

A Computational Interpretation of Context-Free Expressions

Martin Sulzmann¹ and Peter Thiemann²

¹ Faculty of Computer Science and Business Information Systems
Karlsruhe University of Applied Sciences
Moltkestrasse 30, 76133 Karlsruhe, Germany
`martin.sulzmann@hs-karlsruhe.de`

² Faculty of Engineering, University of Freiburg
Georges-Köhler-Allee 079, 79110 Freiburg, Germany
`thiemann@acm.org`

Abstract. We phrase parsing with context-free expressions as a type inhabitation problem where values are parse trees and types are context-free expressions. We first show how containment among context-free and regular expressions can be reduced to a reachability problem by using a canonical representation of states. The proofs-as-programs principle yields a computational interpretation of the reachability problem in terms of a coercion that transforms the parse tree for a context-free expression into a parse tree for a regular expression. It also yields a partial coercion from regular parse trees to context-free ones. The partial coercion from the trivial language of all words to a context-free expression corresponds to a predictive parser for the expression.

1 Introduction

In the context of regular expressions, there have been a number of works which give a *computational* interpretation of regular expressions. For example, Frisch and Cardelli [4] show how to phrase the regular expression parsing problem as a type inhabitation problem. Parsing usually means that for an input string that matches a regular expression we obtain a parse tree which gives a precise explanation which parts of the regular expression have been matched. By interpreting parse trees as values and regular expressions as types, parsing can be rephrased as type inhabitation as shown by Frisch and Cardelli. Henglein and Nielsen [6] as well Lu and Sulzmann [8, 12], formulate containment of regular expressions as a type conversion problem. From a containment proof, they derive a transformation (a type coercion) from parse trees of one regular expression into parse trees of the other regular expression.

This paper extends these ideas to the setting of context-free expressions. Context-free expressions extend regular expressions with a least fixed point operator, so they are effectively equivalent to context-free grammars. An essential new idea is to phrase the containment problem among context-free expressions and regular expressions as a reachability problem [11], where states are represented

by regular expressions and reachable states are Brzozowski-style derivatives [3]. By characterizing the reachability problem in terms of a natural-deduction style proof system, we can apply the proofs-are-programs principle to extract the coercions that implement the desired transformation between parse trees.

In summary, our contributions are:

- an interpretation of context-free expressions as types which are inhabited by valid parse trees (Section 3);
- a reduction of containment among context-free expressions and regular expressions to a reachability problem (Section 4);
- a formal derivation of coercions between context-free and regular parse trees extracted from a natural-deduction style proof of context-free reachability (Section 5).

The optional appendix contains further details such as proofs etc.

2 Preliminaries

This section introduces some basic notations including the languages of regular and context-free expressions and restates some known results for Brzozowski style derivatives.

Let Σ be a finite set of symbols with x, y , and z ranging over Σ . We write Σ^* for the set of finite words over Σ , ε for the empty word, and $v \cdot w$ for the concatenation of words v and w . A language is a subset of Σ^* .

Definition 1 (Regular Expressions). *The set RE of regular expressions is defined inductively by*

$$r, s ::= \phi \mid \varepsilon \mid x \in \Sigma \mid (r + s) \mid (r \cdot s) \mid (r^*)$$

We omit parentheses by assuming that $*$ binds tighter than \cdot and \cdot binds tighter than $+$.

Definition 2 (Regular Languages). *The meaning function L maps a regular expression to a language. It is defined inductively as follows:*

$$L(\phi) = \{\}. \quad L(\varepsilon) = \{\varepsilon\}. \quad L(x) = \{x\}. \quad L(r + s) = L(r) \cup L(s). \quad L(r \cdot s) = \{v \cdot w \mid v \in L(r) \wedge w \in L(s)\}. \quad L(r^*) = \{w_1 \cdot \dots \cdot w_n \mid n \geq 0 \wedge \forall i \in \{1, \dots, n\}. w_i \in L(r)\}.$$

We say that regular expressions r and s are equivalent, $r \equiv s$, if $L(r) = L(s)$.

Definition 3 (Nullability). *A regular expression r is nullable if $\varepsilon \in L(r)$.*

The *derivative* of a regular expression r with respect to some symbol x , written $d_x(r)$, is a regular expression for the left quotient of $L(r)$ with respect to x . That is, $L(d_x(r)) = \{w \in \Sigma^* \mid x \cdot w \in L(r)\}$. A derivative $d_x(r)$ can be computed by recursion over the structure of the regular expression r .

Definition 4 (Brzowski Derivatives [3]).

$$\begin{aligned}
d_x(\phi) &= \phi & d_x(\varepsilon) &= \phi \\
d_x(y) &= \begin{cases} \varepsilon & \text{if } x = y \\ \phi & \text{otherwise} \end{cases} & d_x(r + s) &= d_x(r) + d_x(s) \\
d_x(r \cdot s) &= \begin{cases} d_x(r) \cdot s & \text{if } \varepsilon \notin L(r) \\ d_x(r) \cdot s + d_x(s) & \text{otherwise} \end{cases} & d_x(r^*) &= d_x(r) \cdot r^*
\end{aligned}$$

Example 1. The derivative of $(x+y)^*$ with respect to symbol x is $(\varepsilon + \phi) \cdot (x+y)^*$. The calculation steps are as follows:

$$d_x((x+y)^*) = d_x(x+y) \cdot (x+y)^* = (d_x(x) + d_x(y)) \cdot (x+y)^* = (\varepsilon + \phi) \cdot (x+y)^*$$

Theorem 1 (Expansion [3]). *Every regular expression r can be represented as the sum of its derivatives with respect to all symbols. If $\Sigma = \{x_1, \dots, x_n\}$, then*

$$r \equiv x_1 \cdot d_{x_1}(r) + \dots + x_n \cdot d_{x_n}(r) \quad (+\varepsilon \text{ if } r \text{ nullable})$$

Definition 5 (Descendants and Similarity). *A descendant of r is either r itself or the derivative of a descendant. We say r and s are similar, written $r \sim s$, if one can be transformed into the other by finitely many applications of the rewrite rules (Idempotency) $r + r \sim r$, (Commutativity) $r + s \sim s + r$, (Associativity) $r + (s + t) \sim (r + s) + t$, (Elim1) $\varepsilon \cdot r \sim r$, (Elim2) $\phi \cdot r \sim \phi$, (Elim3) $\phi + r \sim r$, and (Elim4) $r + \phi \sim r$.*

Lemma 1. *Similarity is an equivalence relation that respects regular expression equivalence: $r \sim s$ implies $r \equiv s$.*

Theorem 2 (Finiteness [3]). *The elements of the set of descendants of a regular expression belong to finitely many similarity equivalence classes.*

Similarity rules (Idempotency), (Commutativity), and (Associativity) suffice to achieve finiteness. Elimination rules are added to obtain a compact *canonical representative* for equivalence class of similar regular expressions. The canonical form is obtained by systematic application of the similarity rules in Definition 5. We enforce right-associativity of concatenated expressions, sort alternative expressions according to their size and their first symbol, and concatenations lexicographically, assuming an arbitrary total order on Σ . We further remove duplicates and apply elimination rules exhaustively (the details are standard [5]).

Definition 6 (Canonical Representatives). *For a regular expression r , we write $\text{cnf}(r)$ to denote the canonical representative among all expressions similar to r . We write $D(r)$ for the set of canonical representatives of the finitely many dissimilar descendants of r .*

Example 2. We find that $\text{cnf}((\varepsilon + \phi) \cdot (x+y)^*) = (x+y)^*$ where we assume $x < y$.

Context-free expressions [13] extend regular expressions with a least fixed point operator μ . Our definition elides the Kleene star operator because it can be defined with the fixed point operator: $e^* = \mu\alpha.e \cdot \alpha + \varepsilon$.

Definition 7 (Context-Free Expressions). *Let A be a denumerable set of placeholders disjoint from Σ . The set CFE of context-free expressions is defined inductively by*

$$e, f ::= \phi \mid \varepsilon \mid x \in \Sigma \mid \alpha \in A \mid e + f \mid e \cdot f \mid \mu\alpha.e$$

We only consider closed context-free expressions where (A) all placeholders are bound by some enclosing μ -operator and (B) the placeholder introduced by a μ -operator must be distinct from all enclosing bindings $\mu\alpha$. Requirement (A) guarantees that reduction of a context-free expression does not get stuck whereas requirement (B) ensures that there are no name clashes when manipulating a context-free expression.

While Winter et al [13] define the semantics of a context-free expression by coalgebraic means, we define its meaning with a natural-deduction style big-step semantics.

Definition 8 (Big-Step Semantics). *The reduction relation $\Rightarrow \subseteq CFE \times \Sigma^*$ is defined inductively by the following inference rules.*

$$\varepsilon \Rightarrow \varepsilon \quad x \Rightarrow x \quad \frac{e \Rightarrow w}{e + f \Rightarrow w} \quad \frac{f \Rightarrow w}{e + f \Rightarrow w} \quad \frac{e \Rightarrow v \quad f \Rightarrow w}{e \cdot f \Rightarrow v \cdot w} \quad \frac{[\alpha \mapsto \mu\alpha.e](e) \Rightarrow w}{\mu\alpha.e \Rightarrow w}$$

In the last rule, we write $[\alpha \mapsto \mu\alpha.e](e)$ to denote the expression obtained by replacing all occurrences of placeholder α in e by $\mu\alpha.e$. If $\mu\alpha.e$ is closed, then requirement (B) ensures that there is no inadvertent capture of placeholders.

We further define $L(e) = \{w \in \Sigma^* \mid e \Rightarrow w\}$.

As an immediate consequence of the last rule, we see that unfolding does not affect the language.

Lemma 2. $L(\mu\alpha.e) = L([\alpha \mapsto \mu\alpha.e](e))$.

Definition 9 (Containment). *Let e be a context-free expression or regular expression and let r be a regular expression. We define $e \leq r$ iff $L(e) \subseteq L(r)$.*

We express partial functions as total functions composed with lifting as follows. Let A and B be sets. The set *Maybe B* consists of elements which are either *Nothing* or of the form *Just b*, for $b \in B$. Thus a total function f' of type $A \rightarrow \text{Maybe } B$ corresponds uniquely to a partial function f from A to B : for $a \in A$, if $f(a)$ is not defined, then $f'(a) = \text{Nothing}$; if $f(a) = b$ is defined, then $f'(a) = \text{Just } b$; and vice versa.

3 Parsing as Type Inhabitation

Parsing for regular expressions has been phrased as a type inhabitation problem [4]. We follow suit and generalize this approach to parsing for context-free expressions. For our purposes, parse trees are generated by the following grammar.

Definition 10 (Parse Trees for context-free expressions).

$$p, q ::= \text{EPS} \mid \text{SYM } x \mid \text{INL } p \mid \text{INR } q \mid \text{SEQ } p \ q \mid \text{FOLD } p$$

Like a derivation tree for a context-free grammar, a parse tree is a structured representation of the derivation of a word from some context-free expression. The actual word can be obtained by flattening the parse tree.

Definition 11 (Flattening).

$$\begin{aligned} \text{flatten}(\text{EPS}) &= \varepsilon & \text{flatten}(\text{SYM } x) &= x \\ \text{flatten}(\text{INL } p) &= \text{flatten}(p) & \text{flatten}(\text{INR } q) &= \text{flatten}(q) \\ \text{flatten}(\text{SEQ } p \ q) &= \text{flatten}(p) \cdot \text{flatten}(q) & \text{flatten}(\text{FOLD } p) &= \text{flatten}(p) \end{aligned}$$

Compared to derivation trees whose signatures depend on the underlying grammar, parse trees are generic, but their validity depends on the particular context-free expression. The connection between parse trees and context-free expressions is made via the following typing relation where we interpret context-free expressions as types and parse trees as values.

Definition 12 (Valid Parse Trees, $\vdash p : e$).

$$\begin{aligned} \vdash \text{EPS} : \varepsilon \quad \vdash \text{SYM } x : x & \quad \frac{\vdash p : e \quad \vdash q : f}{\vdash \text{SEQ } p \ q : e \cdot f} \\ \frac{\vdash p : e}{\vdash \text{INL } p : e + f} \quad \frac{\vdash p : f}{\vdash \text{INR } p : e + f} & \quad \frac{\vdash p : [\alpha \mapsto \mu\alpha.e](e)}{\vdash \text{FOLD } p : \mu\alpha.e} \end{aligned}$$

We consider ε as a singleton type with value EPS as its only inhabitant. The concatenation operator \cdot effectively corresponds to a pair where pair values are formed via the binary constructor SEQ . We treat $+$ as a disjoint sum with the respective injection constructors INL and INR . Recursive μ -expressions represent iso-recursive types with FOLD denoting the isomorphism between the unrolling of a recursive type and the recursive type itself.

The following results establish that parse trees obtained via the typing relation can be related to words derivable in the language of context-free expression and vice versa.

Lemma 3. *Let e be a context-free expression and w be a word. If $e \Rightarrow w$, then there exists a parse tree p such that $\vdash p : e$ where $\text{flatten}(p) = w$.*

Lemma 4. *Let e be a context-free expression and p a parse tree. If $\vdash p : e$, then $e \Rightarrow \text{flatten}(p)$.*

Example 3. Let $p = \text{FOLD}(\text{INL}(\text{SEQ}(\text{SYM } x)(\text{SEQ}(\text{INR EPS})(\text{SYM } x))))$ be a parse tree and consider the expression $e = \mu\alpha.x \cdot \alpha + \varepsilon$. We find that $\vdash p : e$ and $\text{flatten}(p) = x \cdot x$.

Instead of tackling the parsing problem, we solve the more general problem of coercing parse trees of context-free expressions into parse trees of regular expressions and vice versa.

4 Containment via Reachability

In this section, we consider the problem of determining containment ($e \leq r$)? between a context-free language represented by some expression e and a regular language represented by regular expression r . This problem is decidable. The standard algorithm constructs a context-free grammar for the intersection $L(e) \cap \overline{L(r)}$ and tests it for emptiness.

We proceed differently to obtain some computational content from the proof of containment. We first rephrase the containment problem ($e \leq r$)? as a reachability problem. Then, in Section 5, we extract computational content by deriving suitable coercions as mappings between the respective parse trees of e and r .

There are coercions in both directions:

1. a total coercion from $L(e)$ to $L(r)$ as a mapping of type $e \rightarrow r$ and
2. a partial coercion from $L(r)$ to $L(e)$ as a mapping of type $r \rightarrow \text{Maybe } e$,

The partial coercion under 2 can be considered as a parser specialized to words from $L(r)$. Thus, the partial coercion from $\Sigma^* \rightarrow \text{Maybe } e$ is a general parser for $L(e)$.

We say that a regular expression r' is reachable from $e \in CFE$ and r if there is some word $w \in L(e)$ such that $L(r') = w/L(r) = \{v \in \Sigma^* \mid w \cdot v \in L(r)\}$. To obtain a finite representation, we define reachability in terms of canonical representatives of derivatives.

