RegEx

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Abstract. BLA BLA Sulzmann and Lu [1]
Keywords:

1 Introduction

Regular exprtessions

$$r ::= \mathit{NULL} \mid \mathit{EMPTY} \mid \mathit{CHAR}\ c \mid \mathit{ALT}\ r_1\ r_2 \mid \mathit{SEQ}\ r_1\ r_2 \mid \mathit{STAR}\ r$$

Values

$$v ::= \mathit{Void} \mid \mathit{Char}\ c \mid \mathit{Left}\ v \mid \mathit{Right}\ v \mid \mathit{Seq}\ v_1\ v_2 \mid \mathit{Stars}\ vs$$

The language of a regular expression

$$\begin{array}{lcl} L\ \mathit{NULL} & \stackrel{\text{def}}{=} & \varnothing \\ L\ \mathit{EMPTY} & \stackrel{\text{def}}{=} & \{[]\} \\ L\ (\mathit{CHAR}\ c) & \stackrel{\text{def}}{=} & \{[c]\} \\ L\ (\mathit{SEQ}\ r_1\ r_2) & \stackrel{\text{def}}{=} & (L\ r_1) \mathbin{@} (L\ r_2) \\ L\ (\mathit{ALT}\ r_1\ r_2) & \stackrel{\text{def}}{=} & (L\ r_1) \cup (L\ r_2) \\ L\ (\mathit{STAR}\ r) & \stackrel{\text{def}}{=} & (L\ r)\star \end{array}$$

The nullable function

$$\begin{array}{lcl} \mathit{nullable}\ \mathit{NULL} & \stackrel{\text{def}}{=} & \mathit{False} \\ \mathit{nullable}\ \mathit{EMPTY} & \stackrel{\text{def}}{=} & \mathit{True} \\ \mathit{nullable}\ (\mathit{CHAR}\ c) & \stackrel{\text{def}}{=} & \mathit{False} \\ \mathit{nullable}\ (\mathit{ALT}\ r_1\ r_2) & \stackrel{\text{def}}{=} & \mathit{nullable}\ r_1 \vee \mathit{nullable}\ r_2 \\ \mathit{nullable}\ (\mathit{SEQ}\ r_1\ r_2) & \stackrel{\text{def}}{=} & \mathit{nullable}\ r_1 \wedge \mathit{nullable}\ r_2 \\ \mathit{nullable}\ (\mathit{STAR}\ r) & \stackrel{\text{def}}{=} & \mathit{True} \end{array}$$

The derivative function for characters and strings

$$
\begin{array}{lcl}
\mathit{der}\ c\ \mathit{NULL} & \stackrel{\text{def}}{=} & \mathit{NULL} \\
\mathit{der}\ c\ \mathit{EMPTY} & \stackrel{\text{def}}{=} & \mathit{NULL} \\
\mathit{der}\ c\ (\mathit{CHAR}\ c') & \stackrel{\text{def}}{=} & \mathit{if}\ c = c'\ \mathit{then}\ \mathit{EMPTY}\ \mathit{else}\ \mathit{NULL} \\
\mathit{der}\ c\ (\mathit{ALT}\ r_1\ r_2) & \stackrel{\text{def}}{=} & \mathit{ALT}\ (\mathit{der}\ c\ r_1)\ (\mathit{der}\ c\ r_2) \\
\mathit{der}\ c\ (\mathit{SEQ}\ r_1\ r_2) & \stackrel{\text{def}}{=} & \mathit{if}\ \mathit{nullable}\ r_1\ \mathit{then}\ \mathit{ALT}\ (\mathit{SEQ}\ (\mathit{der}\ c\ r_1)\ r_2)\ (\mathit{der}\ c\ r_2) \\
 & & \mathit{else}\ \mathit{SEQ}\ (\mathit{der}\ c\ r_1)\ r_2 \\
\mathit{der}\ c\ (\mathit{STAR}\ r) & \stackrel{\text{def}}{=} & \mathit{SEQ}\ (\mathit{der}\ c\ r)\ (\mathit{STAR}\ r) \\
\\
\mathit{ders}\ []\ r & \stackrel{\text{def}}{=} & r \\
\mathit{ders}\ (c::s)\ r & \stackrel{\text{def}}{=} & \mathit{ders}\ s\ (\mathit{der}\ c\ r)
\end{array}
$$

