}}{=} (\lambda b.\ \text{if } a = b \text{ then Some}(t) \text{ else if } b \# t \text{ then Some}((b\,a) \bullet t) \text{ else None})$$

type: $\alpha \to trm$ option

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This is supposed to stand for the α-equivalence class of $\lambda a.t$.

Function $[a].t$ '=' $[\lambda a.t]_\alpha$

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Let's check this for $[a].(a, c)$:

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- d 'applied to' $[a].(a, c)$ 'gives' Some(d, c)

$\vdots$

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$[\lambda a.(a\,c)]_\alpha$:

'$\lambda a.(a\,c)$'

'$\lambda b.(b\,c)$'

'$\lambda d.(d\,c)$'

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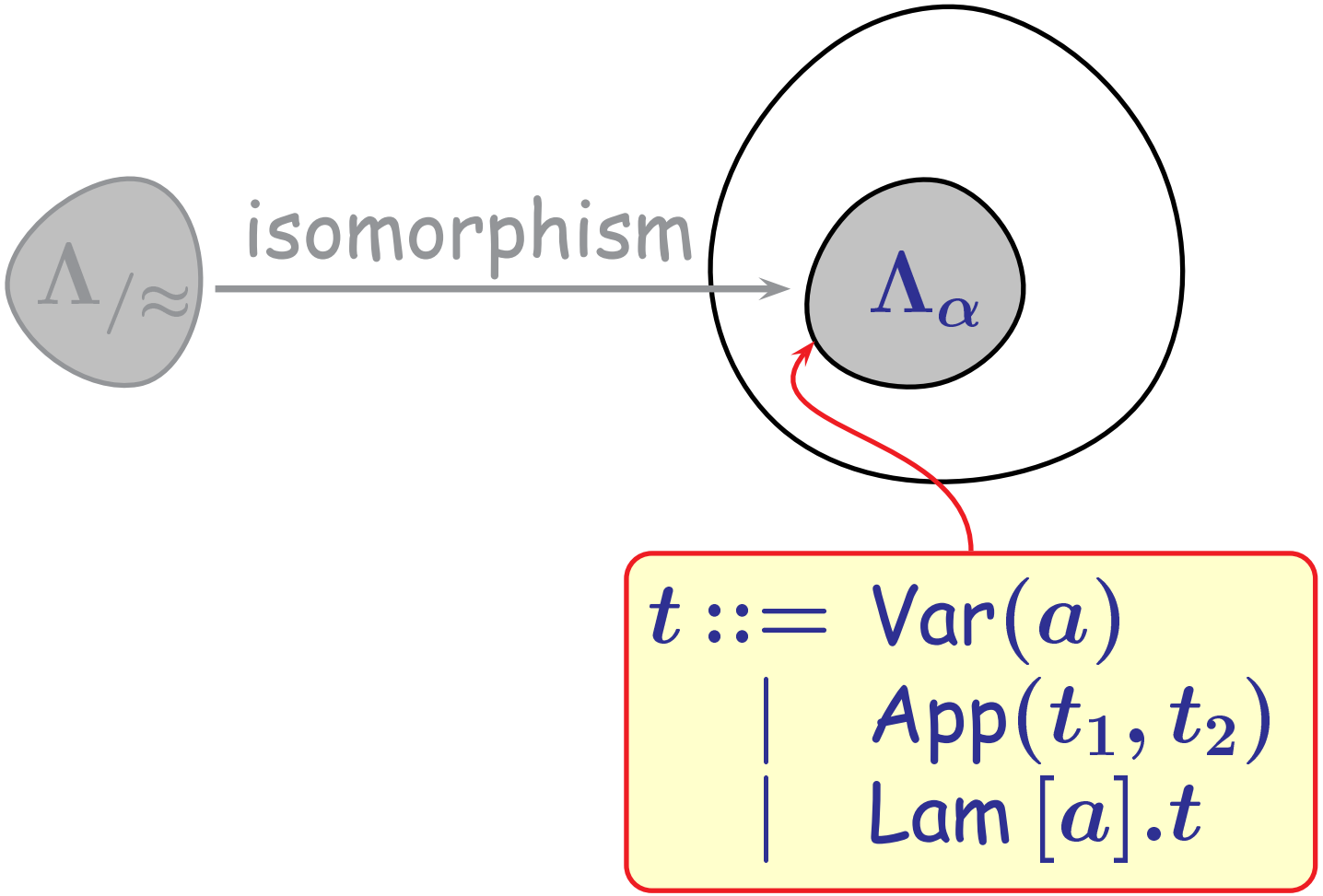
Definition of Small-Set