Definition 13 (Reachability). *Let e be a context-free expression and r a regular expression. We define the set of reachable expressions as $\text{reach}(e, r) = \{\text{cnf}(d_w(r)) \mid w \in \Sigma^*, e \Rightarrow w\}$.*

Theorem 3. *Let e be a context-free expression and r be a regular expression. Then $e \leq r$ iff each expression in $\text{reach}(e, r)$ is nullable.*

By finiteness of dissimilar descendants the set $\text{reach}(e, r)$ is finite and can be computed effectively via a least fixed point construction. Thus, we obtain a new algorithm for containment by reduction to decidable reachability and nullability.

$$\boxed{\Gamma \vdash r \overset{e}{\rightsquigarrow} S}$$

$$\begin{array}{l}
\text{(Eps)} \Gamma \vdash r \overset{e}{\rightsquigarrow} \{cnf(r)\} \quad \text{(Phi)} \Gamma \vdash r \overset{\phi}{\rightsquigarrow} \{\} \quad \text{(Sym)} \Gamma \vdash r \overset{x}{\rightsquigarrow} \{cnf(d_x(r))\} \\
\text{(Alt)} \frac{\Gamma \vdash r \overset{e}{\rightsquigarrow} S_1 \quad \Gamma \vdash r \overset{f}{\rightsquigarrow} S_2}{\Gamma \vdash r \overset{e+f}{\rightsquigarrow} S_1 \cup S_2} \\
\text{(Seq)} \frac{\Gamma \vdash r \overset{e}{\rightsquigarrow} \{r_1, \dots, r_n\} \quad \Gamma \vdash r_i \overset{f}{\rightsquigarrow} S_i \text{ for } i = 1, \dots, n}{\Gamma \vdash r \overset{e+f}{\rightsquigarrow} S_1 \cup \dots \cup S_n} \\
\text{(Rec)} \frac{\Gamma \cup \{r \overset{\mu\alpha.f}{\rightsquigarrow} S\} \vdash r \overset{[\alpha \mapsto \mu\alpha.f](f)}{\rightsquigarrow} S}{\Gamma \vdash r \overset{\mu\alpha.f}{\rightsquigarrow} S} \quad \text{(Hyp)} \frac{r \overset{\mu\alpha.f}{\rightsquigarrow} S \in \Gamma}{\Gamma \vdash r \overset{\mu\alpha.f}{\rightsquigarrow} S}
\end{array}$$

Fig. 1. Reachability proof system

Instead of showing the least fixed point construction, we give a characterization of the set of reachable expressions in terms of a natural-deduction style proof system. The least fixed point construction follows from the proof rules.

The system in Figure 1 defines the judgment $r \overset{e}{\rightsquigarrow} S$ where $e \in CFE$, r a regular expression, and S is a set of regular expressions in canonical form. It makes use of a set Γ of hypothetical proof judgments of the same form. The meaning of a judgment is that S (over)approximates $reach(e, r)$ (see upcoming Lemmas 5 and 6).

The interesting rules are (Rec) and (Hyp). In rule (Hyp), we look up a proof judgment for a context-free expression with topmost operator μ from the assumption set Γ . Such proof judgments are added to Γ in rule (Rec). Hence, we can make use of to be verified proof judgments in subsequent proof steps. Hence, the above proof system is defined coinductively. Soundness of the proof system is guaranteed by the fact that we unfold the fixpoint operator μ in rule (Rec). We can indeed show soundness and completeness: the set $reach(e, r)$ is derivable and any derivable set S is a superset of $reach(e, r)$.

Lemma 5. *Let e be a context-free expression and r be a regular expression. Then, $\vdash r \overset{e}{\rightsquigarrow} reach(e, r)$ is derivable.*

Lemma 6. *Let e be a context-free expression, r be a regular expression and S be a set of expressions such that $\vdash r \overset{e}{\rightsquigarrow} S$. Then, we find that $S \supseteq reach(e, r)$.*

Example 4. Consider $e = \mu\alpha.x \cdot (\alpha \cdot y) + \varepsilon$ and $r = x^* \cdot y^*$. It is easy to see that $\text{reach}(e, r) = \{r, y^*\}$. Indeed, we can verify that $\{\} \vdash r \overset{e}{\rightsquigarrow} \{r, y^*\}$ is derivable.

$$\begin{array}{c}
\text{(Hyp)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{e}{\rightsquigarrow} \{r, y^*\} \checkmark \\
\text{(Sym)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{y}{\rightsquigarrow} \{y^*\} \checkmark \\
\text{(Sym)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash y^* \overset{y}{\rightsquigarrow} \{y^*\} \checkmark \\
\text{(SEQ)} \quad \frac{\text{(Sym)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash y^* \overset{y}{\rightsquigarrow} \{y^*\} \checkmark}{\{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{e \cdot y}{\rightsquigarrow} \{y^*\}} \\
\text{(SEQ)} \quad \frac{\text{(Sym)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{x}{\rightsquigarrow} \{r\} \checkmark}{\{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{x \cdot (e \cdot y)}{\rightsquigarrow} \{y^*\}} \\
\text{(Eps)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{\varepsilon}{\rightsquigarrow} \{r\} \checkmark \\
\text{(ALT)} \quad \frac{\text{(Eps)} \quad \{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{\varepsilon}{\rightsquigarrow} \{r\} \checkmark}{\{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{x \cdot (e \cdot y) + \varepsilon}{\rightsquigarrow} \{r, y^*\}} \\
\text{(REC)} \quad \frac{\{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{x \cdot (e \cdot y) + \varepsilon}{\rightsquigarrow} \{r, y^*\}}{\{\} \vdash r \overset{e}{\rightsquigarrow} \{r, y^*\}}
\end{array}$$

We first apply rule (Rec) followed by (Alt). One of the premises of (Alt) can be verified immediately via (Eps) as indicated by \checkmark . For space reasons, we write premises on top of each other. Next, we apply (Seq) where one of the premises can be verified immediately again. Finally, we find another application of (Seq). $\{r \overset{e}{\rightsquigarrow} \{r, y^*\}\} \vdash r \overset{e}{\rightsquigarrow} \{r, y^*\}$ holds due to (Hyp). Because the reachable set contains two elements, r and y^* , we find two applications of (Sym) and we are done.

Example 5. As a special case, consider $e = \mu\alpha.\alpha$ where $\text{reach}(e, r) = \{\}$ for any regular expression r . The reachability proof system over-approximates and indeed we find that $\vdash r \overset{\mu\alpha.\alpha}{\rightsquigarrow} S$ for any S as shown by the following derivation

$$\text{(REC)} \quad \frac{\text{(Hyp)} \quad \{r \overset{\mu\alpha.\alpha}{\rightsquigarrow} S\} \vdash r \overset{\mu\alpha.\alpha}{\rightsquigarrow} S}{\vdash r \overset{\mu\alpha.\alpha}{\rightsquigarrow} S}$$

5 Coercions

Our proof system for the reachability judgment $r \overset{e}{\rightsquigarrow} S$ in Figure 1 provides a coinductive characterization of the set of reachable expressions. Now we apply the proofs-are-programs principle to derive coercions from derivation trees for reachability. As the proof system is coinductive, we obtain recursive coercions from applications of the rules (Rec) and (Hyp).

Our first step is to define a term language for coercions, which are functions on parse trees. This language turns out to be a lambda calculus (lambda abstraction, function application, variables) with recursion and pattern matching on parse trees.

Definition 14 (Coercion Terms). *Coercion terms c and patterns pat are inductively defined by*

$$\begin{aligned}
c & ::= v \mid k \mid \lambda v.c \mid c \ c \mid \text{rec } x.c \mid \text{case } c \text{ of } [pat_1 \Rightarrow c_1, \dots, pat_n \Rightarrow c_n] \\
pat & ::= v \mid k \ pat_1 \ \dots \ pat_{arity(k)}
\end{aligned}$$

where v range over a denumerable set of variables disjoint from Σ and constructors k are taken from the set $\mathcal{K} = \{\text{EPS}, \text{SEQ}, \text{INL}, \text{INR}, \text{FOLD}, \text{Just}, \text{Nothing}, (-, -)\}$. Constructors $\text{EPS}, \dots, \text{FOLD}$ are employed in the formation of parse trees. Constructors Just and Nothing belong to the *Maybe* type that arises in the construction of partial coercions. The binary constructor $(-, -)$ builds a pair. The function $\text{arity}(k)$ defines the arity of constructor k . Patterns are linear (i.e., all pattern variables are distinct) and we write $\lambda \text{pat}.c$ as a shorthand for $\lambda v.\text{case } v \text{ of } [\text{pat} \Rightarrow c]$.

We give meaning to coercions in terms of a standard denotational semantics where values are elements of a complete partial order formed over the set of parse trees and function space. We write η to denote the mapping from variables to values and $\llbracket c \rrbracket \eta$ to denote the meaning of coercions where η defines the meaning of free variables in c . In case c is closed, we simply write $\llbracket c \rrbracket$.

Earlier work shows how to construct coercions that demonstrate containment among regular expressions [8, 12]. These works use a specialized representation for Kleene star which would require to extend Definitions 10 and 12. We avoid any special treatment of the Kleene star by considering r^* an abbreviation for $\mu \alpha.r \cdot \alpha + \varepsilon$. The representations suggested here is isomorphic to the one used in previous work [8, 12]. We summarize their main results. We adopt the convention that t refers to parse trees of regular expressions, b refers to coercions manipulating regular parse trees. We write $b : r \rightarrow s$ to denote a coercion of type $r \rightarrow s$, and we use $\vdash_r t : r$ for the regular typing judgment.

Definition 15 (Parse Trees for Regular Expressions).

$$t ::= \text{EPS} \mid \text{SYM } x \mid \text{INL } t \mid \text{INR } t \mid \text{SEQ } t_1 t_2 \mid \text{FOLD } t$$

Definition 16 (Valid Regular Parse Trees, $\vdash_r t : r$).

$$\begin{array}{c} \vdash_r \text{EPS} : \varepsilon \quad \vdash_r \text{SYM } x : x \quad \frac{\vdash_r t_1 : r \quad \vdash_r t_2 : s}{\vdash_r \text{SEQ } t_1 t_2 : r \cdot s} \quad \frac{\vdash_r t : r}{\vdash_r \text{INL } t : r + s} \\ \\ \frac{\vdash_r t : s}{\vdash_r \text{INR } t : r + s} \quad \vdash_r \text{FOLD (INR EPS)} : r^* \\ \\ \frac{\vdash_r t_1 : r \quad \vdash_r t_2 : r^*}{\vdash_r \text{FOLD (INL (SEQ } t_1 t_2)) : r^*} \end{array}$$

Lemma 7 (Regular Coercions [8, 12]). *Let r and s be regular expressions such that $r \leq s$. There is an algorithm to obtain coercions $b_1 : r \rightarrow s$ and $b_2 : s \rightarrow \text{Maybe } r$ such that (1) for any $\vdash_r t : r$ we have that $\vdash_r b_1(t) : s$, $\llbracket b_1(t) \rrbracket = t'$ for some t' and $\text{flatten}(t) = \text{flatten}(t')$, and (2) for any $\vdash_r t : s$ where $\text{flatten}(t) \in L(r)$ we have that $\llbracket b_2(t) \rrbracket = \text{Just } t'$ for some t' where $\vdash_r t' : r$ and $\text{flatten}(t) = \text{flatten}(t')$, and (3) for any $\vdash_r t : s$ where $\text{flatten}(t) \notin L(r)$, $b_2(t) = \text{Nothing}$.*

Reachability coercions are derived via the judgment $\Delta \vdash^\uparrow c : \mathbb{U}(e, r)$, which states that under environment Δ an upcast coercion c of type $\mathbb{U}(e, r)$ can be constructed. Environments Δ are defined by $\Delta ::= \{ \} \mid \{v : \mathbb{U}(e, r)\} \mid \Delta \cup \Delta$ and record coercion assumptions, which are needed to construct recursive coercions. We interpret $\mathbb{U}(e, r)$ as the type $(e \times +reach(e, r)) \rightarrow r$. Figure 2 contains the proof rules which are derived from Figure 1 by decorating each rule with an appropriate coercion term. If Δ is empty, we write $\vdash^\uparrow c : \mathbb{U}(e, r)$ for short.

The proof rules in Figure 2 are decidable in the sense that it is decidable if $\Delta \vdash^\uparrow c : \mathbb{U}(e, r)$ can be derived. This property holds because proof rules are syntax-directed and $reach(e, r)$ is decidable. We can also attempt to infer c where we either fail or succeed in a finite number of derivation steps.

Lemma 8 (Upcast Soundness). *Let e be a context-free expression and r be a regular expression such that $\vdash^\uparrow c : \mathbb{U}(e, r)$ for some coercion c . Let p and t be parse trees such that $\vdash p : e$ and $\vdash_r t : +reach(e, r)$ where $flatten(t) \in L(d_{flatten(p)}(r))$. Then, we find that $\llbracket c((p, t)) \rrbracket = t'$ for some t' where $\vdash t' : r$ and $flatten(p) = flatten(t')$.*

The assumption $flatten(t) \in L(d_{flatten(p)}(r))$ guarantees that e 's parse tree p in combination with $+reach(e, r)$'s parse tree t allows us to build a parse tree for r .

For example, consider rule $(Alt)_u$. Suppose $e + f$ parses some input word w because e parses the word w . That is, w 's parse tree has the form $p = \text{INL } p_1$. As we have proofs that $e \leq r$ and $f \leq r$, the downcast $b_1(t)$ cannot fail and yields $\text{Just } t_1$. Formally, we have $\vdash p_1 : e$ and conclude that $flatten(p) = flatten(p_1) \in L(e)$. By Lemma 4, $e \Rightarrow flatten(p_1)$ and therefore we find that $d_{flatten(p_1)}(r)$ is similar to an element of $reach(e, r)$. Because $flatten(t) \in L(d_{flatten(p)}(r))$ we conclude that $flatten(t) \in L(+reach(e, r))$. By Lemma 7, it must be that $b_1(t) = \text{Just } t_1$ for some t_1 where $\vdash t_1 : +reach(e, r)$. By induction the result holds for c_1 and hence we can establish the result for c .

In rule $(Seq)_u$, we exploit the fact that $+reach(e \cdot f, r) = +reach(f, +reach(e, r))$. So, we use coercion c_2 to build a parse tree of $+reach(e, r)$ given parse trees of f and $+reach(e \cdot f, r)$. Then, we build a parse tree of r by applying c_1 to parse trees of e and $+reach(e, r)$.

Due to the coinductive nature of the coercion proof system, coercion terms may be recursive as evidenced by rule $(Rec)_u$. Soundness is guaranteed by the assumption that the set of reachable states is non-empty. As we find a parse tree of that type, progress is made when building the coercion for the unfolded μ -expression. Unfolding must terminate because there are only finitely many combinations of unfolded subterms of the form $\mu\alpha.e$ and regular expressions r . The latter are drawn from the finitely many dissimilar descendant of some r . Hence, resulting coercions must be well-defined as stated in the above result.

