Nominal Isabelle

or, How Not to be Intimidated by the Variable Convention

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Variable Convention:

If $M_1, \ldots, M_n$ occur in a certain mathematical context (e.g. definition, proof), then in these terms all bound variables are chosen to be different from the free variables.

Henk Barendregt in "The Lambda-Calculus: Its Syntax and Semantics"

dinner after my PhD examination

- Aim: develop Nominal Isabelle for reasoning formally about programming languages

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- found an error in an ACM journal paper by Bob Harper and Frank Pfenning about LF (and fixed it in three ways)
- found also fixable errors in my Ph.D.-thesis about cut-elimination (examined by Henk Barendregt and Andy Pitts)
- found that the variable convention can in principle be used for proving false

Nominal Techniques

- Andy Pitts showed me that permutations preserve α-equivalence:

$$t_1 \approx_\alpha t_2 \quad \Rightarrow \quad \pi \bullet t_1 \approx_\alpha \pi \bullet t_2$$

- also permutations and substitutions commute, if you suspend permutations in front of variables

$$\pi \bullet \sigma(t) = \sigma(\pi \bullet t)$$

- this allowed us to define as simple Nominal Unification algorithm

$$\nabla \vdash t \approx_\alpha^? t' \qquad\qquad \nabla \vdash a \,\#^?\, t$$

Nominal Isabelle

- a general theory about atoms and permutations
  - sorted atoms and
  - sort-respecting permutations

- support and freshness

$$\mathit{supp}(x) \stackrel{\text{def}}{=} \{a \mid \mathit{infinite}\ \{b \mid (a\,b)\bullet x \neq x\}\}$$

$$a \# x \stackrel{\text{def}}{=} a \notin \mathit{supp}(x)$$

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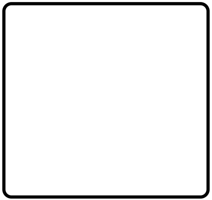
- allow users to reason about alpha-equivalence classes as if they were syntax trees

New Types in HOL



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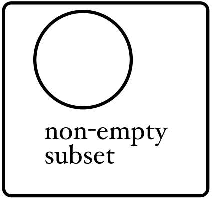

existing
type

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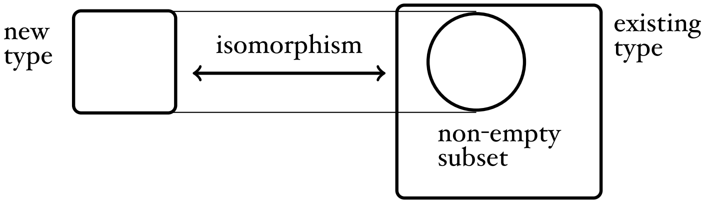

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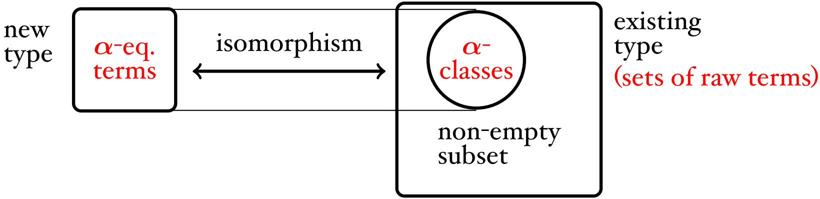

New Types in HOL

new
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α-eq.
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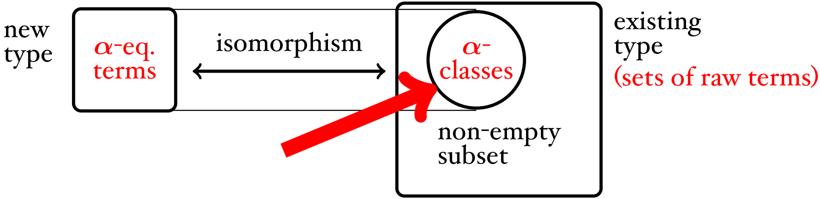

define α-equivalence

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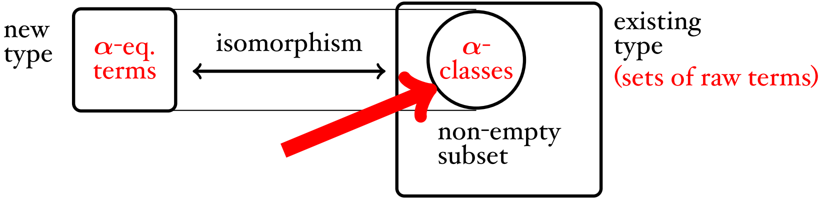

define α-equivalence

The "new types mechanism" is the reason why there is no Nominal Coq.

HOL vs. Nominal

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$$\mathit{finite}(\mathit{supp}(x)) \quad \Rightarrow \quad a \,\#\, a.x$$

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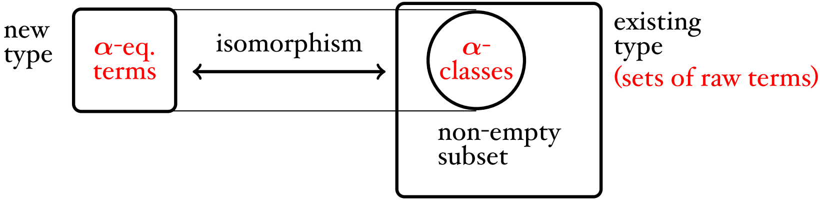
$$a \# a.x$$