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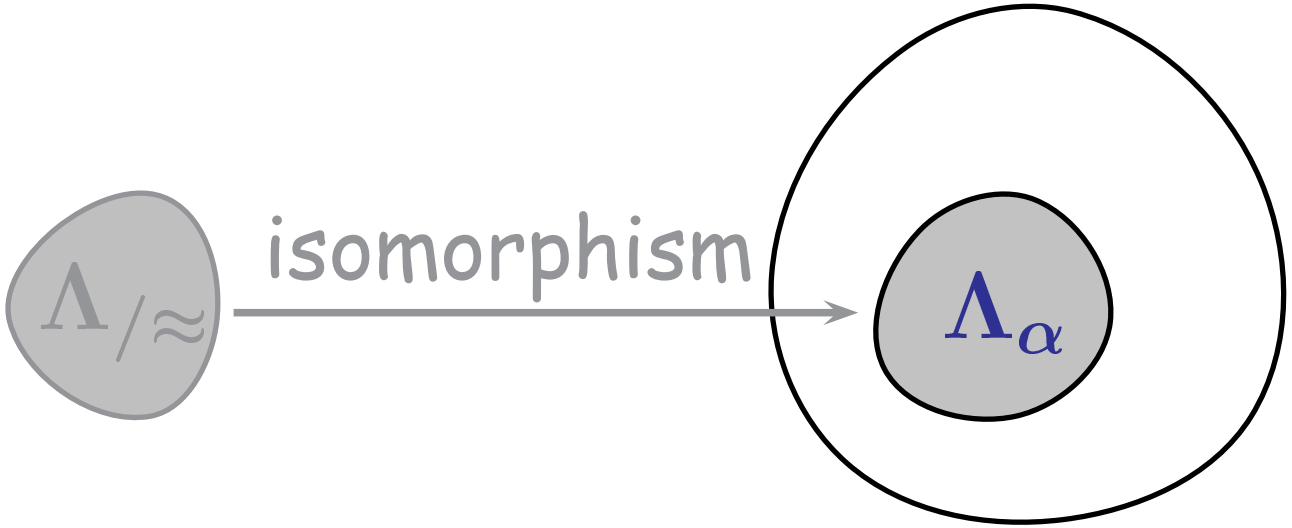

$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}(t_1, t_2) \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\,[a].t \in \Lambda_\alpha}$$

Definition of Small-Set

Which also means that we have a familiar induction principle in place for Λ_α (in a moment). And all terms in Λ_α have finite support.

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$$\begin{array}{l} \mathsf{supp}(\mathsf{Var}(a)) = \{a\} \\ \mathsf{supp}(\mathsf{App}(t, s)) = \mathsf{supp}(t, s) \\ \mathsf{supp}(\mathsf{Lam}\,[a].t) = \mathsf{supp}([a].t) = \mathsf{supp}(t) - \{a\} \end{array}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\,[a].t \in \Lambda_\alpha}$$