Example 6. We show how to derive $\vdash^\uparrow c_0 : \mathbb{U}(e, r)$ where $e = \mu\alpha.x \cdot (\alpha \cdot y) + \varepsilon$, $r = x^* \cdot y^*$ and $reach(e, r) = \{r, y^*\}$. The shape of the derivation tree corresponds

to the derivation we have seen in Example 4.

$$\begin{array}{c}
\text{(Hyp)}_u \Delta \vdash^\uparrow c_7 : \mathbb{U}(e, r) \checkmark \\
\text{(Sym)}_u \Delta \vdash^\uparrow c_6 : \mathbb{U}(y, r + y^*) \checkmark \\
\text{(SEQ)}_u \frac{\Delta \vdash^\uparrow c_5 : \mathbb{U}(e \cdot y, r)}{\Delta \vdash^\uparrow c_4 : \mathbb{U}(x, r) \checkmark} \\
\text{(Sym)}_u \Delta \vdash^\uparrow c_4 : \mathbb{U}(x, r) \checkmark \\
\text{(SEQ)}_u \frac{\Delta \vdash^\uparrow c_3 : \mathbb{U}(x \cdot (e \cdot y), r)}{\Delta \vdash^\uparrow c_2 : \mathbb{U}(\varepsilon, r) \checkmark} \\
\text{(Eps)}_u \Delta \vdash^\uparrow c_2 : \mathbb{U}(\varepsilon, r) \checkmark \\
\text{(ALT)}_u \frac{\Delta \vdash^\uparrow c_1 : \mathbb{U}(x \cdot (e \cdot y) + \varepsilon, r)}{\Delta \vdash^\uparrow c_0 : \mathbb{U}(e, r)} \\
\text{(REC)}_u \frac{\Delta \vdash^\uparrow c_0 : \mathbb{U}(e, r)}{\vdash^\uparrow c_0 : \mathbb{U}(e, r)}
\end{array}$$

We fill in the details by following the derivation tree from bottom to top. We set $\Delta = \{v_{\alpha.e,r} : \mathbb{U}(e, r)\}$. From the first (Rec)_u step we conclude $c_0 = \text{rec } v_{\alpha.e,r} . \lambda(\text{FOLD } p, t) . c_1(p, t)$. Next, we find (Alt)_u which yields

$$\begin{aligned}
c_1 = \lambda(p, t) . \text{case } p \text{ of } [\\
& \text{INL } p_1 \Rightarrow \text{case } (b_1(t)) \text{ of } [\text{Just } t_1 \Rightarrow c_3(p_1, t_1)], \\
& \text{INR } p_2 \Rightarrow \text{case } (b_2(t)) \text{ of } [\text{Just } t_2 \Rightarrow c_2(p_2, t_2)]
\end{aligned}$$

We consider the definition of the auxiliary regular (downcast) coercions b_1 and b_2 . We have that $+reach(x \cdot (e \cdot y) + \varepsilon, r) = r + y^*$, $+reach(\varepsilon, r) = r$ and $+reach(x \cdot (e \cdot y), r) = y^*$. Hence, we need to derive $y^* \leq_{b_1} r + y^*$ and $r \leq_{b_2} r + y^*$.

Recall the requirement (2) for downcast coercions. See Lemma 7. We first consider $y^* \leq_{b_1} r + y^*$. The right component of the sum can be straightforwardly coerced into a parse tree of y^* . For the left component we need to check that the leading part is effectively empty. Recall that Kleene star is represented in terms of μ -expressions. Following Definition 16, an empty parse tree for Kleene star equals FOLD (INR EPS). Thus, we arrive at the following definition for b_1 .

$$\begin{array}{c}
b_1 = \lambda t . \text{case } t \text{ of } [\\
\text{INL } (\text{SEQ } (\text{FOLD } \text{INR } \text{EPS}) v) \Rightarrow \text{Just } v, \\
\text{INL } v \Rightarrow \text{Nothing}, \\
\text{INR } v \Rightarrow \text{Just } v] \\
\hline
y^* \leq_{b_1} r + y^*
\end{array}$$

The derivation of $r \leq_{b_2} r + y^*$ follows a similar pattern. As both expressions r and $r + y^*$ are equal, the downcast never fails here.

$$\begin{array}{c}
b_2 = \lambda t . \text{case } t \text{ of } [\\
\text{INL } v \Rightarrow \text{Just } v, \\
\text{INR } v \Rightarrow \text{Just } (\text{SEQ } (\text{FOLD } \text{INR } \text{EPS}) v)] \\
\hline
r \leq_{b_2} r + y^*
\end{array}$$

Next, consider the premises of the (Alt)_u rule. For $\Delta \vdash^\uparrow c_2 : \mathbb{U}(\varepsilon, r)$ by definition $c_2 = \lambda(\text{EPS}, t) . b_3(t)$ where $r \leq_{b_3} r$ which can be satisfied by $b_3 = \lambda v . v$. For $\Delta \vdash^\uparrow c_3 : \mathbb{U}(x \cdot (e \cdot y), r)$ we find by definition $c_3 = \lambda(\text{SEQ } p_1 p_2, t) . c_4(p_1, c_5(p_2, t))$.

It follows some $(\text{Seq})_u$ step where we first consider $\Delta \vdash \uparrow c_4 : \mathbb{U}(x, r)$. By definition of $(\text{Sym})_u$ and $\text{cnf}(d_r(x)) = r$ we have that $c_4 = \lambda(v, t).b_4 (\text{SEQ } v \ t)$ where $x \cdot r \leq^{b_4} r$. Recall $r = x^* \cdot y^*$. So, upcast b_4 injects x into x^* 's parse tree. Recall the representation of parse trees for Kleene star in Definition 16.

$$b_4 = \lambda(\text{SEQ } v \ (\text{SEQ } t_1 \ t_2)).\text{SEQ} \ (\text{FOLD} \ (\text{INL} \ (\text{SEQ } v \ t_1))) \ t_2$$

Next, we consider $\Delta \vdash \uparrow c_5 : \mathbb{U}(e \cdot y, r)$ where we find another $(\text{Seq})_u$ step. Hence, $c_5 = \lambda(\text{SEQ } p_1 \ p_2, t).c_7 \ (p_1, c_6 \ (p_2, t))$. By $(\text{Hyp})_u$, we have that $c_7 = v_{\alpha.e, r}$. To obtain $\Delta \vdash \uparrow c_6 : \mathbb{U}(y, r + y^*)$ we apply another $(\text{Sym})_u$ step and therefore $c_6 = \lambda(v, t).b_5 (\text{SEQ } v \ t)$. The regular (upcast) coercion b_5 is derived from $y \cdot y^* \leq^{b_5} r + y^*$ because $\text{cnf}(d_y(r + y^*)) = y^*$. Its definition is as follows.

$$b_5 = \lambda(\text{SEQ } v \ t).\text{INR} \ (\text{FOLD} \ (\text{INL} \ (\text{SEQ } v \ t)))$$

This completes the example.

Remark 1 (Ambiguities). Example 6 shows that coercions may be ambiguous in the sense that there are several choices for the resulting parse trees. For example, in the construction of the regular (upcast) coercion $y \cdot y^* \leq^{b_5} x^* \cdot y^* + y^*$ we choose to inject y into the right component of the sum. The alternative is to inject y into the left component by making the x^* part empty.

$$b'_5 = \lambda(\text{SEQ } v \ t).\text{INL} \ (\text{SEQ} \ (\text{FOLD} \ (\text{INR} \ \text{EPS})) \ (\text{FOLD} \ (\text{INL} \ (\text{SEQ } v \ t))))$$

Both are valid choices. To obtain deterministic behavior of coercions we can apply a disambiguation strategy (e.g., favoring left-most alternatives). A detailed investigation of this topic is beyond the scope of the present work.

Based on Lemma 8 we easily obtain an upcast coercion to transform e 's parse tree into a parse tree of r . As $e \leq r$ if all elements in $\text{reach}(e, r)$ are nullable, we simply need to provide an empty parse tree for $+\text{reach}(e, r)$. The upcoming definition of $\text{mkE}()$ supplies such parse trees. It requires to check for nullability of context-free expression. This check is decidable as shown by the following definition.

Definition 17 (CFE Nullability).

$$\begin{array}{ll} \mathcal{N}(\phi) = \mathcal{N}(x) = \text{False} & \mathcal{N}(e + f) = \mathcal{N}(e) \vee \mathcal{N}(f) \\ \mathcal{N}(\varepsilon) = \text{True} & \mathcal{N}(e \cdot f) = \mathcal{N}(e) \wedge \mathcal{N}(f) \\ \mathcal{N}(\alpha) = \text{False} & \mathcal{N}(\mu\alpha.e) = \mathcal{N}(e) \end{array}$$

Lemma 9. *Let e be a context-free expression. Then, we have that $\mathcal{N}(e)$ holds iff $\varepsilon \in L(e)$.*

Based on the nullability check, we can derive empty parse trees (if they exist).

Definition 18 (Empty Parse Tree).

$$\begin{aligned}
mkE(\varepsilon) &= \text{EPS} & mkE(e + f) &= \begin{cases} \text{INL } mkE(e) & \text{if } \mathcal{N}(e) \\ \text{INR } mkE(f) & \text{otherwise} \end{cases} \\
mkE(\mu\alpha.e) &= \text{FOLD } mkE(e) & mkE(e \cdot f) &= \text{SEQ } mkE(e) \text{ } mkE(f)
\end{aligned}$$

Lemma 10. *Let e be a context-free expression such that $\mathcal{N}(e)$. Then, we find that $\vdash mkE(e) : e$ and $\text{flatten}(mkE(e)) = \varepsilon$.*

We summarize the construction of upcast coercions for context-free and regular expressions in containment relation.

Theorem 4 (Upcast Coercions). *Let e be a context-free expression and r be a regular expression such that $e \leq r$ and $\vdash^\uparrow c' : \mathbf{U}(e, r)$ for some coercion c' . Let $c = \lambda x.c'(x, mkE(+reach(e, r)))$. Then, we find that c is well-typed with type $e \rightarrow r$ where for any $\vdash p : e$ we have that $\llbracket c(p) \rrbracket = t'$ for some t' where and $\vdash t' : r$ and $\text{flatten}(p) = \text{flatten}(t')$.*

In analogy to the construction of upcast coercions, we can build a proof system for the construction of downcast coercions. Each such downcast coercion c has type $\mathbf{D}(e, r)$ where $\mathbf{D}(e, r)$ corresponds to $r \rightarrow \text{Maybe}(e \times +reach(e, r))$. That is, a parse tree of r can possibly be coerced into a parse tree of e and some residue which is a parse tree of $+reach(e, r)$. See Figure 3.

Rule $(\text{Eps})_d$ performs a change in representation. The downcast will always succeed. Rule $(\text{Sym})_d$ applies the regular downcast b to split r 's parse tree into x and the parse tree of the (canonical) derivative. The resulting downcast will not succeed if there is no leading x .

In case of a sum, rule $(\text{Alt})_d$ first tests if we can downcast r 's parse tree into a parse tree of the left component e and $+reach(e, r)$. If yes, we upcast $+reach(e, r)$'s parse tree into a parse tree of $+reach(e + f, r)$. Otherwise, we check if a downcast into f and $+reach(f, r)$ is possible.

In rule $(\text{Seq})_d$, we first check if we can obtain parse trees for e and residue $+reach(e, r)$. Otherwise, we immediately reach failure. From $+reach(e, r)$'s parse tree we then attempt to extract f 's parse tree and residue $+reach(f, +reach(e, r))$ which we know is equivalent to $+reach(e \cdot f, r)$. Hence, we combine the parse trees of e and f via SEQ and only need to pass through the residue.

As in case of upcast coercions, downcast coercions may be recursive. See rules $(\text{Rec})_d$ and $(\text{Hyp})_d$. In case the downcast yields the parse tree p' of the unfolding, we apply FOLD . The residue t' can be passed through as we find that $+reach(\mu\alpha.e, r) = +reach([\alpha \mapsto \mu\alpha.e](e), r)$.

Lemma 11 (Downcast Soundness). *Let e be a context-free expression and r be a regular expression such that $\Delta \vdash_\downarrow c : \mathbf{D}(e, r)$ for some coercion c . Let t be such that $\vdash t : r$ and $\llbracket c(t) \rrbracket = \text{Just}(p, t')$ for some p and t' . Then, we have that $\vdash p : e$, $\vdash t' : +reach(e, r)$ and $\text{flatten}(t) = \text{flatten}(p)$.*

$$\boxed{\Delta \vdash_{\Downarrow} c : \mathbf{D}(e, r)}$$

$$(\text{Eps})_d \frac{r \leq^b \text{cnf}(r) \quad c = \lambda t. \text{Just}(\text{EPS}, b(t))}{\Delta \vdash_{\Downarrow} c : \mathbf{D}(\varepsilon, r)}$$

$$(\text{Sym})_d \frac{x \cdot \text{cnf}(d_x(r)) \leq_b r \quad c = \lambda t. \text{case}(b(t)) \text{ of } [\text{Nothing} \Rightarrow \text{Nothing}, \text{Just}(\text{SEQ } x' t') \Rightarrow \text{Just}(x', t')]}{\Delta \vdash_{\Downarrow} c : \mathbf{D}(x, r)}$$

$$(\text{Alt})_d \frac{\begin{array}{c} \Delta \vdash_{\Downarrow} c_1 : \mathbf{D}(e, r) \quad \Delta \vdash_{\Downarrow} c_2 : \mathbf{D}(f, r) \\ +\text{reach}(e, r) \leq^{b_1} +\text{reach}(e + f, r) \quad +\text{reach}(f, r) \leq^{b_2} +\text{reach}(e + f, r) \\ c = \lambda t. \text{case}(c_1(t)) \text{ of} \\ [\text{Nothing} \Rightarrow \text{case}(c_2(t)) \text{ of} \\ [\text{Nothing} \Rightarrow \text{Nothing}, \\ \text{Just}(p_2, t_2) \Rightarrow \text{Just}(\text{INR } p_2, b_2(t_2))], \\ \text{Just}(p_1, t_1) \Rightarrow \text{Just}(\text{INL } p_1, b_1(t_1))] \end{array}}{\Delta \vdash_{\Downarrow} c : \mathbf{D}(e + f, r)}$$

$$(\text{Seq})_d \frac{\begin{array}{c} \Delta \vdash_{\Downarrow} c_1 : \mathbf{D}(e, r) \quad \Delta \vdash_{\Downarrow} c_2 : \mathbf{D}(f, +\text{reach}(e, r)) \\ c = \lambda t. \text{case}(c_1(t)) \text{ of} \\ [\text{Nothing} \Rightarrow \text{Nothing}, \\ \text{Just}(p_1, t_1) \Rightarrow \text{case}(c_2(t_1)) \text{ of} \\ [\text{Nothing} \Rightarrow \text{Nothing}, \\ \text{Just}(p_2, t_2) \Rightarrow \text{Just}(\text{SEQ } p_1 p_2, t_2)] \end{array}}{\Delta \vdash_{\Downarrow} c : \mathbf{D}(e \cdot f, r)}$$

$$(\text{Rec})_d \frac{\begin{array}{c} v_{\alpha.e,r} \notin \Delta \quad \Delta \cup \{(v_{\alpha.e,r} : \mathbf{D}(\mu\alpha.e, r))\} \vdash_{\Downarrow} c' : \mathbf{D}([\alpha \mapsto \mu\alpha.e](e), r) \\ c = \text{rec } v_{\alpha.e,r}. \lambda t. \text{case}(c'(t)) \text{ of} \\ [\text{Nothing} \Rightarrow \text{Nothing}, \\ \text{Just}(p', t') \Rightarrow \text{Just}(\text{FOLD } p', t')] \end{array}}{\Delta \vdash_{\Downarrow} c : \mathbf{D}(\mu\alpha.e, r)}$$

$$(\text{Hyp})_d \frac{(v_{\alpha.e,r} : \mathbf{D}(\mu\alpha.e, r)) \in \Delta}{\Delta \vdash_{\Downarrow} v_{\alpha.e,r} : \mathbf{D}(\mu\alpha.e, r)}$$

Fig. 3. Reachability downcast coercions

Example 7. We consider the derivation of $\vdash_{\Downarrow} c_0 : \mathsf{D}(e, r)$ where $e = \mu\alpha.x \cdot (\alpha \cdot y) + \varepsilon$, $r = x^* \cdot y^*$ and $\mathit{reach}(e, r) = \{r, y^*\}$. The downcast coercion attempts to turn a parse of r into a parse tree of e and some residual parse tree of $+reach(e, r)$. The construction is similar to Example 6. We consider the downcast coercions resulting from $(\mathsf{Rec})_d$ and $(\mathsf{Alt})_d$.