Our Work

- define fv and α



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- define fv and α
- build quotient / new type
- derive a reasoning infrastructure (#, distinctness, injectivity, cases,...)
- derive a **stronger** cases lemma
- from this, a **stronger** induction principle (Barendregt variable convention built in)

$$\text{Foo}\ (\lambda x.\lambda y.t)\ (\lambda u.\lambda v.s)$$

Nominal Isabelle

- Users can for example define lambda-terms as:

atom_decl name

nominal_datatype lam =
  Var name
 | App lam lam
 | Lam x::name t::lam **binds** x **in** t ("Lam _. _")

- These are **named** alpha-equivalence classes, for example

Lam a.(Var a) = Lam b.(Var b)

(Weak) Induction Principles

- The usual induction principle for lambda-terms is as follows:

$$\frac{\begin{array}{l}\forall x.\ P\,x \\ \forall t_1\,t_2.\ P\,t_1 \wedge P\,t_2 \Rightarrow P\,(t_1\ t_2) \\ \forall x\,t.\ P\,t \Rightarrow P\,(\lambda x.t)\end{array}}{P\,t}$$

- It requires us in the lambda-case to show the property P for all binders x.

  (This nearly always requires renamings and they can be tricky to automate.)

Strong Induction Principles

- Therefore we introduced the following strong induction principle:

$$\frac{\begin{array}{l}\forall x\, c.\ P\, c\, x\\ \forall t_1\, t_2\, c.\ (\forall d.\ P\, d\, t_1) \wedge (\forall d. P\, d\, t_2) \Rightarrow P\, c\, (t_1\, t_2)\\ \forall x\, t\, c.\ x \# c \wedge (\forall d. P\, d\, t) \Rightarrow P\, c\, (\lambda x. t)\end{array}}{P\ c\ t}$$

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The variable over which the induction proceeds:

"...By induction over the structure of M..."

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The **context** of the induction; i.e. what the binder should be fresh for $\Rightarrow (x, y, N, L)$:

"...By the variable convention we can assume $z \not\equiv x, y$ and z not free in N, L..."

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The property to be proved by induction:

$$\begin{array}{l}\lambda(x,y,N,L).\,\lambda M.\ x \neq y \wedge x \,\#\, L \Rightarrow \\ \qquad M[x:=N][y:=L] = M[y:=L][x:=N[y:=L]]\end{array}$$

Binding Sets of Names

- binding sets of names has some interesting properties:

$$\forall\{x, y\}.\, x \rightarrow y \quad \approx_\alpha \quad \forall\{y, x\}.\, y \rightarrow x$$

* x, y, z are assumed to be distinct

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Binding Sets of Names

- bind ... prop...

For type-schemes the order of bound names does not matter, and α-equivalence is preserved under vacuous binders.

$$\forall\{x, y\}.\, x \to y \quad \not\approx_\alpha \quad \forall\{z\}.\, z \to z$$

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Other Binding Modes

- alpha-equivalence being preserved under vacuous binders is <u>not</u> always wanted:

$$\text{let } x = 3 \text{ and } y = 2 \text{ in } x - y \text{ end}$$

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$$\not\approx_\alpha \mathrm{let}\ y = 2\ \mathrm{and}\ x = 3\ \mathrm{and}\ z = \mathrm{loop}\ \mathrm{in}\ x - y\ \mathrm{end}$$

Even Another Binding Mode

- sometimes one wants to abstract more than one name, but the order <u>does</u> matter

$$\mathrm{let}\ (\boldsymbol{x},\boldsymbol{y}) = (3,2)\ \mathrm{in}\ \boldsymbol{x} - \boldsymbol{y}\ \mathrm{end}$$

$$\not\approx_\alpha \mathrm{let}\ (\boldsymbol{y},\boldsymbol{x}) = (3,2)\ \mathrm{in}\ \boldsymbol{x} - \boldsymbol{y}\ \mathrm{end}$$

Specification of Binding

```
nominal_datatype trm =
    Var name
  | App trm trm
  | Lam x::name t::trm    bind x in t
  | Let as::assns t::trm  bind bn(as) in t
and assns =
    ANil
  | ACons name trm assns
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binder bn where
    bn(ANil) = []
  | bn(ACons a t as) = [a] @ bn(as)
```

So Far So Good

- A Faulty Lemma with the Variable Convention?

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Inductive Definitions:

$$\frac{\text{prem}_1 \ldots \text{prem}_n \ \ \text{scs}}{\text{concl}}$$

Rule Inductions:

1.) Assume the property for the premises. Assume the side-conditions.

2.) Show the property for the conclusion.

Faulty Reasoning

- Consider the two-place relation foo:

$$\frac{}{x \mapsto x} \qquad \frac{}{t_1\ t_2 \mapsto t_1\ t_2} \qquad \frac{t \mapsto t'}{\lambda x.t \mapsto t'}$$

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- Cases 1 and 2 are trivial:
  - If $y \# x$ then $y \# x$.
  - If $y \# t_1\ t_2$ then $y \# t_1\ t_2$.

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$$y \not\in \mathrm{fv}(\lambda x.t) \iff y \not\in \mathrm{fv}(t) - \{x\} \overset{x \neq y}{\iff} y \not\in \mathrm{fv}(t)$$

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Conclusions

- The user does not see anything of the “raw” level.
- The Nominal Isabelle automatically derives the strong structural induction principle for <u>**all**</u> nominal datatypes (not just the lambda-calculus)
- They are easy to use: you just have to think carefully what the variable convention should be.
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- We can explore the dark corners of the variable convention: when and where it can be used safely.
- **Main Point:** Actually these proofs using the variable convention are all trivial / obvious / routine...**provided** you use Nominal Isabelle. ;o)

Thank you very much!
Questions?