Isomorphism

In order to show that $\Lambda_{/\approx}$ and Λ_α are isomorphic we define a function q from Λ to Λ_α:

$$\begin{array}{lcl} q(a) & \stackrel{\text{def}}{=} & \mathsf{Var}(a) \\ q(t_1\, t_2) & \stackrel{\text{def}}{=} & \mathsf{App}(q(t_1), q(t_2)) \\ q(\lambda a.t) & \stackrel{\text{def}}{=} & \mathsf{Lam}\,[a].q(t) \end{array}$$

with the property

$$t_1 \approx t_2 \;\Leftrightarrow\; q(t_1) = q(t_2)$$

"Term-Constructors"

We did not define α-equivalence classes in a constructors-based way, but we can "fake" that: we can make the elements in the new **type** be of the form:

$$\mathsf{Var}(a),\ \mathsf{App}(s, t),\ \mathsf{Lam}\,[a].t$$

and have the following "injectivity" properties:

$$\frac{a = b}{\mathsf{Var}(a) = \mathsf{Var}(b)} \qquad \frac{t_1 = s_1 \quad t_2 = s_2}{\mathsf{App}(t_1, t_2) = \mathsf{App}(s_1, s_2)}$$

$$\frac{t = s}{\mathsf{Lam}\,[a].t = \mathsf{Lam}\,[a].s} \qquad \frac{a \neq b \quad t = (a\,b) \bullet s \quad a \# s}{\mathsf{Lam}\,[a].t = \mathsf{Lam}\,[b].s}$$

Struct. Induction on Λ_α

$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t \in \Lambda_\alpha \quad s \in \Lambda_\alpha}{\mathsf{App}(t, s) \in \Lambda_\alpha}$$

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Structural induction principle:

$$\frac{\forall a.\ P\,(\mathsf{Var}(a)) \qquad \forall t\ s.\ P\ t \Longrightarrow P\ s \Longrightarrow P\,(\mathsf{App}(t, s)) \qquad \forall a\ t.\ P\ t \Longrightarrow P\,(\mathsf{Lam}\,[a].t)}{\forall t.\ P\ t}$$

Substitution Lemma: If $x \not\equiv y$ and $x \notin FV(L)$, then

$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]].$$

Proof: By induction on the structure of M.

- **Case 1:** M is a variable.

  Case 1.1. $M \equiv x$. Then both sides equal $N[y := L]$ since $x \not\equiv y$.

  Case 1.2. $M \equiv y$. Then both sides equal L, for $x \notin FV(L)$ implies $L[x := \ldots] \equiv L$.

  Case 1.3. $M \equiv z \not\equiv x, y$. Then both sides equal z.

- **Case 2:** $M \equiv \lambda z.M_1$. By the variable convention we may assume that $z \not\equiv x, y$ and z is not free in N, L. Then by induction hypothesis

$$\begin{aligned}
&(\lambda z.M_1)[x := N][y := L] \\
\equiv\ &\lambda z.(M_1[x := N][y := L]) \\
\equiv\ &\lambda z.(M_1[y := L][x := N[y := L]]) \\
\equiv\ &(\lambda z.M_1)[y := L][x := N[y := L]].
\end{aligned}$$

- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis.

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  Case 1.3. $M \equiv$

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Can we do better?

Answer: yes.

Can We Do Better?

We consider induction hypotheses of the form $P\ x\ t$ with x **finitely supported**. In the case of the substitution lemma:

$$P = \lambda(x, y, N, L).\ \ \lambda M.$$
$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]]$$
$$x = (x, y, N, L)$$

$$\frac{\forall a\, x.\ P\, x\, (\mathsf{Var}(a)) \quad \forall t\, s\, x.\ (\forall z.\ P\, z\, t) \Longrightarrow (\forall z.\ P\, z\, s) \Longrightarrow P\, x\, (\mathsf{App}(t, s)) \quad \forall a\, t\, x.\ a \# x \Longrightarrow (\forall z.\ P\, z\, t) \Longrightarrow P\, x\, (\mathsf{Lam}\, [a].t)}{P\, x\, t}$$