$$c_0 : \mathsf{D}(e, r) = \mathit{rec} \ v_{\alpha.e,r} \ \lambda t. \mathit{case} \ (c_1 \ (t)) \ \mathit{of} \\ \quad [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \\ \quad \mathit{Just} \ (p', t') \Rightarrow \mathit{Just} \ (\mathsf{FOLD} \ p', t')]$$

$$c_1 : \mathsf{D}(x \cdot (e \cdot y) + \varepsilon, r) = \lambda t. \mathit{case} \ (c_2 \ (t)) \ \mathit{of} \\ \quad [\mathit{Nothing} \Rightarrow \mathit{case} \ (c_3 \ (t)) \ \mathit{of} \\ \quad \quad [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \\ \quad \quad \mathit{Just} \ (p_2, t_2) \Rightarrow \mathit{Just} \ (\mathsf{INR} \ p_2, b_2 \ (t_2))], \\ \quad \mathit{Just} \ (p_1, t_1) \Rightarrow \mathit{Just} \ (\mathsf{INL} \ p_1, b_1 \ (t_1))]$$

$$\text{where } r \leq^{b_1} r + y^* \quad r \leq^{b_2} r + y^* \quad b_1 = \mathsf{INR} \quad b_2 = \mathsf{INL}$$

The auxiliary coercion c_2 greedily checks for a leading symbol x . Otherwise, we pick the base case $(\mathsf{Eps})_d$ where the entire input becomes the residue. This is dealt with by coercion c_3 .

$$c_3 : \mathsf{D}(\varepsilon, r) = \lambda t. \mathit{Just} \ (\mathsf{EPS}, b_3 \ (t)) \\ \text{where } \mathit{cnf}(r) = r \quad b_3 = \lambda x.x \quad r \leq^{b_3} \mathit{cnf}(r)$$

Coercion c_2 first checks for x , then recursively calls (in essence) c_0 , followed by a check for y . Here are the details.

$$c_2 : \mathsf{D}(x \cdot (e \cdot y), r) = \lambda t. \mathit{case} \ (c_4 \ (t)) \ \mathit{of} \\ \quad [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \\ \quad \mathit{Just} \ (p_1, t_1) \Rightarrow \mathit{case} \ (c_5 \ (t_1)) \ \mathit{of} \\ \quad \quad [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \\ \quad \quad \mathit{Just} \ (p_2, t_2) \Rightarrow \mathit{Just} \ (\mathsf{SEQ} \ p_1 \ p_2, t_2)]]$$

Auxiliary coercion c_4 checks for x and any residue is passed on to coercion c_5 .

$$c_4 : \mathsf{D}(x, r) = \lambda t. \mathit{case} \ (b_4 \ (t)) \ \mathit{of} \ [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \ \mathit{Just} \ (\mathsf{SEQ} \ x' \ t') \Rightarrow \mathit{Just} \ (x', t')] \\ \text{where } \mathit{cnf}(d_x(r)) = r \quad x \cdot r \leq_{b_4} r$$

$$b_4 = \lambda \mathsf{SEQ} \ t_1 \ t_2. \mathit{case} \ t_1 \ \mathit{of} \ [\\ \quad \mathsf{FOLD} \ \mathsf{INR} \ \mathsf{EPS} \Rightarrow \mathit{Nothing}, \\ \quad \mathsf{FOLD} \ \mathsf{INL} \ (\mathsf{SEQ} \ t_3 \ t_4) \Rightarrow \mathit{Just} \ (\mathsf{SEQ} \ t_3 \ (\mathsf{SEQ} \ t_4 \ t_3))]$$

In coercion c_5 , we check for e which then leads to the recursive call.

$$c_5 : \mathsf{D}(e \cdot y, r) = \lambda t. \mathit{case} \ (c_7 \ (t)) \ \mathit{of} \\ \quad [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \\ \quad \mathit{Just} \ (p_1, t_1) \Rightarrow \mathit{case} \ (c_6 \ (t_1)) \ \mathit{of} \\ \quad \quad [\mathit{Nothing} \Rightarrow \mathit{Nothing}, \\ \quad \quad \mathit{Just} \ (p_2, t_2) \Rightarrow \mathit{Just} \ (\mathsf{SEQ} \ p_1 \ p_2, t_2)]] \\ c_7 : \mathsf{D}(e, r) = v_{\alpha.e,r}$$

Finally, coercion c_6 checks for y

$$\begin{aligned}
c_6 : \mathbb{D}(y, r + y^*) &= \lambda t. \text{case } (b_6(t)) \text{ of } [\\
&\quad \text{Nothing} \Rightarrow \text{Nothing}, \\
&\quad \text{Just } (\text{SEQ } x' t') \Rightarrow \text{Just } (x', t')] \\
\text{where } \text{cnf}(d_y(r + y^*)) &= y^* y \cdot y^* \leq_{b_6} r + y^* \\
b_6 &= \lambda t. \text{case } t \text{ of } [\\
&\quad \text{INL } \text{SEQ } t_1 t_2 \Rightarrow b'_6(t_2), \\
&\quad \text{INR } t \Rightarrow b'_6(t)] \\
y \cdot y^* \leq_{b'_6} y^* & \\
b'_6 &= \lambda t. \text{case } t \text{ of } [\\
&\quad \text{FOLD } \text{INR } \text{EPS} \Rightarrow \text{Nothing}, \\
&\quad \text{FOLD } \text{INL } (\text{SEQ } t_1 t_2) \Rightarrow \text{Just } (\text{SEQ } t_1 t_2)]
\end{aligned}$$

Consider input $t = \text{SEQ } t_1 t_2$ where $t_1 = \text{FOLD } (\text{INL } \text{SEQ } x (\text{FOLD } (\text{INR } \text{EPS})))$, $t_2 = \text{FOLD } (\text{INL } \text{SEQ } y (\text{FOLD } (\text{INR } \text{EPS})))$, $\vdash t : r$ and $\text{flatten}(t) = x \cdot y$. Then $\llbracket c_0(t) \rrbracket = \text{Just } (p, t')$ where $p = \text{FOLD } (\text{INL } \text{SEQ } x (\text{SEQ } (\text{FOLD } (\text{INR } \text{EPS})) y))$ and $\vdash p : e$ and $\text{flatten}(p) = x \cdot y$. For residue t' we find $\text{flatten}(t') = \varepsilon$. This completes the example.

Any context-free expression e is contained in the regular language Σ^* . We wish to derive a downcast coercion for this containment which effectively represents a parser for $L(e)$. That is, the parser maps a parse tree for $w \in \Sigma^*$ (which is isomorphic to w) to a parse tree $\vdash p : e$ with $w = \text{flatten}(p)$ if $w \in L(e)$. However, our parser, like any predictive parser, is sensitive to the shape of context-free expressions. So, we need to syntactically restrict the class of context-free expressions on which our parser can be applied.

Definition 19 (Guarded Context-Free Expressions). *A context-free expression is guarded if the expression is of the following shape:*

$$\begin{aligned}
e, f &::= \phi \mid \varepsilon \mid x \in \Sigma \mid \alpha \in A \mid e + f \mid e \cdot f \mid \mu \alpha. g \\
g &::= x \cdot e \mid \varepsilon \mid x \cdot e + g
\end{aligned}$$

where for each symbol x there exists at most one guard $x \cdot e$ in g .

For any context-free expression we find an equivalent guarded variant. This follows from the fact that guarded expressions effectively correspond to context-free grammars in Greibach Normal Form. We additionally impose the conditions that guards x are unique and ε appears last. This ensures that the parser leaves no residue behind.

Theorem 5 (Predictive Guarded Parser). *Let e be a guarded context-free expression and r be a regular expression such that $e \leq r$ and $\Delta \vdash_{\Downarrow} c' : \mathbb{D}(e, r)$ for some coercion c' . Let $c = \lambda x. \text{case } (c'(x)) \text{ of } [\text{Nothing} \Rightarrow \text{Nothing}, \text{Just } (p, t') \Rightarrow \text{Just } p]$ Then, we find that c is well-typed with type $r \rightarrow \text{Maybe } e$ and terminates for any input $t \vdash t : r$ If $\llbracket c(t) \rrbracket = \text{Just } p$ for some p , then we have that $\vdash p : e$ and $\text{flatten}(t) = \text{flatten}(p)$.*

Guardedness is essential as shown by the following examples. Consider $e' = \mu\alpha.\varepsilon + x \cdot (\alpha \cdot y)$, $r = x^* \cdot y^*$. The difference to e from Example 7 is that subexpression ε appears in leading position. Hence, the guardedness condition is violated. The downcast coercion for this example (after some simplifications) has the form $c_0 = \lambda t.(\text{FOLD}(\text{INL EPS}), t)$. As we can see no input is consumed at all. We return the trivial parse term and the residue t contains the unconsumed input. As an example for a non-terminating parser consider $e' = \mu\alpha.\alpha \cdot x + \varepsilon$ and $r = (x + y)^*$. Again the guardedness condition is violated because subexpression α is not guarded. The coercion resulting from $\vdash_{\Downarrow} c'_0 : \mathbb{D}(e', r)$ has (after some simplifications) the following form $c'_0 = \text{rec } v.\lambda t.\text{case } v(t) \text{ of } \dots$. Clearly, this parser is non-terminating which is no surprise as the context-free expression is left-recursive.

6 Related Work and Conclusion

Our work builds upon prior work in the setting of regular expressions by Frisch and Cardelli [4], Henglein and Nielsen [6] and Lu and Sulzmann [8, 12], as well as Brandt and Henglein’s coinductive characterization of recursive type equality and subtyping [2]. We extend these ideas to the case of context-free expressions and their parse trees.

There are simple standard methods to construct predictive parsers (e.g., recursive descent etc) contained in any textbook on compiler construction [1]. But the standard methods are tied to parse from a single regular input language, Σ^* , whereas our approach provides specialized parsers from an arbitrary regular language. These parsers will generally be more deterministic, fail earlier, etc. because they are exploiting knowledge about the input.

Based on our results we obtain a predictive parser for guarded context-free expressions. Earlier works in this area extend Brzozowski-style derivatives [3] to the context-free setting while we use plain regular expression derivatives in combination with reachability. See the works by Krishnaswami [7], Might, Darais and Spiewak [10] and Winter, Bonsangue, and Rutten [13]. Krishnaswami [7] shows how to elaborate general context-free expressions into some equivalent guarded form and how to transform guarded parse trees into their original representation. We could integrate this elaboration/transformation step into our approach to obtain a general, predictive parser for context-free expressions.

Marriott, Stuckey, and Sulzmann [9] show how containment among context-free languages and regular languages can be reduced to a reachability problem [11]. While they represent languages as context-free grammars and DFAs, we rely on context-free expressions, regular expressions, and specify reachable states in terms of Brzozowski-style derivatives [3]. This step is essential to obtain a characterization of the reachability problem in terms of a natural-deduction style proof system. By applying the proofs-are-programs principle we derive up-cast and downcast coercions to transform parse trees of context-free and regular expressions. These connections are not explored in any prior work.

Acknowledgments

We thank the APLAS'17 reviewers for their constructive feedback.

References

1. A. V. Aho, R. Sethi, and J. D. Ullman. *Compilers: Principles, Techniques, and Tools*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1986.
2. M. Brandt and F. Henglein. Coinductive axiomatization of recursive type equality and subtyping. *Fundam. Inf.*, 33(4):309–338, Apr. 1998.
3. J. A. Brzozowski. Derivatives of regular expressions. *J. ACM*, 11(4):481–494, 1964.
4. A. Frisch and L. Cardelli. Greedy regular expression matching. In *Proc. of ICALP'04*, pages 618–629. Springer-Verlag, 2004.
5. C. Grabmayer. Using proofs by coinduction to find "traditional" proofs. In *Proc. of CALCO'05*, pages 175–193. Springer-Verlag, 2005.
6. F. Henglein and L. Nielsen. Regular expression containment: Coinductive axiomatization and computational interpretation. In *Proc. of POPL'11*, pages 385–398. ACM, 2011.
7. N. R. Krishnaswami. A typed, algebraic approach to parsing,. <https://www.cl.cam.ac.uk/~nk480/parsing.pdf>, 2017.
8. K. Z. M. Lu and M. Sulzmann. An implementation of subtyping among regular expression types. In *Proc. of APLAS'04*, volume 3302 of *LNCS*, pages 57–73. Springer, 2004.
9. K. Marriott, P. J. Stuckey, and M. Sulzmann. Resource usage verification. In *Proc. of APLAS'03*, volume 2895 of *LNCS*, pages 212–229. Springer, 2003.
10. M. Might, D. Darais, and D. Spiewak. Parsing with derivatives: a functional pearl. In *Proc. of ICFP'11*, pages 189–195. ACM, 2011.
11. T. Reps. Program analysis via graph reachability. In *Proc. of ILPS'97*, pages 5–19, Cambridge, MA, USA, 1997. MIT Press.
12. M. Sulzmann and K. Z. M. Lu. A type-safe embedding of XDuce into ML. *Electron. Notes Theor. Comput. Sci.*, 148(2):239–264, 2006.
13. J. Winter, M. M. Bonsangue, and J. J. M. M. Rutten. Coalgebraic characterizations of context-free languages. *Logical Methods in Computer Science*, 9(3), 2013.

Appendix

A Notation

Definition 20 (Substitution). We write $[\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n]$ to denote a substitution mapping variables α_i to expressions e_i . We maintain the invariant that the free variables (if any) in e_i are disjoint from α_i . That is, the substitutions we consider here are idempotent.

Let $\psi = [\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n]$. Then, we define $\psi(\alpha) = e$ if there exists i such that $\alpha_i = \alpha$ and $e_i = e$. This extends to expressions in the natural way.

Let $\psi = [\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n]$. Then, we define $\psi_{\setminus \alpha_1} = [\alpha_2 \mapsto e_2, \dots, \alpha_n \mapsto e_n]$.

We write id to denote the empty substitution $[]$. Let $\psi_1 = [\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n]$ and $\psi_2 = [\beta_1 \mapsto f_1, \dots, \beta_m \mapsto f_m]$ be two substitutions where α_i and β_j are distinct and the free variables in e_i and f_j are disjoint from α_i and β_j . Then, we define $\psi_1 \sqcup \psi_2 = [\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n, \beta_1 \mapsto f_1, \dots, \beta_m \mapsto f_m]$.

B Coercion Semantics

Values are elements of a complete partial order \mathcal{V} which is defined as the least solution of the following domain equation, where “+” and “ Σ ” stand for the lifted sum of domains. The distinguished element \mathbf{W} (wrong) will be used to indicate errors. The resulting domain yields a non-strict interpretation.

Definition 21 (Values).

$$\mathcal{V} = \{\mathbf{W}\} + (\mathcal{V} \rightarrow \mathcal{V}) + \sum_{k \in \mathcal{K}} (\{k\} \times \mathcal{V}_1 \times \dots \times \mathcal{V}_{arity(k)})$$

We write a to denote values, i.e. elements in \mathcal{V} . We abuse notation by writing $_{\perp}$ for the injections into the sum on the left-hand side (the summand is clear from the argument) and \downarrow (drop) for their right-inverses.

To define the meaning of coercions, we first establish a semantic match relation among values and patterns to obtain the binding of pattern variables. We write \sqcup for the disjoint union of environments η which map variables to values.