The *flat* function for values

$$
\begin{array}{lcl}
|\mathit{Void}| & \stackrel{\text{def}}{=} & [] \\
|\mathit{Char}\ c| & \stackrel{\text{def}}{=} & [c] \\
|\mathit{Left}\ v| & \stackrel{\text{def}}{=} & |v| \\
|\mathit{Right}\ v| & \stackrel{\text{def}}{=} & |v| \\
|\mathit{Seq}\ v_1\ v_2| & \stackrel{\text{def}}{=} & |v_1| \mathbin{@} |v_2| \\
|\mathit{Stars}\ []| & \stackrel{\text{def}}{=} & [] \\
|\mathit{Stars}\ (v::vs)| & \stackrel{\text{def}}{=} & |v| \mathbin{@} |\mathit{Stars}\ vs|
\end{array}
$$

The *mkeps* function

$$
\begin{array}{lcl}
\mathit{mkeps}\ \mathit{EMPTY} & \stackrel{\text{def}}{=} & \mathit{Void} \\
\mathit{mkeps}\ (\mathit{SEQ}\ r_1\ r_2) & \stackrel{\text{def}}{=} & \mathit{Seq}\ (\mathit{mkeps}\ r_1)\ (\mathit{mkeps}\ r_2) \\
\mathit{mkeps}\ (\mathit{ALT}\ r_1\ r_2) & \stackrel{\text{def}}{=} & \mathit{if}\ \mathit{nullable}\ r_1\ \mathit{then}\ \mathit{Left}\ (\mathit{mkeps}\ r_1)\ \mathit{else}\ \mathit{Right}\ (\mathit{mkeps}\ r_2) \\
\mathit{mkeps}\ (\mathit{STAR}\ r) & \stackrel{\text{def}}{=} & \mathit{Stars}\ []
\end{array}
$$

The *inj* function

$$
\begin{array}{lcl}
\mathit{inj}\ (\mathit{CHAR}\ d)\ c\ \mathit{Void} & \stackrel{\text{def}}{=} & \mathit{Char}\ d \\
\mathit{inj}\ (\mathit{ALT}\ r_1\ r_2)\ c\ (\mathit{Left}\ v_1) & \stackrel{\text{def}}{=} & \mathit{Left}\ (\mathit{inj}\ r_1\ c\ v_1) \\
\mathit{inj}\ (\mathit{ALT}\ r_1\ r_2)\ c\ (\mathit{Right}\ v_2) & \stackrel{\text{def}}{=} & \mathit{Right}\ (\mathit{inj}\ r_2\ c\ v_2) \\
\mathit{inj}\ (\mathit{SEQ}\ r_1\ r_2)\ c\ (\mathit{Seq}\ v_1\ v_2) & \stackrel{\text{def}}{=} & \mathit{Seq}\ (\mathit{inj}\ r_1\ c\ v_1)\ v_2 \\
\mathit{inj}\ (\mathit{SEQ}\ r_1\ r_2)\ c\ (\mathit{Left}\ (\mathit{Seq}\ v_1\ v_2)) & \stackrel{\text{def}}{=} & \mathit{Seq}\ (\mathit{inj}\ r_1\ c\ v_1)\ v_2 \\
\mathit{inj}\ (\mathit{SEQ}\ r_1\ r_2)\ c\ (\mathit{Right}\ v_2) & \stackrel{\text{def}}{=} & \mathit{Seq}\ (\mathit{mkeps}\ r_1)\ (\mathit{inj}\ r_2\ c\ v_2) \\
\mathit{inj}\ (\mathit{STAR}\ r)\ c\ (\mathit{Seq}\ v\ (\mathit{Stars}\ vs)) & \stackrel{\text{def}}{=} & \mathit{Stars}\ ((\mathit{inj}\ r\ c\ v)::vs)
\end{array}
$$

The inhabitation relation:

$$\frac{\vdash v_1 : r_1 \qquad \vdash v_2 : r_2}{\vdash \mathit{Seq}\ v_1\ v_2 : \mathit{SEQ}\ r_1\ r_2}$$

$$\frac{\vdash v_1 : r_1}{\vdash (\mathit{Left}\ v_1) : \mathit{ALT}\ r_1\ r_2} \qquad \frac{\vdash v_2 : r_1}{\vdash (\mathit{Right}\ v_2) : \mathit{ALT}\ r_2\ r_1}$$

$$\frac{}{\vdash \mathit{Void} : \mathit{EMPTY}} \qquad \frac{}{\vdash (\mathit{Char}\ c) : \mathit{CHAR}\ c}$$

$$\frac{}{\vdash \mathit{Stars}\ [] : \mathit{STAR}\ r} \qquad \frac{\vdash v : r \qquad \vdash \mathit{Stars}\ vs : \mathit{STAR}\ r}{\vdash \mathit{Stars}\ (v::vs) : \mathit{STAR}\ r}$$

We have also introduced a slightly restricted version of this relation where the last rule is restricted so that $|v| \neq []$. This relation for *non-problematic* is written $\models v : r$.

