POSIX Lexing with Derivatives of Regular Expressions (Proof Pearl)

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Brzozowski's Derivatives of Regular Expressions

Idea: If r matches the string $c::s$, what is a regular expression that matches just s?

chars:

$$\begin{array}{lcl}
\mathbf{0}\backslash c & \stackrel{\text{def}}{=} & \mathbf{0}\\
\mathbf{1}\backslash c & \stackrel{\text{def}}{=} & \mathbf{0}\\
d\backslash c & \stackrel{\text{def}}{=} & \textit{if}\; d = c\; \textit{then}\; \mathbf{1}\; \textit{else}\; \mathbf{0}\\
r_1 + r_2\backslash c & \stackrel{\text{def}}{=} & r_1\backslash c \;+\; r_2\backslash c\\
r_1 \cdot r_2\backslash c & \stackrel{\text{def}}{=} & \textit{if nullable}\; r_1\\
 & & \textit{then}\; r_1\backslash c \cdot r_2 \;+\; r_2\backslash c\; \textit{else}\; r_1\backslash c \cdot r_2\\
r^*\backslash c & \stackrel{\text{def}}{=} & r\backslash c \;\cdot\; r^*
\end{array}$$

strings:

$$\begin{array}{lcl}
r\backslash [] & \stackrel{\text{def}}{=} & r\\
r\backslash c::s & \stackrel{\text{def}}{=} & (r\backslash c)\backslash s
\end{array}$$

Brzozowski's Matcher

Does r_1 match string abc?

$$r_1 \xrightarrow{\_\backslash a} r_2 \xrightarrow{\_\backslash b} r_3 \xrightarrow{\_\backslash c} r_4 \quad nullable?$$

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It is an easy exercise to formally prove (e.g. Coq, HOL, Isabelle):

$$\mathit{matches}(r, s) \text{ if and only if } s \in L(r)$$

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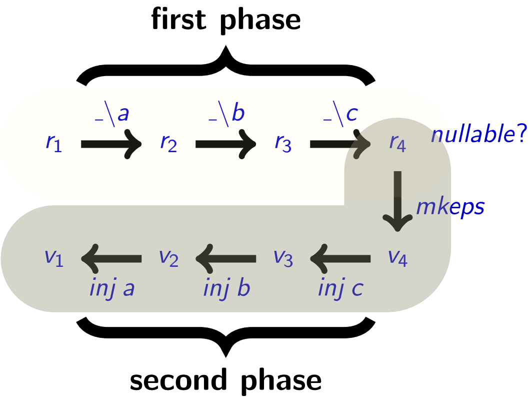
But Brzozowski's matcher gives only a yes/no-answer.

Sulzmann and Lu added a second phase in order to answer **how** the regular expression matched the string.



There are several possible answers for **how**: POSIX, GREEDY, ...

POSIX Matching (needed for Lexing)

Longest Match Rule: The longest initial substring matched by any regular expression is taken as the next token.

Rule Priority: For a particular longest initial substring, the first regular expression that can match determines the token.

For example: $r_{keywords} + r_{identifiers}$

- `iffoo␣bla`
- `if␣bla`

Problems with POSIX

Grathwohl, Henglein and Rasmussen wrote:

"The POSIX strategy is more complicated than the greedy because of the dependence on information about the length of matched strings in the various subexpressions."

Also Kuklewicz maintains a unit-test repository for POSIX matching, which indicates that most POSIX matchcers are buggy.

`http://www.haskell.org/haskellwiki/Regex_Posix`

Sulzmann and Lu Matcher

We want to match the string *abc* using r_1

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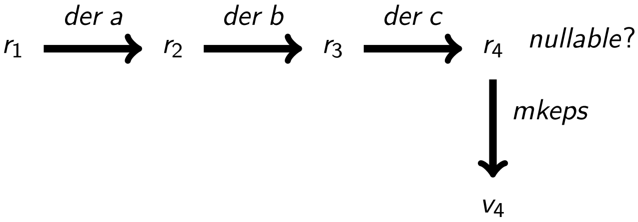