Definition 22 (Pattern Matching, $\vdash_m a : pat \rightsquigarrow \eta$).

$$\vdash_m a : v \rightsquigarrow [v \mapsto a] \quad \frac{\vdash_m \downarrow a_1 : pat_1 \rightsquigarrow \eta_1 \dots \vdash_m \downarrow a_n : pat_n \rightsquigarrow \eta_n \quad n = arity(k)}{\vdash_m (k, a_1, \dots, a_n) : k \ pat_1 \dots pat_n \rightsquigarrow \eta_1 \sqcup \dots \sqcup \eta_n}$$

Pattern matching may fail, for example, in case of differences in constructors and number of arguments. We write $Just \eta = match(v, pat)$ to indicate that $\vdash_m v : pat \rightsquigarrow \eta$ is derivable. Otherwise $match(v, pat) = Nothing$.

Definition 23 (Coercion Semantics, $\llbracket c \rrbracket \eta$).

$$\begin{aligned}
\llbracket v \rrbracket \eta &= \eta(v) \\
\llbracket \lambda v. c \rrbracket \eta &= (\lambda y. \llbracket c \rrbracket (\eta \sqcup [v \mapsto y]))_{\perp} \\
\llbracket k \ c_1 \dots c_{\text{arity}(k)} \rrbracket \eta &= (k, \llbracket c_1 \rrbracket \eta, \dots, \llbracket c_{\text{arity}(k)} \rrbracket \eta)_{\perp} \\
\llbracket c_1 \ c_2 \rrbracket \eta &= \text{if } \downarrow \llbracket c_1 \rrbracket \eta \in \mathcal{V} \rightarrow \mathcal{V} \text{ then } \downarrow (\llbracket c_1 \rrbracket \eta) (\llbracket c_2 \rrbracket \eta) \text{ else } \mathbf{W}_{\perp} \\
\llbracket \text{case } c \text{ of } [pat_1 \Rightarrow c_1, \dots, pat_n \Rightarrow c_n] \rrbracket \eta &= \\
&\text{let } y = \downarrow \llbracket c \rrbracket \eta \text{ in} \\
&\text{if Just } \eta_1 = \text{match}(y, pat_1) \text{ then } \llbracket c_1 \rrbracket (\eta \sqcup \eta_1) \\
&\dots \\
&\text{else if Just } \eta_n = \text{match}(y, pat_n) \text{ then } \llbracket c_n \rrbracket (\eta \sqcup \eta_n) \\
&\text{else } \mathbf{W}_{\perp}
\end{aligned}$$

C Downcast Example

C.1 $\vdash_{\downarrow} c_0 : \mathbf{D}(\mu\alpha.x \cdot \alpha + \varepsilon, (x + y)^*)$

Consider $e = \mu\alpha.x \cdot \alpha + \varepsilon$ and $r = (x + y)^*$. We have that $\text{reach}(e, r) = \{r\}$. Recall that $\text{cnf}(d_x((x + y)^*)) = (x + y)^*$.

The following derivation tree verifies that $\{\} \vdash r \overset{e}{\rightsquigarrow} \{r\}$.

$$\begin{array}{c}
\text{(Sym)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{x}{\rightsquigarrow} \{r\} \checkmark \\
\text{(Hyp)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{e}{\rightsquigarrow} \{r\} \checkmark \\
\text{(SEQ)} \quad \frac{\text{(Sym)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{x}{\rightsquigarrow} \{r\} \checkmark \quad \text{(Hyp)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{e}{\rightsquigarrow} \{r\} \checkmark}{\{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{x \cdot e}{\rightsquigarrow} \{r\}} \\
\text{(Eps)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{\varepsilon}{\rightsquigarrow} \{r\} \checkmark \\
\text{(ALT)} \quad \frac{\text{(Eps)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{\varepsilon}{\rightsquigarrow} \{r\} \checkmark}{\{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{x \cdot e + \varepsilon}{\rightsquigarrow} \{r\}} \\
\text{(REC)} \quad \frac{\text{(ALT)} \quad \{r \overset{e}{\rightsquigarrow} \{r\}\} \vdash r \overset{x \cdot e + \varepsilon}{\rightsquigarrow} \{r\}}{\{\} \vdash r \overset{e}{\rightsquigarrow} \{r\}}
\end{array}$$

The derivation of $\vdash_{\downarrow} c_0 : \mathbf{D}(e, r)$ follows the shape of the above derivation tree. We find the following coercions where for clarity we label them with the corresponding rule names. Definitions of auxiliary regular coercions are found at the end.

(Rec)_d $c_0 = \text{rec } v_\alpha. \lambda t. \text{case } c_1(t) \text{ of}$
 $[Nothing \Rightarrow Nothing,$
 $Just(p', t') \Rightarrow Just(\text{FOLD } p', t')]$

(Alt)_d $c_1 = \lambda t. \text{case } c_2(t) \text{ of}$
 $[Nothing \Rightarrow \text{case } (c_3(t)) \text{ of}$
 $[Nothing \Rightarrow Nothing,$
 $Just(p_2, t_2) \Rightarrow Just(\text{INR } p_2, b_2(t_2))],$
 $Just(p_1, t_1) \Rightarrow Just(\text{INL } p_1, b_1(t_1))]$

(Seq)_d $c_2 = \lambda t. \text{case } (c_3(t)) \text{ of}$
 $[Nothing \Rightarrow Nothing,$
 $Just(p_1, t_1) \Rightarrow \text{case } (c_4(t_1)) \text{ of}$
 $[Nothing \Rightarrow Nothing,$
 $Just(p_2, t_2) \Rightarrow Just(\text{SEQ } p_1 p_2, t_2)]]]$

(Sym)_d $c_3 = \lambda t. \text{case } (b_4(t)) \text{ of } [Nothing \Rightarrow Nothing, Just(\text{SEQ } x' t') \Rightarrow Just(x', t')]$

(Hyp)_d $c_4 = v_\alpha$

(Eps)_d $c_5 = \lambda t. Just(\text{EPS}, b_3(t))$

$$\frac{\text{reach}(e \cdot x, r) = r \quad \text{reach}(e \cdot x + \varepsilon, r) = r \quad b_1 = \lambda t. t}{+\text{reach}(e \cdot x, r) \leq^{b_1} +\text{reach}(e \cdot x + \varepsilon, r)}$$

$$\frac{\text{reach}(\varepsilon, r) = r \quad \text{reach}(e \cdot x + \varepsilon, r) = r \quad b_2 = \lambda t. t}{+\text{reach}(\varepsilon, r) \leq^{b_2} +\text{reach}(e \cdot x + \varepsilon, r)}$$

$$\frac{b_3 = \lambda t. t \quad \text{cnf}(r) = r}{r \leq^{b_3} \text{cnf}(r)}$$

$\text{cons} = \text{rec } v. \lambda x. \lambda xs. \text{case } xs \text{ of } [$
 $\text{FOLD}(\text{INR } \text{EPS}) \Rightarrow \text{FOLD}(\text{INL}(\text{SEQ } x(\text{FOLD}(\text{INR } \text{EPS}))))),$
 $\text{FOLD}(\text{INL}(\text{SEQ } y \text{ } ys)) \Rightarrow \text{FOLD}(\text{INL}(\text{SEQ } x(\text{FOLD}(\text{INL}(\text{SEQ } y \text{ } ys)))))]]$

$$\begin{array}{l}
b_4 = \text{rec } v. \lambda t. \text{case } t \text{ of } [\\
\quad \text{FOLD (INR EPS)} \Rightarrow \text{Nothing}, \\
\quad \text{FOLD (INL (SEQ (INR (SYM } y)) t_2))} \Rightarrow \text{Nothing}, \\
\quad \text{FOLD (INL (SEQ (INL (SYM } x)) (\text{FOLD (INR EPS)}))} \Rightarrow \\
\quad \quad \text{Just (SEQ (SYM } x) (\text{FOLD (INR EPS)})), \\
\quad \text{FOLD (INL (SEQ (INL (SYM } x)) t_2)} \Rightarrow \\
\quad \quad \text{case } (v (t_2)) \text{ of } [\\
\quad \quad \quad \text{Nothing} \Rightarrow \text{Nothing}, \\
\quad \quad \quad \text{Just (SEQ (SYM } x_2) t_3)} \Rightarrow \text{Just (SEQ (SYM } x) (\text{cons (SYM } x_2) t_3))] \\
\hline
x \cdot r \leq_{b_4} r
\end{array}$$

C.2 $\vdash_{\Downarrow} c'_0 : D(\mu\alpha.\varepsilon + x \cdot \alpha, (x + y)^*)$

By reusing the above calculations we obtain

$$\begin{array}{l}
(\text{Rec})_d c_0 = \text{rec } v_\alpha. \lambda t. \text{case } c'_1 (t) \text{ of} \\
\quad [\text{Nothing} \Rightarrow \text{Nothing}, \\
\quad [\text{Just } (p', t') \Rightarrow \text{Just (FOLD } p', t')]] \\
(\text{Alt})_d c'_1 = \lambda t. \text{case } c'_2 (t) \text{ of} \\
\quad [\text{Nothing} \Rightarrow \text{case } (c'_3 (t)) \text{ of} \\
\quad \quad [\text{Nothing} \Rightarrow \text{Nothing}, \\
\quad \quad \quad \text{Just } (p_2, t_2) \Rightarrow \text{Just (INR } p_2, b_1 (t_2))], \\
\quad \quad \text{Just } (p_1, t_1) \Rightarrow \text{Just (INL } p_1, b_2 (t_1))] \\
(\text{Eps})_d c'_2 = \lambda t. \text{Just (EPS, } b_3 (t)) \\
\dots
\end{array}$$

So, by unfolding and removing dead code we find that

$$c'_0 = \lambda t. (\text{EPS}, t)$$

C.3 $\vdash_{\Downarrow} c''_0 : D(\mu\alpha.\alpha \cdot x + \varepsilon, (x + y)^*)$

Via similar reasoning we find that

$$\begin{aligned}
(\text{Rec})_d \ c'_0 &= \text{rec } v_\alpha. \lambda t. \text{case } c'_1(t) \text{ of} \\
&\quad [\text{Nothing} \Rightarrow \text{Nothing}, \\
&\quad [\text{Just } (p', t') \Rightarrow \text{Just } (\text{FOLD } p', t')]] \\
(\text{Alt})_d \ c'_1 &= \lambda t. \text{case } c'_2(t) \text{ of} \\
&\quad [\text{Nothing} \Rightarrow \text{case } (c'_3(t)) \text{ of} \\
&\quad \quad [\text{Nothing} \Rightarrow \text{Nothing}, \\
&\quad \quad \text{Just } (p_2, t_2) \Rightarrow \text{Just } (\text{INR } p_2, b_1(t_2))], \\
&\quad \text{Just } (p_1, t_1) \Rightarrow \text{Just } (\text{INL } p_1, b_2(t_1))] \\
(\text{Hyp})_d \ c'_2 &= v_\alpha \\
&\dots
\end{aligned}$$

D Least Fixed Point Construction for $\text{reach}(e, r)$

To compute $\text{reach}(e, r)$, we need to compute $\text{reach}(e', r')$ for all subterms of e and for certain r . To capture this notion exactly, we define a function to compute the set of subterms of a context-free expressions.

Definition 24 (Subterms).

$$\begin{aligned}
\mathcal{T}(\varepsilon) &= \{\varepsilon\} \quad \mathcal{T}(\phi) = \{\phi\} \quad \mathcal{T}(x) = \{x\} \\
\mathcal{T}(e + f) &= \{e + f\} \cup \mathcal{T}(e) \cup \mathcal{T}(f) \quad \mathcal{T}(e \cdot f) = \{e \cdot f\} \cup \mathcal{T}(e) \cup \mathcal{T}(f) \\
\mathcal{T}(\alpha) &= \{\alpha\} \quad \mathcal{T}(\mu\alpha.e) = \{\mu\alpha.e\} \cup \mathcal{T}(e)
\end{aligned}$$

Lemma 12. *For any context-free expression e , the set $\mathcal{T}(e)$ is finite.*

We write R and S to denote sets of regular expressions. We write E to denote an equation of the form $(e, r) = R$. We can view a set \mathcal{E} of such equations as a mapping from pairs (e, r) to R . If $(e, r) = R \in \mathcal{E}$ then we write $\mathcal{E}(e, r)$ to denote R . If no such equation exists in \mathcal{E} , then we set $\mathcal{E}(e, r) = \emptyset$.

We define $\mathcal{E}_{e,r}$ as the set of equations where the pairs range over subterms of e and derivatives of r and map to sets of descendants of r .

$$\mathcal{E}_{e,r} = \{(f, s) = S \mid f \in \mathcal{T}(e), s \in D(r), S \subseteq D(r)\}$$

For two sets of equations $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}_{e,r}$, we define $\mathcal{E}_1 \leq \mathcal{E}_2$ if for each $(f, s) \in \mathcal{T}(e) \times D(r)$ we have that $\mathcal{E}_1(f, s) \subseteq \mathcal{E}_2(f, s)$. This definition makes $(\mathcal{E}_{e,r}, \leq)$ a complete, finite lattice with least element $\{\}$ and greatest element $\{(f, s) = D(r) \mid (f, s) \in \mathcal{T}(e) \times D(r)\}$.

Next, we define the reachability step function $\mathcal{R}(\cdot, \cdot)$ which operates on $\mathcal{T}(e) \times D(r) \times \wp \mathcal{E}_{e,r}$ and yields a subset of $D(r)$.

Definition 25 (Reachability Step).

$$\begin{aligned}
\mathcal{R}(\phi, r, \mathcal{E}) &= \{\} \\
\mathcal{R}(\varepsilon, r, \mathcal{E}) &= \{\text{cnf}(r)\} \\
\mathcal{R}(x, r, \mathcal{E}) &= \{\text{cnf}(d_x(r))\} \\
\mathcal{R}(e + f, r, \mathcal{E}) &= \mathcal{R}(e, r, \mathcal{E}) \cup \mathcal{R}(f, r, \mathcal{E}) \\
\mathcal{R}(e \cdot f, r, \mathcal{E}) &= \bigcup_{s \in \mathcal{R}(e, r, \mathcal{E})} \mathcal{R}(f, s, \mathcal{E}) \\
\mathcal{R}(\alpha, r, \mathcal{E}) &= \mathcal{E}(\mu\alpha.e, r) \\
\mathcal{R}(\mu\alpha.e, r, \mathcal{E}) &= \mathcal{R}(e, r, \mathcal{E})
\end{aligned}$$

For the second last case, we assume that each variable α can be linked to its surrounding scope $\mu\alpha.e$. This is guaranteed by the fact that we consider a fixed set of subterms.

We sometimes write $\mathcal{R}(f, \mathcal{R}(e, r, \mathcal{E}), \mathcal{E})$ as a shorthand for $\bigcup_{s \in \mathcal{R}(e, r, \mathcal{E})} \mathcal{R}(f, s, \mathcal{E})$.

Definition 26 (Reachability Function). Let e be a context-free expression and r be a regular expression. We define $\mathcal{F}_{e,r} : \mathcal{E}_{e,r} \rightarrow \mathcal{E}_{e,r}$ as follows:

$$\mathcal{F}_{e,r}(\mathcal{E}) = \{(f, s) = \mathcal{E}(f, s) \cup \mathcal{R}(f, s, \mathcal{E}) \mid (f, s) \in \mathcal{T}(e) \times D(r)\}$$

Lemma 13. Function $\mathcal{F}_{e,r}$ is well-defined and monotonic with respect to the ordering \leq .

Proof. All calls to $\mathcal{F}_{e,r}$ yield well-defined calls to $\mathcal{R}(\cdot, \cdot)$ as elements in $\mathcal{E}_{e,r}$ cover all cases on which $\mathcal{R}(\cdot, \cdot)$. Furthermore, the range of function $\mathcal{R}(\cdot, \cdot)$ is the set $D(r)$. Hence, computation will never get stuck.