Our Posix relation $s \in r \to v$

$$\frac{}{[] \in \mathit{EMPTY} \to \mathit{Void}} \qquad \frac{}{[c] \in \mathit{CHAR}\ c \to (\mathit{Char}\ c)}$$

$$\frac{s \in r_1 \to v}{s \in \mathit{ALT}\ r_1\ r_2 \to (\mathit{Left}\ v)} \qquad \frac{s \in r_2 \to v \qquad s \notin (L\ r_1)}{s \in \mathit{ALT}\ r_1\ r_2 \to (\mathit{Right}\ v)}$$

$$\frac{\begin{array}{c} s_1 \in r_1 \to v_1 \qquad s_2 \in r_2 \to v_2 \\ \nexists s_3\ s_4.\ s_3 \neq [] \wedge s_3 @ s_4 = s_2 \wedge s_1 @ s_3 \in (L\ r_1) \wedge s_4 \in (L\ r_2) \end{array}}{(s_1 @ s_2) \in \mathit{SEQ}\ r_1\ r_2 \to \mathit{Seq}\ v_1\ v_2}$$

$$\frac{\begin{array}{c} s_1 \in r \to v \qquad s_2 \in \mathit{STAR}\ r \to \mathit{Stars}\ vs \\ |v| \neq [] \qquad \nexists s_3\ s_4.\ s_3 \neq [] \wedge s_3 @ s_4 = s_2 \wedge s_1 @ s_3 \in (L\ r) \wedge s_4 \in (L\ (\mathit{STAR}\ r)) \end{array}}{(s_1 @ s_2) \in \mathit{STAR}\ r \to \mathit{Stars}\ (v::vs)}$$

$$\frac{}{[] \in \mathit{STAR}\ r \to \mathit{Stars}\ []}$$

Our version of Sulzmann's ordering relation

$$\frac{v_1 \succeq_{r_1} v_1' \qquad v_1 \neq v_1'}{Seq\ v_1\ v_2 \succeq_{SEQ\ r_1\ r_2} Seq\ v_1'\ v_2'} \qquad \frac{v_2 \succeq_{r_2} v_2'}{Seq\ v_1\ v_2 \succeq_{SEQ\ r_1\ r_2} Seq\ v_1\ v_2'}$$

$$\frac{len\ (|v_1|) \leq len\ (|v_2|)}{Left\ v_2 \succeq_{ALT\ r_1\ r_2} Right\ v_1} \qquad \frac{len\ (|v_2|) < len\ (|v_1|)}{Right\ v_1 \succeq_{ALT\ r_1\ r_2} Left\ v_2}$$

$$\frac{v_2 \succeq_{r_2} v_2'}{Right\ v_2 \succeq_{ALT\ r_1\ r_2} Right\ v_2'} \qquad \frac{v_1 \succeq_{r_1} v_1'}{Left\ v_1 \succeq_{ALT\ r_1\ r_2} Left\ v_1'}$$

$$\frac{}{Void \succeq_{EMPTY} Void} \qquad \frac{}{Char\ c \succeq_{CHAR\ c} Char\ c}$$

$$\frac{|Stars\ (v::vs)| = []}{Stars\ [] \succeq_{STAR\ r} Stars\ (v::vs)} \qquad \frac{|Stars\ (v::vs)| \neq []}{Stars\ (v::vs) \succeq_{STAR\ r} Stars\ []}$$

$$\frac{v_1 \succeq_r v_2 \qquad v_1 \neq v_2}{Stars\ (v_1::vs_1) \succeq_{STAR\ r} Stars\ (v_2::vs_2)}$$

$$\frac{Stars\ vs_1 \succeq_{STAR\ r} Stars\ vs_2}{Stars\ (v::vs_1) \succeq_{STAR\ r} Stars\ (v::vs_2)} \qquad \frac{}{Stars\ [] \succeq_{STAR\ r} Stars\ []}$$

A prefix of a string s

$$s_1 \sqsubseteq s_2 \stackrel{def}{=} \exists s3.\ s_1 \mathbin{@} s3 = s_2$$

Values and non-problematic values

$$Values\ r\ s \stackrel{def}{=} \{v \mid \vdash v : r \wedge (|v|) \sqsubseteq s\}$$

$$NValues\ r\ s \stackrel{def}{=} \{v \mid \models v : r \wedge (|v|) \sqsubseteq s\}$$

The point is that for a given s and r there are only finitely many non-problematic values.