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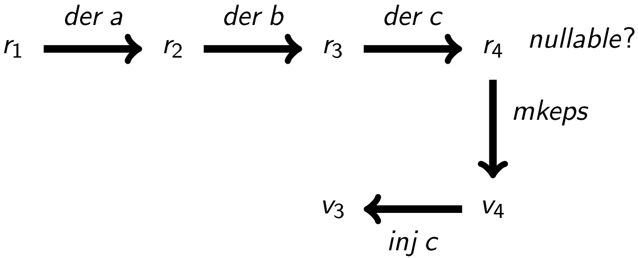

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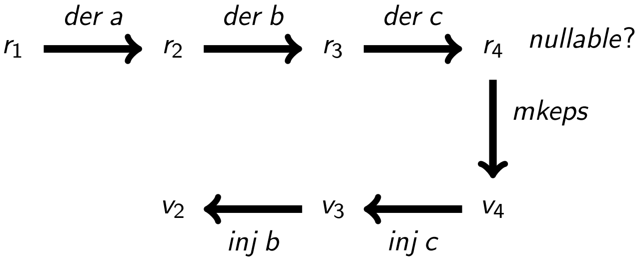

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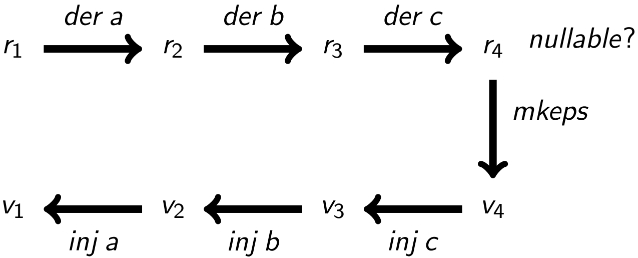

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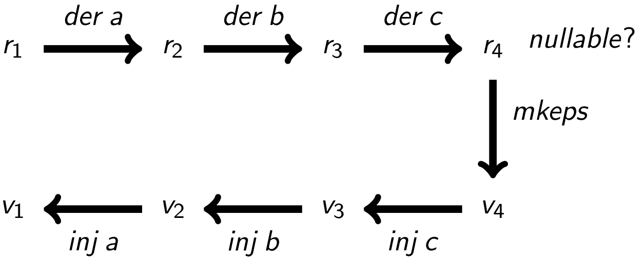

Regular Expressions and Values

Regular expressions and their corresponding values (for how a regular expression matched string):

$$
\begin{array}{lll}
r & ::= & \mathbf{0} \\
 & \mid & \mathbf{1} \\
 & \mid & c \\
 & \mid & r_1 \cdot r_2 \\
 & \mid & r_1 + r_2 \\
 & & \\
 & \mid & r^*
\end{array}
\qquad
\begin{array}{lll}
v & ::= & \\
 & \mid & Empty \\
 & \mid & Char(c) \\
 & \mid & Seq(v_1 \cdot v_2) \\
 & \mid & Left(v) \\
 & \mid & Right(v) \\
 & \mid & [v_1, \ldots, v_n]
\end{array}
$$

There is also a notion of a string behind a value $\mid v \mid$

Mkeps and Injection Functions

Mkeps Function

$$
\begin{array}{lcl}
mkeps\ (\mathbf{1}) & \stackrel{\text{def}}{=} & Empty \\
mkeps\ (r_1 \cdot r_2) & \stackrel{\text{def}}{=} & Seq\ (mkeps\ r_1)\ (mkeps\ r_2) \\
mkeps\ (r_1 + r_2) & \stackrel{\text{def}}{=} & if\ nullable\ r_1\ then\ Left\ (mkeps\ r_1) \\
 & & else\ Right\ (mkeps\ r_2) \\
mkeps\ (r^*) & \stackrel{\text{def}}{=} & Stars\ []
\end{array}
$$

