

We already proved that

If $\text{nullable}(r)$ then $\text{POSIX} (\text{mkeps } r) r$

holds. This is essentially the “base case” for the correctness proof of the algorithm. For the “induction case” we need the following main theorem, which we are currently after:

If (*) $\text{POSIX } v (\text{der } c r)$ and $\vdash v : \text{der } c r$
then $\text{POSIX} (\text{inj } r c v) r$

That means a POSIX value v is still *POSIX* after injection. I am not sure whether this theorem is actually true in this full generality. Maybe it requires some restrictions.

If we unfold the *POSIX* definition in the then-part, we arrive at

$\forall v'. \text{ if } \vdash v' : r \text{ and } |\text{inj } r c v| = |v'| \text{ then } |\text{inj } r c v| \succ_r v'$

which is what we need to prove assuming the if-part (*) in the theorem above. Since this is a universally quantified formula, we just need to fix a v' . We can then prove the implication by assuming

(a) $\vdash v' : r$ and (b) $|\text{inj } r c v| = |v'|$

and our goal is

(goal) $|\text{inj } r c v| \succ_r v'$

There are already two lemmas proved that can transform the assumptions (a) and (b) into

(a*) $\vdash \text{proj } r c v' : \text{der } c r$ and (b*) $c \# |v| = |v'|$

Another lemma shows that

$|v'| = c \# |\text{proj } r c v|$

Using (b*) we can therefore infer

(b**) $|v| = |\text{proj } r c v|$

The main idea of the proof is now a simple instantiation of the assumption $\text{POSIX } v (\text{der } c r)$. If we unfold the *POSIX* definition, we get

$\forall v'. \text{ if } \vdash v' : \text{der } c r \text{ and } |v| = |v'| \text{ then } v \succ_{\text{der } c r} v'$

We can instantiate this v' with $\text{proj } r c v'$ and can use (a*) and (b**) in order to infer

$$v \succ_{\text{der } c r} \text{proj } r c v'$$

The point of the side-lemma below is that we can “add” an inj to both sides to obtain

$$\text{inj } r c v \succ_r \text{inj } r c (\text{proj } r c v')$$

Finally there is already a lemma proved that shows that an injection and projection is the identity, meaning

$$\text{inj } r c (\text{proj } r c v') = v'$$

With this we have shown our goal (pending a proof of the side-lemma next).

Side-Lemma

A side-lemma needed for the theorem above which might be true, but can also be false, is as follows:

- If
- (1) $v_1 \succ_{\text{der } c r} v_2$,
 - (2) $\vdash v_1 : \text{der } c r$, and
 - (3) $\vdash v_2 : \text{der } c r$ holds,
- then $\text{inj } r c v_1 \succ_r \text{inj } r c v_2$ also holds.

It essentially states that if one value v_1 is bigger than v_2 then this ordering is preserved under injections. This is proved by induction (on the definition of $\text{der} \dots$ this is very similar to an induction on r).

The case that is still unproved is the sequence case where we assume $r = r_1 \cdot r_2$ and also r_1 being nullable. The derivative $\text{der } c r$ is then

$$\text{der } c r = ((\text{der } c r_1) \cdot r_2) + (\text{der } c r_2)$$

or without the parentheses

$$\text{der } c r = (\text{der } c r_1) \cdot r_2 + \text{der } c r_2$$

In this case the assumptions are

- (a) $v_1 \succ_{(der\ c\ r_1) \cdot r_2 + der\ c\ r_2} v_2$
- (b) $\vdash v_1 : (der\ c\ r_1) \cdot r_2 + der\ c\ r_2$
- (c) $\vdash v_2 : (der\ c\ r_1) \cdot r_2 + der\ c\ r_2$
- (d) $nullable(r_1)$

The induction hypotheses are

- (IH1) $\forall v_1 v_2. v_1 \succ_{der\ c\ r_1} v_2 \wedge \vdash v_1 : der\ c\ r_1 \wedge \vdash v_2 : der\ c\ r_1$
 $\longrightarrow inj\ r_1\ c\ v_1 \succ_{r_1} inj\ r_1\ c\ v_2$
- (IH2) $\forall v_1 v_2. v_1 \succ_{der\ c\ r_2} v_2 \wedge \vdash v_1 : der\ c\ r_2 \wedge \vdash v_2 : der\ c\ r_2$
 $\longrightarrow inj\ r_2\ c\ v_1 \succ_{r_2} inj\ r_2\ c\ v_2$

The goal is

$$(goal) \quad inj\ (r_1 \cdot r_2)\ c\ v_1 \succ_{r_1 \cdot r_2} inj\ (r_1 \cdot r_2)\ c\ v_2$$

If we analyse how (a) could have arisen (that is make a case distinction), then we will find four cases:

- LL $v_1 = Left(w_1), v_2 = Left(w_2)$
- LR $v_1 = Left(w_1), v_2 = Right(w_2)$
- RL $v_1 = Right(w_1), v_2 = Left(w_2)$
- RR $v_1 = Right(w_1), v_2 = Right(w_2)$

We have to establish our goal in all four cases.