A Formal Proof in Isabelle

```
lemma forget:
assumes a: "x # L"
shows "L[x ::= N] = L"
using a by (nominal_induct L avoiding: x N rule: lam_induct)
           (auto simp add: abs_fresh fresh_atm)

lemma fresh_fact:
fixes x::"name"
assumes a: "x # M" and b: "x # N"
shows "x # M[y ::= N]"
using a b by (nominal_induct M avoiding: x y N rule: lam_induct)
             (auto simp add: abs_fresh fresh_atm)

lemma subst_lemma:
assumes a: "x ≠ y" and b: "x # L"
shows "M[x ::= N][y ::= L] = M[y ::= L][x ::= N[y ::= L]]"
using a b by (nominal_induct M avoiding: x y N L rule: lam_induct)
             (auto simp add: forget fresh_fact)
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lemma fresh_fact:
fixes x::"nam
assumes a: "x
shows "$x \#$
using a b by (
(

lemma subst_
assumes a: "$x \neq y$" and b: "$x \# L$"
shows "$M[x ::= N][y ::= L] = M[y ::= L][x ::= N[y ::= L]]$"
using a b by (nominal_induct M avoiding: x y N L rule: lam_induct)
(auto simp add: forget fresh_fact)

abs_fresh:

$$(x \# \text{Lam}\,[y].M) = (x = y \vee x \# M)$$

fresh_atm:

$$(x \# y) = (x \neq y)$$

provided y is an atom

A Formal Proof in Isabelle

```
lemma forget:
assumes a: "x # L"
shows "L[x ::= N] = L"
using a by (nominal_induct L avoiding: x N rule: lam_induct)
           (auto simp add: abs_fresh fresh_atm)

lemma fresh_fact:
fixes x::"name"
assumes a: "x # M" and b: "x # N"
shows "x # M[y ::= N]"
using a b by (nominal_induct M avoiding: x y N rule: lam_induct)
             (auto simp add: abs_fresh fresh_atm)

lemma subst_lemma:
assumes a: "x ≠ y" and b: "x # L"
shows "M[x ::= N][y ::= L] = M[y ::= L][x ::= N[y ::= L]]"
using a b by (nominal_induct M avoiding: x y N L rule: lam_induct)
             (auto simp add: forget fresh_fact)
```

Weakening Lemma

Weakening Lemma: If $\Gamma_1 \vdash t : \sigma$ and Γ_2 is a valid super-context of Γ_1, then $\Gamma_2 \vdash t : \sigma$.

- **valid**: no variable has two assigned types
- **super-context**: Γ_2 has all the assignments of Γ_1

Proof: The literature is not helpful, because either too complicated[1] (if it is done right, e.g. Gallier, McKinna & Pollack, Pitts) or not helpful[2] (labelled as obvious, routine, trivial, e.g. Hindley).

[1] We can do better.
[2] I am sure they would need lots of fixing, if they were spelled out.

Formal Proof

```
lemma weakening_lemma:
assumes a: "Γ₁ ⊢ t : σ" and b: "Γ₁ ≪ Γ₂" and c: "valid Γ₂"
shows "Γ₂ ⊢ t : σ"
using a b c
apply (nominal_induct Γ₁ t σ avoiding: Γ₂ rule: typing_induct)
apply (auto simp add: sc_def)
done
```

Reducibility Lemma: Let t be any term (not assumed to be reducible), and suppose all the free variables of t are among $x_1,\ldots,x_n$ of types $U_1,\ldots,U_n$. If $u_1,\ldots,u_n$ are reducible terms of types $U_1,\ldots,U_n$ then $t[x_1 := u_1,\ldots,x_n := u_n]$ is reducible.

Proof: By induction on t. We write $t[\underline{x} := \underline{u}]$ for the substitution.

- **Case 1** t is x_i:
  one has to check the tautology "if u_i is reducible, then u_i is reducible"; details are left to the reader.
- **Case 2** t is $w\,v$:
  by induction hypothesis $w[\underline{x} := \underline{u}]$ and $v[\underline{x} := \underline{u}]$ are reducible, and so (by definition) $w[\underline{x} := \underline{u}](v[\underline{x} := \underline{u}])$; but this term is nothing other than $t[\underline{x} := \underline{u}]$.
- **Case 3** t is $\lambda y.w$ of type $V \to W$:
  by induction hypothesis, $w[\underline{x} := \underline{u}, y := v]$ is reducible for all v of type V. Lemma 6.3.2 says that $t[\underline{x} := \underline{u}] = \lambda y.(w[\underline{x} := \underline{u}])$ is reducible. "□"

Reducibility Lemma: Let t be any term (not assumed to be reducible), and suppose all the free variables of t are among $x_1,\ldots,x_n$ of types $U_1,\ldots,U_n$. If $u_1,\ldots,u_n$ are reducible terms of types $U_1,\ldots,U_n$ then $t[x_1 := u_1, \ldots, x_n := u_n]$ is reducible.

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Lemma 6.3.2: If for all reducible u of type U, $v[x := u]$ is reducible, then so is $\lambda x.v$.