Mkeps and Injection Functions

Injection Function

$$
\begin{array}{lcl}
inj\ d\ c\ () & \stackrel{\text{def}}{=} & Char\ d \\
inj\ (r_1 + r_2)\ c\ (Left\ v_1) & \stackrel{\text{def}}{=} & Left\ (inj\ r_1\ c\ v_1) \\
inj\ (r_1 + r_2)\ c\ (Right\ v_2) & \stackrel{\text{def}}{=} & Right\ (inj\ r_2\ c\ v_2) \\
inj\ (r_1 \cdot r_2)\ c\ (Seq\ v_1\ v_2) & \stackrel{\text{def}}{=} & Seq\ (inj\ r_1\ c\ v_1)\ v_2 \\
inj\ (r_1 \cdot r_2)\ c\ (Left\ (Seq\ v_1\ v_2)) & \stackrel{\text{def}}{=} & Seq\ (inj\ r_1\ c\ v_1)\ v_2 \\
inj\ (r_1 \cdot r_2)\ c\ (Right\ v_2) & \stackrel{\text{def}}{=} & Seq\ (mkeps\ r_1)\ (inj\ r_2\ c\ v_2) \\
inj\ (r^*)\ c\ (Seq\ v\ (Stars\ vs)) & \stackrel{\text{def}}{=} & Stars\ (inj\ r\ c\ v{::}vs)
\end{array}
$$

POSIX Ordering Relation by Sulzmann & Lu

- Introduce an inductive defined ordering relation $v \succ_r v'$ which captures the idea of POSIX matching.
- The algorithm returns the maximum of all possible values that are possible for a regular expression.
- The idea is from a paper by Frisch & Cardelli about GREEDY matching (GREEDY $=$ preferring instant gratification to delayed repletion)

Problems

- Sulzmann: …Let's assume v is not a *POSIX* value, then there must be another one …contradiction.
- Exists ?

$$L(r) \neq \varnothing \;\Rightarrow\; \exists v.\ POSIX(v, r)$$

- In the sequence case $Seq(v_1, v_2) \succ_{r_1 \cdot r_2} Seq(v_1', v_2')$, the induction hypotheses require $|v_1| = |v_1'|$ and $|v_2| = |v_2'|$, but you only know

$$|v_1| @ |v_2| = |v_1'| @ |v_2'|$$

- Although one begins with the assumption that the two values have the same flattening, this cannot be maintained as one descends into the induction (alternative, sequence)

Our Solution

- A direct definition of what a POSIX value is, using the relation $s \in r \to v$ (our specification)

$$\frac{}{[] \in \mathbf{1} \to \mathit{Empty}} \qquad \frac{}{[c] \in c \to \mathit{Char}(c)}$$

$$\frac{s \in r_1 \to v}{s \in r_1 + r_2 \to \mathit{Left}(v)} \qquad \frac{s \in r_2 \to v \quad s \notin L(r_1)}{s \in r_1 + r_2 \to \mathit{Right}(v)}$$

$$\frac{\begin{array}{l} s_1 \in r_1 \to v_1 \\ s_2 \in r_2 \to v_2 \\ \neg(\exists s_3\, s_4.\ s_3 \neq [] \wedge s_3 @ s_4 = s_2 \wedge s_1 @ s_3 \in L(r_1) \wedge s_4 \in L(r_2)) \end{array}}{s_1 @ s_2 \in r_1 \cdot r_2 \to \mathit{Seq}(v_1, v_2)}$$

...

Properties

It is almost trival to prove:

- Uniqueness

  If $s \in r \rightarrow v_1$ and $s \in r \rightarrow v_2$ then $v_1 = v_2$

- Correctness

  $lexer(r, s) = v$ if and only if $s \in r \rightarrow v$

Properties

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  If $s \in r \rightarrow v_1$ and $s \in r \rightarrow v_2$ then $v_1 = v_2$

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  $lexer(r, s) = v$ if and only if $s \in r \rightarrow v$

You can now start to implement optimisations and derive correctness proofs for them.

Conclusions

- Sulzmann and Lu's informal proof contained small gaps (acknowledged) but we believe it had also fundamental flaws
- We replaced the POSIX definition of Sulzmann & Lu by a new definition (ours is inspired by work of Vansummeren, 2006)
- Now, its a nice exercise for theorem proving
- Some optimisations need to be applied to the algorithm in order to become fast enough
- Can be used for lexing, is a small beautiful functional program