For monotonicity, we need to show that if $\mathcal{E}_1 \leq \mathcal{E}_2$ then $\mathcal{F}_{e,r}(\mathcal{E}_1) \leq \mathcal{F}_{e,r}(\mathcal{E}_2)$, which holds if $\mathcal{R}(\cdot, \cdot)$ is monotonic in the last parameter. The proof for monotonicity of $\mathcal{R}(\cdot, \cdot)$ is by easy induction over the first parameter. \square

The Knaster-Tarski Theorem guarantees that the least fixpoint of $\mathcal{F}_{e,r}$ exists. Let $\mathcal{X}_{e,r}$ denote the least fixpoint. That is, $\mathcal{X}_{e,r} = \bigcup_{i=0}^{\infty} \mathcal{F}_{e,r}^i(\perp)$ where $\perp = \{(f, s) = \{\} \mid (f, s) \in \mathcal{T}(e) \times D(r)\}$. Hence, in the sequence of elements $X^0_{e,r} = \perp$ and $X^{n+1}_{e,r} = \mathcal{F}_{e,r}(X^n_{e,r})$, we find $\mathcal{X}_{e,r} = X^m_{e,r}$ for some $m \geq 0$ where $X^m_{e,r} = X^{m+k}_{e,r}$ for all $k \geq 0$.

For sets R and S of regular expressions, we define $R \sim S$ if for each $r \in R$ we find $s \in S$ where $r \sim s$ and vice versa.

For the proof of $\mathcal{X}_{e,r}(e, r) \sim \text{reach}(e, r)$ to go through, we need to include subterms in $\mathcal{T}(e)$. As these subterms contain free variables α , we need to map these variables to their corresponding definition.

Definition 27 (Binding of μ -Expressions). Let e be a context-free expression. We build a substitution which maps bound variables α in e to their corresponding definition.

$$\mathcal{S}(\varepsilon) = id \quad \mathcal{S}(\phi) = id \quad \mathcal{S}(x) = id \quad \mathcal{S}(\alpha) = id$$

$$\mathcal{S}(e + f) = \mathcal{S}(e) \sqcup \mathcal{S}(f) \quad \mathcal{S}(e \cdot f) = \mathcal{S}(e) \sqcup \mathcal{S}(f)$$

$$\begin{aligned}
\mathcal{S}(\mu\alpha.e) &= [\alpha_1 \mapsto \psi(e_1), \dots, \alpha_n \mapsto \psi(e_n), \alpha \mapsto \mu\alpha.e] \\
\text{where } \mathcal{S}(e) &= [\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n] \quad \psi = [\alpha \mapsto \mu\alpha.e]
\end{aligned}$$

The substitution $\mathcal{T}(e)$ is well-defined as variables α introduced by $\mu\alpha$ are distinct by assumption. See the cases for concatenation and alternation. In case of $\mu\alpha.e$, we first build $\mathcal{S}(e) = [\alpha_1 \mapsto e_1, \dots, \alpha_n \mapsto e_n]$ where e_i may only refer to α or other variables but not to α_i . Hence, the application of $\psi(e_i)$ to maintain the invariant.

Some helper statements which follow by definition.

Lemma 14. *Let e be a context-free expression and $\mu\alpha.f \in \mathcal{T}(e)$. Let $\psi = \mathcal{S}(e)$. Then, we have that $[\alpha \mapsto \mu\alpha.\psi_{\setminus\alpha}(f)](\psi_{\setminus\alpha}(f)) = \psi(f)$.*

Lemma 15. *Let e be a context-free expression and r be a regular expression. Let $(\mu\alpha.f, s) \in \mathcal{T}(e) \times D(r)$. Then, we have that $\mathcal{X}_{e,r}(\mu\alpha.f, s) = \mathcal{X}_{e,r}(f, s) = \mathcal{X}_{e,r}(\alpha, s)$.*

Lemma 16. *Let e be a context-free expression and r be a regular expression. Let $(f, s) \in \mathcal{T}(e) \times D(r)$. Then, we have that $\mathcal{R}(f, s, \mathcal{X}_{e,r}) = \mathcal{X}_{e,r}(f, s)$.*

The generalized statement.

Lemma 17. *Let e be a context-free expression and r be a regular expression. Let $\psi = \mathcal{T}(e)$, w be a word, $(f, s) \in \mathcal{T}(e) \times D(r)$ such that $\psi(f) \Rightarrow w$. Then, there exists $t \in \mathcal{X}_{e,r}(f, s)$ such that $t \sim d_w(s)$.*

Proof. By induction on the derivation $\psi(f) \Rightarrow w$ and observing the various shapes of $\psi(f)$.

Case $\psi(\mu\alpha.f)$:

By definition $\psi(\mu\alpha.f) = \mu\alpha.\psi_{\setminus\alpha}(f)$. Hence, we find that $\frac{[\alpha \mapsto \psi_{\setminus\alpha}](\psi_{\setminus\alpha}(f)) \Rightarrow w}{\mu\alpha.\psi_{\setminus\alpha}(f) \Rightarrow w}$

By Lemma 14, we find that $[\alpha \mapsto \mu\alpha.\psi_{\setminus\alpha}(f)](\psi_{\setminus\alpha}(f)) = \psi(f)$.

By induction, there exists $t \in \mathcal{X}_{e,r}(f, s)$ where $t \sim d_w(s)$. By Lemma 15, $\mathcal{X}_{e,r}(f, s) = \mathcal{X}_{e,r}(\mu\alpha.f, s)$ and thus we are done.

Case $\psi(\alpha)$:

There must exist $\mu\alpha.f \in \mathcal{T}(e)$. Hence, this case can be reduced to the one above and we find that $t \in \mathcal{X}_{e,r}(\mu\alpha, s)$ where $t \sim d_w(s)$. By Lemma 15, $\mathcal{X}_{e,r}(\mu\alpha.f, s) = \mathcal{X}_{e,r}(\alpha, s)$ and thus we are done again.

Case $\psi(f_1 + f_2)$:

Suppose $\frac{\psi(f_1) \Rightarrow w}{\psi(f_1) + \psi(f_2) \Rightarrow w}$. By induction, there exists $t \in \mathcal{X}_{e,r}(f_1, s)$ where $t \sim d_w(s)$. By construction $\mathcal{X}_{e,r}(f_1 + f_2, s) \supseteq \mathcal{X}_{e,r}(f_1, s)$ (recall the definition of \mathcal{F}). Thus, we are done. Same reasoning applies for the (sub)case $\frac{\psi(f_2) \Rightarrow w}{\psi(f_1) + \psi(f_2) \Rightarrow w}$.

Case $\psi(f_1 \cdot f_2)$:

Consider $\frac{\psi(f_1) \Rightarrow w_1 \quad \psi(f_2) \Rightarrow w_2}{\psi(f_1) \cdot \psi(f_2) \Rightarrow w_1 \cdot w_2}$. By induction on the case (f_1, s) , there exists $t_1 \in \mathcal{X}_{e,r}(f_1, s)$ where $t_1 \sim d_{w_1}(s)$. By induction on the case (f_2, t_1) ,

there exists $t \in \mathcal{X}_{e,r}(f_2, t_1)$ where $t \sim d_{w_2}(t_1) \sim d_{w_1 \cdot w_2}(s)$. We have that $t \in \mathcal{X}_{e,r}(f_1 \cdot f_2, s)$ based on the following reasoning.

$$\begin{aligned} & \mathcal{X}_{e,r}(f_1 \cdot f_2, s) \\ & \supseteq \\ & \mathcal{F}_{f_2, \mathcal{F}_{f_1, s} \mathcal{X}_{e,r}} \mathcal{X}_{e,r} \\ & \supseteq \\ & \mathcal{F}_{f_2, t_1} \mathcal{X}_{e,r} \\ & = \\ & \mathcal{X}_{e,r}(f_2, t_1) \ni t_2 \end{aligned}$$

Thus, we are done for this case.

The remaining cases are straightforward. \square

Lemma 18. *Let e be a context-free expression and r be a regular expression. Let $\psi = \mathcal{T}(e)$. Let $n \geq 0$, $(f, s) \in \mathcal{T}(e) \times D(r)$ and t be a regular expression such that $t \in X^n_{e,r}(f, s)$. Then, there exists word w such that $\psi(f) \Rightarrow w$ and $t \sim d_w(s)$.*

Proof. By induction over n .

Case $n = 0$: Statement holds trivially as $X^0_{e,r}(f, s) = \{\}$.

Case $n \implies n + 1$:

We proceed by induction over the structure of f .

Subcase ϕ : Trivial.

Subcase x :

Consider $t \in X^{n+1}_{e,r}(x, s) = \{cnf(d_x(s))\}$. Take $w = x$ and the statement is satisfied.

Subcase ε :

Consider $t \in X^{n+1}_{e,r}(x, s) = \{cnf(s)\}$. Take $w = \varepsilon$ to satisfy the statement.

Subcase α :

Consider

$$\begin{aligned} t & \in X^{n+1}_{e,r}(\alpha, s) \\ & = (\mathcal{F}_{e,r}(X^n_{e,r}))(\alpha, s) \\ & = X^n_{e,r}(\alpha, s) \cup \mathcal{R}(\alpha, s, X^n_{e,r}) \\ & = X^n_{e,r}(\alpha, s) \cup X^n_{e,r}(\mu\alpha.f, s) \end{aligned}$$

Suppose $t \in X^n_{e,r}(\alpha, s)$. By induction, there exists w such that $\psi(\alpha) \Rightarrow w$ and $t \sim d_w(s)$ and thus we can establish the statement. Otherwise, $t \in X^n_{e,r}(\mu\alpha.f, s)$. By induction, there exists w such that $\psi(\mu\alpha.f) \Rightarrow w$ and $t \sim d_w(s)$. By construction of ψ we have that $\psi(\alpha) = \psi(\mu\alpha.f)$ and we are done again.

Subcase $\mu\alpha.f$:

Consider

$$\begin{aligned} t & \in X^{n+1}_{e,r}(\mu\alpha.f, s) \\ & = X^n_{e,r}(\mu\alpha.f, s) \cup \mathcal{R}(f, s, X^n_{e,r}) \\ & \subseteq X^n_{e,r}(\mu\alpha.f, s) \cup X^{n+1}_{e,r}(f, s) \end{aligned}$$

Suppose $t \in X^n_{e,r}(\mu\alpha.f, s)$. By induction, there exists w such that $\psi(\mu\alpha.f) \Rightarrow w$ and $t \sim d_w(s)$. Hence, we can establish the statement. Otherwise, $t \in X^{n+1}_{e,r}(f, s)$.

By induction, there exists w such that $\psi(f) \Rightarrow w$ and $t \sim d_w(s)$. By Lemma 14 we find that $[\alpha \mapsto \mu\alpha.\psi_{\setminus\alpha}(f)](\psi_{\setminus\alpha}(f)) = \psi(f)$. By definition $\psi(\mu\alpha.f) = \mu\alpha.\psi_{\setminus\alpha}(f)$. Hence, we can conclude that $\psi(\mu\alpha.f) \Rightarrow w$ and we are done for this subcase.

Subcase $f_1 + f_2$:

Consider

$$\begin{aligned} t &\in X^{n+1}_{e,r}(f_1 + f_2, s) \\ &= (\mathcal{F}_{e,r}(X^n_{e,r}))(f_1 + f_2, s) \\ &= X^n_{e,r}(f_1 + f_2, s) \cup \mathcal{R}(f_1, s, X^n_{e,r}) \cup \mathcal{R}(f_2, s, X^n_{e,r}) \end{aligned}$$

Suppose $t \in X^n_{e,r}(f_1 + f_2, s)$. By induction, there exists w such that $\psi(f_1 + f_2)$ and $t \sim d_w(s)$. Hence, the statement holds. Suppose $t \in \mathcal{R}(f_1, s, X^n_{e,r})$. By definition $\mathcal{R}(f_1, s, X^n_{e,r}) \subseteq X^{n+1}_{e,r}(f_1, s)$. By induction, there exists w such that $\psi(f_1) \Rightarrow w$ and $t \sim d_w(s)$. We can conclude that $\psi(f_1 + f_2) \Rightarrow w$ and are done. Otherwise, $t \in \mathcal{R}(f_2, s, X^n_{e,r})$. Similar reasoning applies as in the previous case.

Subcase $f_1 \cdot f_2$:

Consider

$$\begin{aligned} t &\in X^{n+1}_{e,r}(f_1 \cdot f_2, s) \\ &= X^n_{e,r}(f_1 \cdot f_2, s) \cup \mathcal{R}(f_2, \mathcal{R}(f_1, s, X^n_{e,r}), X^n_{e,r}) \end{aligned}$$

Suppose $t \in X^n_{e,r}(f_1 \cdot f_2, s)$. By induction there exists w such that $\psi(f_1 \cdot f_2)$ and $t \sim d_w(s)$. Hence, the statement holds. Otherwise, $t \in \mathcal{R}(f_2, \mathcal{R}(f_1, s, X^n_{e,r}), X^n_{e,r})$. There exists $t_1 \in \mathcal{R}(f_1, s, X^n_{e,r})$ such that $t \in \mathcal{R}(f_2, t_1, X^n_{e,r})$. By induction on $t_1 \in \mathcal{R}(f_1, s, X^n_{e,r})$, there exists w_1 such that $\psi(f_1) \Rightarrow w_1$ and $t_1 \sim d_{w_1}(s)$. by induction on $t \in \mathcal{R}(f_2, t_1, X^n_{e,r})$, there exists w_2 where $\psi(f_2) \Rightarrow w_2$ and $t_2 \sim d_{w_2}(t_1)$. We can conclude that $\psi(f_1 \cdot f_2) \Rightarrow w_1 \cdot w_2$ and $t \sim d_{w_1 \cdot w_2}(s)$ and are thus done. \square

Lemma 19. *Let e be a context-free expression and r be a regular expression. Then, we have that $\mathcal{X}_{e,r}(e, r) \sim \text{reach}(e, r)$*

Proof. Follows from Lemmas 17 and 18 and the fact that $\mathcal{T}(e)(e) = e$.

E Proofs

For some proofs we make use of the terminology and results introduced in the above.

E.1 Proof of Lemma 3

Proof. By induction on the derivation $e \Rightarrow w$. \square

E.2 Proof of Lemma 4

Proof. By induction on the derivation $\vdash p : e$. For brevity, we consider some selected cases.

Case $\mu\alpha.e$:

By assumption $\frac{\vdash p : [\alpha \mapsto \mu\alpha.e](e)}{\vdash \text{FOLD } p : \mu\alpha.e}$. By induction, $[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \text{flatten}(p)$.

By definition, $\text{flatten}(\text{FOLD } p) = \text{flatten}(p)$. Hence, $\mu\alpha.e \Rightarrow \text{flatten}(\text{FOLD } p)$.

Case $e + f$:

Subcase $p = \text{INL } p_1$:

By induction, $e \Rightarrow \text{flatten}(p_1)$. By definition, $\text{flatten}(\text{INL } p_1) = \text{flatten}(p_1)$.

Hence, $e + f \Rightarrow \text{flatten}(p)$.