Case LR

The corresponding rule (instantiated) is:

$$\frac{len\ |w_1| \geq len\ |w_2|}{Left(w_1) \succ_{(der\ c\ r_1) \cdot r_2 + der\ c\ r_2} Right(w_2)}$$

This means we can also assume in this case

$$(e) \quad len\ |w_1| \geq len\ |w_2|$$

which is the premise of the rule above. Instantiating v_1 and v_2 in the assumptions (b) and (c) gives us

- (b*) $\vdash Left(w_1) : (der\ c\ r_1) \cdot r_2 + der\ c\ r_2$
- (c*) $\vdash Right(w_2) : (der\ c\ r_1) \cdot r_2 + der\ c\ r_2$

Since these are assumptions, we can further analyse how they could have arisen according to the rules of $\vdash _ : _$. This gives us two new assumptions

$$\begin{aligned} (\mathbf{b}^{**}) \quad & \vdash w_1 : (\mathit{der} \ c \ r_1) \cdot r_2 \\ (\mathbf{c}^{**}) \quad & \vdash w_2 : \mathit{der} \ c \ r_2 \end{aligned}$$

Looking at (\mathbf{b}^{**}) we can further analyse how this judgement could have arisen. This tells us that w_1 must have been a sequence, say $u_1 \cdot u_2$, with

$$\begin{aligned} (\mathbf{b}^{***}) \quad & \vdash u_1 : \mathit{der} \ c \ r_1 \\ & \vdash u_2 : r_2 \end{aligned}$$

Instantiating the goal means we need to prove

$$\mathit{inj} \ (r_1 \cdot r_2) \ c \ (\mathit{Left}(u_1 \cdot u_2)) \succ_{r_1 \cdot r_2} \mathit{inj} \ (r_1 \cdot r_2) \ c \ (\mathit{Right}(w_2))$$

We can simplify this according to the rules of inj :

$$(\mathit{inj} \ r_1 \ c \ u_1) \cdot u_2 \succ_{r_1 \cdot r_2} (\mathit{mkeps} \ r_1) \cdot (\mathit{inj} \ r_2 \ c \ w_2)$$

This is what we need to prove. There are only two rules that can be used to prove this judgement:

$$\frac{v_1 = v'_1 \quad v_2 \succ_{r_2} v'_2}{v_1 \cdot v_2 \succ_{r_1 \cdot r_2} v'_1 \cdot v'_2} \quad \frac{v_1 \succ_{r_1} v'_1}{v_1 \cdot v_2 \succ_{r_1 \cdot r_2} v'_1 \cdot v'_2}$$

Using the left rule would mean we need to show that

$$\mathit{inj} \ r_1 \ c \ u_1 = \mathit{mkeps} \ r_1$$

but this can never be the case.¹ Lets assume it would be true, then also if we flat each side, it must hold that

$$|\mathit{inj} \ r_1 \ c \ u_1| = |\mathit{mkeps} \ r_1|$$

But this leads to a contradiction, because the right-hand side will be equal to the empty list, or empty string. This is because we assumed $\mathit{nullable}(r_1)$ and there is a lemma called `mkeps_flat` which shows this. On the other side we know by assumption (\mathbf{b}^{***}) and lemma `v4` that the other side needs to be a string starting with c (since we inject c into u_1). The empty string can never be equal to something starting with c ... therefore there is a contradiction.

¹Actually Isabelle found this out after analysing its argument. ;o)

That means we can only use the rule on the right-hand side to prove our goal. This implies we need to prove

$$inj\ r_1\ c\ u_1 \succ_{r_1} mkeps\ r_1$$

Case RL

The corresponding rule (instantiated) is:

$$\frac{len\ |w_1| > len\ |w_2|}{Right(w_1) \succ_{(der\ c\ r_1) \cdot r_2 + der\ c\ r_2} Left(w_2)}$$

Problems in the paper proof

I cannot verify the claim on page 17, in the *SEQ*-case under point (1):

By assumption $v_1 \cdot v_2$ is *POSIX*. Hence, we can follow that $\vdash v_1 : der\ c\ r_1$ is *POSIX* as well.