```
lemma all_Rd:
assumes a: "Γ ⊢ t : τ"
      and b: "∀x σ. (x, σ) ∈ set Γ ⟶ x ∈ dom(θ) ∧ θ⟨x⟩ ∈ Rd_σ"
shows "t[θ] ∈ Rd_τ"
using a b proof (nominal_induct t avoiding: Γ τ θ rule: lam_induct)
case (Lam a t)
  have ih: "⋀Γ τ θ. Γ ⊢ t : τ ⟹
```

Reducibility Lemma: Let t be any term (not assumed to be reducible), and suppose all the free variables of t are among $x_1,\ldots,x_n$ of types $U_1,\ldots,U_n$. If $u_1,\ldots,u_n$ are reducible terms of types $U_1,\ldots,U_n$ then $t[x_1 := u_1,\ldots,x_n := u_n]$ is reducible.

```
  hence "∃τ1 τ2. τ = τ1→τ2 ∧ (a, τ1)::Γ ⊢ t : τ2" using t3_elim fr by simp
  then obtain τ1 τ2 where τ_inst: "τ = τ1→τ2"
                      and ty: "(a, τ1)#Γ ⊢ t : τ2" by blast
  from ih have "∀s ∈ Rd_τ1. t[θ][a ::= s] ∈ Rd_τ2" using fr ty θ_cond
     by (force dest: fresh_context simp add: psubts_subst)
  hence "Lam [a].(t[θ]) ∈ Rd_τ1→τ2" by (simp only: abs_Rd) -"Lem. 6.3.2"
thus "(Lam [a].t)[θ] ∈ Rd_τ" using fr τ_inst by simp
qed (force dest!: t1_elim t2_elim)+
```

```
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      and b: "∀x σ. (x, σ) ∈ set Γ ⟶ x ∈ dom(θ) ∧ θ⟨x⟩ ∈ Rd_σ"
shows "t[θ] ∈ Rd_τ"
using a b proof (nominal_induct t avoiding: Γ τ θ rule: lam_induct)
case (Lam a t)
  have ih: "⋀Γ τ θ. Γ ⊢ t : τ ⟹
  (∀x σ. (x, σ) ∈ set Γ ⟶ x ∈ dom(θ) ∧ θ⟨x⟩ ∈ Rd_σ) ⟹ t[θ] ∈ Rd_τ"
  and θ_cond: "∀x σ. (x, σ) ∈ set Γ ⟶ x ∈ dom(θ) ∧ θ⟨x⟩ ∈ Rd_σ"
  and fr: "a # Γ" "a # θ"
  and "Γ ⊢ Lam [a].t : τ" by fact
  hence "∃τ1 τ2.τ = τ1→τ2 ∧ (a, τ1) :: Γ ⊢ t : τ2" using t3_elim fr by simp
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thus "(Lam [a].t)[θ] ∈ Rd_τ" using fr τ_inst by simp
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```