Subcase $p = \text{INR } p_2$: Similar reasoning as above. \square

E.3 Proof of Theorem 3

Proof. By definition $e \leq r$ iff $(\forall w \in \Sigma^*, e \Rightarrow w \text{ implies } r \Rightarrow w)$ iff $(\forall w \in \Sigma^*, e \Rightarrow w \text{ implies } d_w(r) \Rightarrow \varepsilon)$ iff each expression in $\text{reach}(e, r)$ is nullable. \square

E.4 Proof of Lemma 5

Proof. We generalize the statement as follows. Consider e and r fixed. For $f, f' \in \mathcal{T}(e)$ we write $f < f'$ to denote that f is a subexpression in f' where $f \neq f'$. Let $\psi = \mathcal{S}(e)$. Consider $(f, s) \in \mathcal{T}(e) \times D(r)$. Let $\Gamma = \{s \xrightarrow{\psi(\mu\alpha.f')} \text{reach}(\psi(\mu\alpha.f'), s) \mid \mu\alpha.f' \in \mathcal{T}(e) \wedge f < \mu\alpha.f'\}$. So, the environment Γ consists of all assumptions which are in the surrounding scope of f .

We claim that $\Gamma \vdash s \xrightarrow{\psi(f)} \text{reach}(\psi(s), s)$ is derivable. The statement follows for e and r from the fact that for e the environment Γ is empty and $\psi(e) = e$.

We verify that $\Gamma \vdash s \xrightarrow{\psi(f)} \text{reach}(\psi(s), s)$ is derivable by induction on f .

Case $\mu\alpha.f$: We observe that $\psi(\mu\alpha.f) = \mu\alpha.\psi_{\lambda\alpha}(f)$. Hence, the desired statement

$$\Gamma \vdash s \xrightarrow{\psi(\mu\alpha.f)} \text{reach}(\psi(\mu\alpha.f), s)$$

is equal to

$$\Gamma \vdash s \xrightarrow{\mu\alpha.\psi_{\lambda\alpha}(f)} \text{reach}(\mu\alpha.\psi_{\lambda\alpha}(f), s).$$

By rule inversion,

$$\Gamma \vdash s \xrightarrow{\mu\alpha.\psi_{\lambda\alpha}(f)} \text{reach}(\mu\alpha.\psi_{\lambda\alpha}(f), s)$$

if

$$\Gamma \cup \{s \xrightarrow{\mu\alpha.\psi_{\lambda\alpha}(f)} \text{reach}(\mu\alpha.\psi_{\lambda\alpha}(f), s)\} \vdash s \xrightarrow{[\alpha \mapsto \mu\alpha.\psi_{\lambda\alpha}(f)](\psi_{\lambda\alpha}(f))} \text{reach}(\mu\alpha.\psi_{\lambda\alpha}(f), s) \quad (1).$$

By assumption Γ has the proper form for $\psi(\mu\alpha.f)$. Hence, $\Gamma \cup \{s \stackrel{\psi(\mu\alpha.f)}{\rightsquigarrow} reach(\psi(\mu\alpha.f), s)\}$ has the proper form for f . By induction,

$$\Gamma \cup \{s \stackrel{\psi(\mu\alpha.f)}{\rightsquigarrow} reach(\psi(\mu\alpha.f), s)\} \vdash s \stackrel{\psi(f)}{\rightsquigarrow} reach(\psi(f), s) \quad (2).$$

We observe that $\psi(f) = [\alpha \mapsto \mu\alpha.\psi_{\setminus\alpha}(f)](\psi_{\setminus\alpha}(f))$ and $reach(\mu\alpha.\psi_{\setminus\alpha}(f), s) = reach([\alpha \mapsto \mu\alpha.\psi_{\setminus\alpha}(f)](\psi_{\setminus\alpha}(f)), s) = reach(\psi(f), s)$. Hence, (1) and (2) are equal and therefore the desired statement can be derived.

Case $e + f$: Expressions e and f share the same Γ . By induction, $\Gamma \vdash s \stackrel{\psi(e)}{\rightsquigarrow} reach(\psi(e), s)$ and $\Gamma \vdash s \stackrel{\psi(f)}{\rightsquigarrow} reach(\psi(f), s)$. By rule (Alt), $\Gamma \vdash s \stackrel{\psi(e)+\psi(f)}{\rightsquigarrow} reach(\psi(e), s) \cup reach(\psi(f), s)$. We observe that $\psi(e + f) = \psi(e) + \psi(f)$ and $reach(\psi(e + f), s) = reach(\psi(e), s) \cup reach(\psi(f), s)$ and are done for this case.

Case $e \cdot f$: Expressions e and f share the same Γ . By induction, $\Gamma \vdash s \stackrel{\psi(e)}{\rightsquigarrow} reach(\psi(e), s)$. Suppose $reach(\psi(e), s) = \{\}$. Then, $\Gamma \vdash s \stackrel{\psi(e)\cdot\psi(f)}{\rightsquigarrow} \{\}$. Under the assumption, $reach(\psi(e) \cdot \psi(f), s) = \{\}$ and we are done. Otherwise, $reach(\psi(e), s) = \{s_1, \dots, s_n\}$ for $n > 0$. By induction, for each combination (f, s_i) , $\Gamma \vdash s_i \stackrel{\psi(f)}{\rightsquigarrow} reach(\psi(f), s_i)$. By rule (Seq), $\Gamma \vdash s \stackrel{\psi(e)\cdot\psi(f)}{\rightsquigarrow} reach(\psi(f), s_1) \cup \dots \cup reach(\psi(f), s_n)$. By the fact that $reach(\psi(f), reach(\psi(e), s)) = reach(\psi(f), s_1) \cup \dots \cup reach(\psi(f), s_n)$ we reach the desired conclusion.

Cases x, ε, ϕ : Straightforward. \square

E.5 Proof of Lemma 6

The proof requires a couple of technical statements.

Lemma 20 (Strengthening). *Let e be a context-free expression, r be a regular expression, S be a set and Γ', Γ be two environments such that $\Gamma' \supseteq \Gamma$ and $\Gamma' \vdash r \stackrel{e}{\rightsquigarrow} S$ where in the derivation tree the extra assumptions $\Gamma' - \Gamma$ are not used. Then, we also find that $\Gamma \vdash r \stackrel{e}{\rightsquigarrow} S$.*

Proof. By induction on the derivation.

Lemma 21 (Weakening). *Let e be a context-free expression, r be a regular expression, S be a set and Γ', Γ be two environments such that $\Gamma' \supseteq \Gamma$ and $\Gamma \vdash r \stackrel{e}{\rightsquigarrow} S$. Then, we also find that $\Gamma' \vdash r \stackrel{e}{\rightsquigarrow} S$.*

Proof. By induction on the derivation.

Lemma 22 (Substitution). *Let $\mu\alpha.f$ be a context-free expression, r be a regular expression and S a set such that $\vdash r \stackrel{\mu\alpha.f}{\rightsquigarrow} S$. Then, we find that $\vdash r \stackrel{[\alpha \mapsto \mu\alpha.f](f)}{\rightsquigarrow} S$.*

Proof. We generalize the statement and include some environment Γ . We write D to denote the derivation tree for $\Gamma \vdash r \stackrel{\mu\alpha.f}{\rightsquigarrow} S$. The shape of D is as follows.

$$\text{(REC)} \frac{\dots}{\frac{\Gamma \cup \{r \stackrel{\mu\alpha.f}{\rightsquigarrow} S\} \vdash r \stackrel{[\alpha \mapsto \mu\alpha.f](f)}{\rightsquigarrow} S}{\Gamma \vdash r \stackrel{\mu\alpha.f}{\rightsquigarrow} S}}$$

Suppose, in the upper derivation tree (denoted by ...), there are no applications of (Hyp) for $\mu\alpha.f$. By Lemma 20, we can immediately conclude that $\Gamma \vdash r \overset{[\alpha \mapsto \mu\alpha.f]}{\rightsquigarrow}(f) S$ is derivable as well. Otherwise, we consider all applications of (Hyp) for $\mu\alpha.f$. In the below, we show only one such application.

$$\text{(HYP)} \frac{\Gamma' \cup \{r \overset{\mu\alpha.f}{\rightsquigarrow} S\} \vdash r \overset{\mu\alpha.f}{\rightsquigarrow} S}{\dots}$$

$$\text{(REC)} \frac{\Gamma \cup \{r \overset{\mu\alpha.f}{\rightsquigarrow} S\} \vdash r \overset{[\alpha \mapsto \mu\alpha.f]}{\rightsquigarrow}(f) S}{\Gamma \vdash r \overset{\mu\alpha.f}{\rightsquigarrow} S}$$

where by construction $\Gamma' \supseteq \Gamma$.

Each such (Hyp) rule application can be replaced by the derivation tree D where we make use of $\Gamma' \cup \{r \overset{\mu\alpha.f}{\rightsquigarrow} S\}$ instead of Γ (justified by Lemma 21). In fact, we can argue that the extra assumption $r \overset{\mu\alpha.f}{\rightsquigarrow} S$ is no longer required due to the elimination of rule (Hyp). Hence, we can argue that $\Gamma \vdash r \overset{[\alpha \mapsto \mu\alpha.f]}{\rightsquigarrow}(f) S$ is derivable. \square

We write e^k to denote that all recursive constructs in e have been unfolded at least k -times.

Lemma 23. *Let e be a context-free expression, r be a regular expression. Then, for any $n \geq 0$ there exists a k such that $S \supseteq X^n_{e,r}(e, r)$ where $\vdash r \overset{e^k}{\rightsquigarrow} S$.*

Proof. We define

$$\begin{aligned} \mathcal{R}'(\phi, r) &= \{\} \\ \mathcal{R}'(\varepsilon, r) &= \{cnf(r)\} \\ \mathcal{R}'(x, r) &= \{cnf(d_x(r))\} \\ \mathcal{R}'(e + f, r) &= \mathcal{R}'(e, r) \cup \mathcal{R}'(f, r) \\ \mathcal{R}'(e \cdot f, r) &= \bigcup_{s \in \mathcal{R}'(e, r)} \mathcal{R}'(f, s) \\ \mathcal{R}'(\alpha, r) &= \{\} \\ \mathcal{R}'(\mu\alpha.e, r) &= \{\} \end{aligned}$$

S1: For a fixed e and r , for any $(f, s) \in \mathcal{T}(e) \times D(r)$ and $n \geq 0$, there exists k such that $\mathcal{R}'((\psi(f))^k, s) \supseteq X^n_{e,r}(f, s)$. Like the proof of Lemma 18, we verify the statement by applying induction over n and observing the structure of f .

Case $n = 0$: Straightforward.

Case $n \implies n + 1$:

We proceed by induction over the structure of f .

Subcase $\mu\alpha.f$. We have that $X^{n+1}_{e,r}(\mu\alpha.f, s) \subseteq X^n_{e,r}(\mu\alpha.f, s) \cup X^{n+1}_{e,r}(f, s)$. By induction on n , $\mathcal{R}'((\psi(\mu\alpha.f))^{k_1}, s) \supseteq X^n_{e,r}(\mu\alpha.f, s)$ for some k_1 . By induction on f , $\mathcal{R}'((\psi(f))^{k_2}, s) \supseteq X^{n+1}_{e,r}(f, s)$ for some k_2 . Recall that $[\alpha \mapsto \mu\alpha.\psi_{\lambda\alpha}(f)](\psi_{\lambda\alpha}(f)) = \psi(f)$ and $\psi(\mu\alpha.f) = \mu\alpha.\psi_{\lambda\alpha}(f)$. Hence, $(\psi(\mu\alpha.f))^1 = \psi(f)$. Function $\mathcal{R}'(\cdot)$ is a monotone function respect to unfoldings. We set

$k = k_1 + k_2$. Then, $\mathcal{R}'((\psi(\mu\alpha.f))^k, s) \supseteq X^{n+1}_{e,r}(\mu\alpha.f, s)$ and we are done for this case.

S2: For $\vdash r \xrightarrow{e^k} S$, we have that $S \supseteq \mathcal{R}'(e^k, r)$. By induction on k and observing the structure of e .

Desired statement follows from S1 and S2. \square

We are in the position to proof Lemma 6. We recall the statement of this proposition: Let e be a context-free expression, r be a regular expression and S be a set of expressions such that $\vdash r \xrightarrow{e} S$. Then, we find that $S \supseteq \text{reach}(e, r)$.

Proof. Assume the contrary. Then, there exists $s \in S$ and $s \notin X^n_{e,r}(e, r)$ for some $n \geq 0$. By Lemma 22, we find that $\vdash r \xrightarrow{e^k} S$ for any k . By Lemma 23, $S \supseteq X^n_{e,r}(e, r)$ which contradicts the assumption. \square

Based on the above, we obtain a greatest fixpoint method to compute $\text{reach}(e, r)$. We consider e and r fixed. For each combination $(f, s) \in \mathcal{T}(e) \times D(r)$, we set the respective S to $D(r)$. In each greatest fixpoint step, we pick a combination where we remove one of the elements in S . Check if $\vdash f \xrightarrow{s} S^3$ is still derivable. If yes, continue the process of eliminating elements.

E.6 Proof of Lemma 24

Definition 28 (Well-Behaved Upcast). Let e be a context-free expression, r be a regular expression, and c be a coercion of type $\mathbb{U}(e, r)$, where we write $\mathbb{U}(e, r)$ for the type $(e, +\text{reach}(e, r)) \rightarrow r$.

We say c is a well-behaved upcast iff for any $\vdash p : e$ and $\vdash_r t : +\text{reach}(e, r)$ we find that $\vdash_r c(p, t) : r$.

We further define environments Δ by

$$\Delta ::= \{ \} \mid \{ v : \mathbb{U}(e, r) \} \mid \Delta \cup \Delta$$

We say that Δ is a well-behaved upcast environment iff each $(v : \mathbb{U}(e, r)) \in \Delta$ is a well-behaved upcast coercion.

Lemma 24 (Soundness). Let Δ be a well-behaved upcast environment. Let e be a context-free expression and r be a regular expression such that $\Delta \vdash \uparrow c : \mathbb{U}(e, r)$ for some coercion c . Let p and t be parse trees such that $\vdash p : e$ and $\vdash_r t : +\text{reach}(e, r)$ where $\text{flatten}(t) \in L(d_{\text{flatten}(p)}(r))$. Then, we find that $\Delta \vdash c((p, t)) : r$.

Proof. By induction on the derivation to construct coercions. For brevity, we sometimes omit Γ in case it is not relevant.

Case ε : By assumption $\vdash p : \varepsilon$ and $\vdash t : +\text{reach}(\varepsilon, r)$.

Thus, $p = \text{EPS}$ and $t : \text{cnf}(r)$.

³ Need to include the environment, apply ψ , as we already start if with some environment, can only apply (Hyp) after one application of (Rec) ...

Inversion yields a regular coercion $c_1 : \text{cnf}(r) \rightarrow r$.

Hence $(\lambda(\text{EPS}, q).c_1(t))(\varepsilon, t) = c_1(t)$ with $\vdash c_1(t) : r$.

Case x : By assumption $\vdash p : x$ and $\vdash t : +\text{reach}(x, r)$.

Thus $p = \text{SYM } x$ and $\vdash t : +\text{reach}(x, r)$ where $+\text{reach}(x, r) = \text{cnf}(d_x(r))$.

Inversion yields a regular coercion $c : (x, \text{cnf}(d_x(r))) \rightarrow r$.

Hence $\vdash c((p, t)) : r$

Case $e \cdot f$: By assumption $\vdash p : e \cdot f$ and $\vdash t : +\text{reach}(e \cdot f, r)$ and $\text{flatten}(t) \in L(d_{\text{flatten}(p)}(r))$.

Inversion for p yields $p = \text{SEQ } p_1 p_2$ such that $\vdash p_1 : e$ and $\vdash p_2 : f$. It holds that $\text{flatten}(p) = \text{flatten}(p_1) \cdot \text{flatten}(p_2)$.