Some lemmas we have proved:

$(L\ r) = \{|v| \mid \vdash v : r\}$
$(L\ r) = \{|v| \mid \models v : r\}$
If $nullable\ r$ *then* $\vdash mkeps\ r : r$.
If $nullable\ r$ *then* $|mkeps\ r| = []$.
If $\vdash v : der\ c\ r$ *then* $\vdash (inj\ r\ c\ v) : r$.
If $\vdash v : der\ c\ r$ *then* $|inj\ r\ c\ v| = c::(|v|)$.
If $nullable\ r$ *then* $[] \in r \to mkeps\ r$.
If $s \in r \to v$ *then* $|v| = s$.
If $s \in r \to v$ *then* $\models v : r$.
If $s \in r \to v_1$ *and* $s \in r \to v_2$ *then* $v_1 = v_2$.

This is the main theorem that lets us prove that the algorithm is correct according to $s \in r \to v$:

If $s \in der\ c\ r \to v$ *then* $(c::s) \in r \to (inj\ r\ c\ v)$.

Proof The proof is by induction on the definition of *der*. Other inductions would go through as well. The interesting case is for $SEQ\ r_1\ r_2$. First we analyse the case where *nullable* r_1. We have by induction hypothesis

$$\begin{array}{ll}(IH1) & \forall s\ v.\ \text{if } s \in der\ c\ r_1 \to v \text{ then } (c{::}s) \in r_1 \to (inj\ r_1\ c\ v)\\ (IH2) & \forall s\ v.\ \text{if } s \in der\ c\ r_2 \to v \text{ then } (c{::}s) \in r_2 \to (inj\ r_2\ c\ v)\end{array}$$

and have

$$s \in ALT\ (SEQ\ (der\ c\ r_1)\ r_2)\ (der\ c\ r_2) \to v$$

There are two cases what v can be: (1) *Left* v' and (2) *Right* v'.

(1) We know $s \in SEQ\ (der\ c\ r_1)\ r_2 \to v'$ holds, from which we can infer that there are s_1, s_2, v_1, v_2 with

$$s_1 \in der\ c\ r_1 \to v_1 \qquad \text{and} \qquad s_2 \in r_2 \to v_2$$

and also

$$\nexists s_3\ s_4.\ s_3 \neq [] \wedge s_3 \mathbin{@} s_4 = s_2 \wedge s_1 \mathbin{@} s_3 \in (L\ (der\ c\ r_1)) \wedge s_4 \in (L\ r_2)$$

and have to prove

$$(c{::}s_1 \mathbin{@} s_2) \in SEQ\ r_1\ r_2 \to Seq\ (inj\ r_1\ c\ v_1)\ v_2$$

The two requirements $(c{::}s_1) \in r_1 \to (inj\ r_1\ c\ v_1)$ and $s_2 \in r_2 \to v_2$ can be proved by the induction hypothese (IH1) and the fact above.
This leaves to prove

$$\nexists s_3\ s_4.\ s_3 \neq [] \wedge s_3 \mathbin{@} s_4 = s_2 \wedge c{::}s_1 \mathbin{@} s_3 \in (L\ r_1) \wedge s_4 \in (L\ r_2)$$

which holds because $c{::}s_1 \mathbin{@} s_3 \in (L\ r_1)$ implies $s_1 \mathbin{@} s_3 \in (L\ (der\ c\ r_1))$

(2) This case is similar.

The final case is that $\neg$ *nullable* r_1 holds. This case again similar to the cases above.

Things we have proved about our version of the Sulzmann ordering

$$If \vdash v : r\ then\ v \succeq_r v.$$

Things we like to prove, but cannot:

If $s \in r \to v_1, \vdash v_2 : r$, then $v_1 \succeq_r v_2$

References

1. M. Sulzmann and K. Lu. POSIX Regular Expression Parsing with Derivatives. In *Proc. of the 12th International Conference on Functional and Logic Programming (FLOPS)*, volume 8475 of *LNCS*, pages 203–220, 2014.

2 Roy's Rules

$$Void \lhd \epsilon \qquad Char\ c \lhd Lit\ c$$

$$\frac{v_1 \lhd r_1}{Left\ v_1 \lhd r_1 + r_2} \qquad \frac{v_2 \lhd r_2 \qquad |v_2| \notin L(r_1)}{Right\ v_2 \lhd r_1 + r_2}$$

$$\frac{v_1 \lhd r_1 \qquad v_2 \lhd r_2 \qquad s \in L(r_1 \backslash |v_1|) \wedge |v_2| \backslash s \notin L(r_2) \Rightarrow s = []}{(v_1, v_2) \lhd r_1 \cdot r_2}$$

$$\frac{v \lhd r \qquad vs \lhd r^* \qquad |v| \neq []}{(v :: vs) \lhd r^*} \qquad [] \lhd r^*$$