Further,

$$\begin{aligned} \text{reach}(e \cdot f, r) &= \bigcup \{ \text{reach}(f, s) \mid s \in \text{reach}(e, r) \} \\ &= \text{reach}(f, +\text{reach}(e, r)) \end{aligned}$$

so that $\vdash t : +\text{reach}(f, +\text{reach}(e, r))$.

Now

$$\begin{aligned} \text{flatten}(t) \in L(d_{\text{flatten}(p)}(r)) &= L(d_{\text{flatten}(p_1) \cdot \text{flatten}(p_2)}(r)) \\ &\subseteq L(d_{\text{flatten}(p_2)}(+\text{reach}(e, r))) \end{aligned}$$

Inversion on the coercion derivation yields

$$\begin{aligned} \Gamma \vdash \uparrow c_1 : \mathbb{U}(e, r) \\ \Gamma \vdash \uparrow c_2 : \mathbb{U}(f, +\text{reach}(e, r)) \end{aligned}$$

Induction on the derivation of c_2 yields $\vdash c_2((p_2, t)) : +\text{reach}(e, r)$.

Induction on the derivation of c_2 using p_1 for p and $c_2((p_2, t))$ for t yields $\vdash c_1((p_1, c_2((p_2, t)))) : r$ as desired.

Case $e + f$: By assumption $\vdash p : e + f$ and $\vdash t : +\text{reach}(e + f, r)$ and $\text{flatten}(t) \in L(d_{\text{flatten}(p)}(r))$. We distinguish among the following subcases.

Subcase $p = \text{INL } p_1$: At this point, we have $\vdash p_1 : e$ by inversion of the assumption. We conclude that $\text{flatten}(p) = \text{flatten}(p_1) \in L(e)$. By Lemma 4, $e \Rightarrow \text{flatten}(p_1)$ and therefore we find that $d_{\text{flatten}(p_1)}(r)$ is similar to an element of $\text{reach}(e, r)$. Because $\text{flatten}(t) \in L(d_{\text{flatten}(p)}(r))$ we conclude that $\text{flatten}(t) \in L(+\text{reach}(e, r))$. By Lemma 7, it must be that $b_1(t) = \text{Just } t_1$ for some t_1 where $\vdash t_1 : +\text{reach}(e, r)$. By induction we find that $\vdash c_1((p_1, t_1)) : r$. By combining the above results, we conclude that $c((p, t)) : r$, too.

Subcase $p = \text{INR } p_2$: Analogously.

Case $\mu\alpha.e$:

By assumption $\vdash p : \mu\alpha.e$ and $\vdash t : +\text{reach}(\mu\alpha.e, r)$ and $\text{flatten}(t) \in L(d_{\text{flatten}(p)}(r))$.

By inversion of the assumption, $p = \text{FOLD } p'$, $\vdash p' : [\alpha \mapsto \mu\alpha.e](e)$, and $\text{flatten}(p) = \text{flatten}(p')$.

Further, as $\text{reach}(\mu\alpha.e, r) = \text{reach}([\alpha \mapsto \mu\alpha.e](e), r)$ (by Lemma 2), we obtain $\vdash t : +\text{reach}([\alpha \mapsto \mu\alpha.e](e), r)$.

Subcase: If $(\mu\alpha.e, r)$ is not in Γ , then inversion yields some c' such that

$$\Gamma \cup \{v : (\mu\alpha.e, r)\} \vdash^\uparrow c' : \mathbb{U}([\alpha \mapsto \mu\alpha.e](e), r)$$

Hence, induction is applicable observing $\vdash p' : [\alpha \mapsto \mu\alpha.e](e)$, $\vdash t : +reach([\alpha \mapsto \mu\alpha.e](e), r)$, and that the flattening assumption holds. Thus, $\vdash c' ((p', t)) : r$ and we need to show that

$$\begin{aligned} & \vdash (\text{rec } v.\lambda(\text{FOLD } p', t).c' ((p', t)))(\text{FOLD } p', t) : r \\ \Leftrightarrow & \quad \text{By subject reduction} \\ & \vdash (\lambda(\text{FOLD } p', t).c' ((p', t)))[v \mapsto (\text{rec } v.\lambda(\text{FOLD } p', t).c' ((p', t)))](\text{FOLD } p', t) : r \\ \Leftrightarrow & \quad \text{By subject reduction} \\ & \vdash c' ((p', t))[v \mapsto (\text{rec } v.\lambda(\text{FOLD } p', t).c' ((p', t)))] : r \\ \Leftrightarrow & \quad \text{Substitution lemma backwards} \\ & v : (\mu\alpha.e, +reach(\mu\alpha.e, r)) \rightarrow r \vdash c' ((p', t)) : r \end{aligned}$$

The last statement holds and thus we are done.

Subcase: If $v : (\mu\alpha.e, r) \in \Gamma$, then the statement holds immediately. \square

E.7 Proof of Lemma 9

Proof. The direction from left to right follows immediately.

Suppose $\varepsilon \in L(e)$ which implies $e \Rightarrow \varepsilon$. Consider the case $\frac{[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \varepsilon}{\mu\alpha.e \Rightarrow \varepsilon}$.

We argue that if $[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \varepsilon$ then $e \Rightarrow \varepsilon$. This can be verified by the number of unfolding steps applied on $\mu\alpha.e$.

Consider $[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \varepsilon$ where no further unfolding steps are executed on $\mu\alpha.e$. Then, by induction on the derivation $[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \varepsilon$ we obtain a derivation $e \Rightarrow \varepsilon$. For clarity, we write $f \stackrel{\mu\alpha}{\Rightarrow} \varepsilon$ to denote a derivation in which no unfolding step takes place for $\mu\alpha.e$.

Consider the induction step. By induction, if $[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \varepsilon$ then $[\alpha \mapsto \mu\alpha.e](e) \stackrel{\mu\alpha}{\Rightarrow} \varepsilon$. By induction on the derivation $\stackrel{\mu\alpha}{\Rightarrow}$ we can argue that we obtain $e \stackrel{\mu\alpha}{\Rightarrow} \varepsilon$. Hence, if $[\alpha \mapsto \mu\alpha.e](e) \Rightarrow \varepsilon$ then $e \stackrel{\mu\alpha}{\Rightarrow} \varepsilon$.

The above reasoning considers the elimination of the unfolding step for a specific $\mu\alpha.e$. By induction, we can argue that any unfolding step can be eliminated. We write $\stackrel{\mu\alpha}{\Rightarrow}$ to denote the derivation where no unfolding steps occur. Hence, if $e \Rightarrow \varepsilon$ then $e \stackrel{\mu\alpha}{\Rightarrow} \varepsilon$. By induction on $\stackrel{\mu\alpha}{\Rightarrow}$ we can argue that $\mathcal{N}(e)$ holds. \square

E.8 Proof of Lemma 10

Proof. Based on the observations in the proof of Lemma 9, from $\varepsilon \in L(e)$ we can conclude $e \stackrel{\mu\alpha}{\Rightarrow} \varepsilon$. Then, by induction on $\stackrel{\mu\alpha}{\Rightarrow}$ we can derive the desired statements. \square

E.9 Proof of Theorem 4

Proof. Follows immediately from Lemmas 24 and 10. Note that all elements in $\text{reach}(e, r)$ are nullable. Hence, $\text{flatten}(\text{mkE}(\text{+reach}(e, r))) \in d_{\text{flatten}(p)}(r)$.

E.10 Proof of Lemma 25

Definition 29 (Well-Behaved Downcast). *Let e be a context-free expression and r be a regular expression. We write $\text{D}(e, r)$ to denote the type $r \rightarrow \text{Maybe}(e, \text{+reach}(e, r))$. Let c be a coercion of type $\text{D}(e, r)$. We say c is a well-behaved downcast coercion iff (1) for any $\vdash_r t : r$ we find that $\vdash c(t) : \text{Maybe}(e, \text{+reach}(e, r))$. Moreover, (2) if $c(t) = \text{Just}(p, t')$, then $\text{flatten}(t) = \text{flatten}(p) \cdot \text{flatten}(t')$. (3) If $c(t) = \text{Nothing}$, then there exist no $\vdash p : e$ and $\vdash_r t' : \text{+reach}(e, r)$ such that $\text{flatten}(t) = \text{flatten}(p) \cdot \text{flatten}(t')$.*

We say that Δ is a well-behaved downcast environment iff each $(v : \text{D}(e, r)) \in \Delta$ is a well-behaved downcast coercion.

In case (2) holds, we can conclude that $\text{flatten}(u) \in L(d_{\text{flatten}(p)}(r))$ due to $\vdash t : r$ and Lemma 4 and the fact that derivatives denote left quotients.

Lemma 25. *Let Δ be a well-behaved downcast environment, e be a context-free expression, r a regular expression, and c a coercion such that $\Delta \vdash_{\downarrow} c : \text{D}(e, r)$. (1) For each regular parse tree $\vdash_r t : r$ we find that $\Delta \vdash c(t) : \text{Maybe}(e, \text{+reach}(e, r))$.*

Moreover, (2) if $c(t) = \text{Just}(p, t')$, then $\text{flatten}(t) = \text{flatten}(p) \cdot \text{flatten}(t')$ and $\text{flatten}(t') \in d_{\text{flatten}(p)}(r)$. (3) If $c(t) = \text{Nothing}$, then there exist no $\vdash p : e$ and $\vdash_r u : \text{+reach}(e, r)$ such that $\text{flatten}(t) = \text{flatten}(p) \cdot \text{flatten}(u)$ and $\text{flatten}(u) \in d_{\text{flatten}(p)}(r)$.

Proof. By induction on the downcast coercion derivation.

Case EPS. (1) follows easily. For (2), we find that $\text{flatten}(\text{EPS}) \cdot \text{flatten}(b(t)) = \varepsilon \cdot \text{flatten}(t) = \text{flatten}(t)$. Case (3) never arises here.

Case SYM x . (1) is again straightforward. Case (2), suppose $c(t) = \text{Just}(\text{SYM } x, t')$, then $\text{flatten}(\text{SYM } x) \cdot \text{flatten}(t') = x \cdot \text{flatten}(t') = \text{flatten}(t)$. Case (3), suppose $c(t) = \text{Nothing}$. Then, $b(t) = \text{Nothing}$ which implies $d_x(r)$ denotes the empty language and therefore no p, t' with the desired property can exist.

Case $e+f$. By induction we obtain c_1 such that $\Gamma \vdash c_1(t) : \text{Maybe}(e, \text{+reach}(e, r))$; if $c_1(t) = \text{Just}(p_1, t_1)$, then $\text{flatten}(t) = \text{flatten}(p_1) \cdot \text{flatten}(t_1)$. In this case, $c(t) = \text{Just}(\text{INL } p_1, b_1(t_1))$. By property of regular coercions (see Lemma 7), we find that $\text{flatten}(t) = \text{flatten}(p_1) \cdot \text{flatten}(b_1(t_1)) = \text{flatten}(p_1) \cdot \text{flatten}(t_1)$. If $c_1(t) = \text{Nothing}$, then no such p_1 and t_1 exist and therefore also no t'_1 of the form $\vdash t'_1 : \text{+reach}(e+f, r)$ exists.

Similar reasoning applies to c_2 . On the other hand, if $c(t) = \text{Just}(p_1, t'_1)$ then one of the respective cases of either c_1 or c_2 applies. Similar reasoning applies in case of $c(t) = \text{Nothing}$ as any $\vdash p : e+f$ has either form $\text{INL } p_1$ or $\text{INR } p_2$, where neither suitable p_1 nor p_2 exist, no such p can exist either.

Case $e \cdot f$. By induction, we obtain c_1, c_2 such that $\Gamma \vdash c_1(t) : \text{Maybe}(e, +\text{reach}(e, r))$ and $\Gamma \vdash c_2(t_1) : \text{Maybe}(f, +\text{reach}(f, +\text{reach}(e, r)))$. Hence, $\Gamma \vdash c(t) : \text{Maybe}(e \cdot f, +\text{reach}(e \cdot f, r))$.

Suppose $c(t) = \text{Just}(p, t_2)$. Then, $c_1(t) = \text{Just}(p_1, t_1)$ and $c_2(t_1) = \text{Just}(p_2, t_2)$. By induction, $\text{flatten}(t) = \text{flatten}(p_1) \cdot \text{flatten}(t_1)$ and $\text{flatten}(t_1) = \text{flatten}(p_2) \cdot \text{flatten}(t_2)$. Hence, $\text{flatten}(t) = \text{flatten}(\text{SEQ } p_1 \ p_2) \cdot \text{flatten}(t_2)$.

Also, $\text{flatten}(t_1) \in d_{\text{flatten}(p_1)}(r)$ and $\text{flatten}(t_2) \in d_{\text{flatten}(p_2)}(+\text{reach}(e, r))$, by induction, but as $\text{flatten}(t_1) = \text{flatten}(p_2) \cdot \text{flatten}(t_2)$, we know that $\text{flatten}(t_2) \in d_{\text{flatten}(p_2)}(d_{\text{flatten}(p_1)}(r)) = d_{\text{flatten}(\text{SEQ } p_1 \ p_2)}(r)$.

Suppose $c(t) = \text{Nothing}$. Then, either $c_1(t) = \text{Nothing}$ or if $c_1(t) = \text{Just}(p_1, t_1)$ then $c_2(t_1) = \text{Nothing}$.

If $c_1(t) = \text{Nothing}$, then there exist no $\vdash p_1 : e$ and $\vdash t_1 : +\text{reach}(e, r)$ such that $\text{flatten}(t) = \text{flatten}(p_1) \cdot \text{flatten}(t_1)$. If there was some suitable $p = \text{SEQ } p_1 \ p_2$ with $\vdash p : e \cdot f$, then we can construct suitable t_1 for p_1 . Contradiction.

If $c_1(t) = \text{Just}(p_1, t_1)$ and $c_2(t_1) = \text{Nothing}$, then we can derive a similar contradiction for p_2 and t_2 .

Case $\mu\alpha.e$. By induction, $\Gamma \cup \{v : (\mu\alpha.e, r)\} \vdash c'(t) : \text{Maybe}([\alpha \mapsto \mu\alpha.e](e), +\text{reach}([\alpha \mapsto \mu\alpha.e](e), r))$. By subjection reduction, $\Gamma \vdash c(t) : \text{Maybe}(\mu\alpha.e, +\text{reach}(\mu\alpha.e, r))$.

Suppose $c(t) = \text{Just}(p, t')$. We find that $p = \text{FOLD } p'$. We have that $+\text{reach}([\alpha \mapsto \mu\alpha.e](e), r) = +\text{reach}(\mu\alpha.e, r)$. Hence, $\vdash t' : +\text{reach}(\mu\alpha.e, r)$ implies $\vdash t' : +\text{reach}([\alpha \mapsto \mu\alpha.e](e), r)$ and vice versa. By induction, $\text{flatten}(t) = \text{flatten}(p') \cdot \text{flatten}(t')$ and this establishes (2). Suppose $c(t) = \text{Nothing}$. Then, $c'(t) = \text{Nothing}$. By induction, no suitable p' and t' exists where $\vdash p' : [\alpha \mapsto \mu\alpha.e](e)$ and $\vdash t' : +\text{reach}([\alpha \mapsto \mu\alpha.e](e), r)$. Hence, there can be no suitable p and t' either where $\vdash p : \mu\alpha.e$ and $\vdash t' : +\text{reach}(\mu\alpha.e, r)$ and we are done. \square