Compilers and Formal Languages (2)

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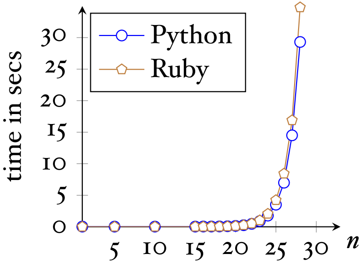
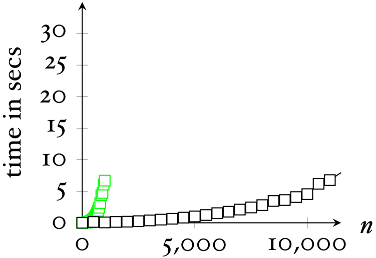
An Efficient Regular Expression Matcher

Graphs: $a^{?\{n\}} \cdot a^{\{n\}}$ and strings $\underbrace{a \ldots a}_{n}$





In the handouts is a similar graph with $(a^*)^* \cdot b$ for Java.

Languages

- A **Language** is a set of strings, for example

$$\{[], hello, foobar, a, abc\}$$

- **Concatenation** of strings and languages

$$foo \mathbin{@} bar \;=\; foobar$$

$$A \mathbin{@} B \;\stackrel{\text{def}}{=}\; \{s_1 \mathbin{@} s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

For example $A = \{foo, bar\}$, $B = \{a, b\}$

$$A \mathbin{@} B = \{fooa, foob, bara, barb\}$$

The Power Operation

- The **Power** of a language:

$$A^0 \stackrel{\text{def}}{=} \{[]\}$$
$$A^{n+1} \stackrel{\text{def}}{=} A \,@\, A^n$$

For example

$$A^4 = A \,@\, A \,@\, A \,@\, A$$
$$A^1 = A$$
$$A^0 = \{[]\}$$

Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in A^4?

Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in A^4?

What if $A = \{[a], [b], [c], []\}$;
how many strings are then in A^4?

The Star Operation

- The **Star** of a language:

$$A\star \stackrel{\text{def}}{=} \bigcup_{0 \leq n} A^n$$

This expands to

$$A^0 \cup A^1 \cup A^2 \cup A^3 \cup A^4 \cup \ldots$$

$$\{[]\} \cup A \cup A\,@\,A \cup A\,@\,A\,@\,A \cup A\,@\,A\,@\,A\,@\,A \cup \ldots$$

The Meaning of a Regular Expression

$$
\begin{array}{rcl}
L(\mathbf{0}) & \stackrel{\text{def}}{=} & \{\} \\
L(\mathbf{1}) & \stackrel{\text{def}}{=} & \{[]\} \\
L(c) & \stackrel{\text{def}}{=} & \{[c]\} \\
L(r_1 + r_2) & \stackrel{\text{def}}{=} & L(r_1) \cup L(r_2) \\
L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \{s_1 @ s_2 \mid s_1 \in L(r_1) \wedge s_2 \in L(r_2)\} \\
L(r^*) & \stackrel{\text{def}}{=} & (L(r))\star \quad \stackrel{\text{def}}{=} \bigcup_{0 \leq n} L(r)^n
\end{array}
$$

L is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

The Specification of Matching

A regular expression r matches a string s provided

$$s \in L(r)$$

...and the point of the this lecture is to decide this problem as fast as possible (unlike Python, Ruby, Java etc)

Semantic Derivative

- The **Semantic Derivative** of a language wrt to a character c:

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c :: s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \{\} \end{aligned}$$

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For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\, f\, A &= \{oo, rak\} \\ Der\, b\, A &= \{ar\} \\ Der\, a\, A &= \{\} \end{aligned}$$

We can extend this definition to strings

$$Ders\, s\, A = \{s' \mid s @ s' \in A\}$$

Regular Expressions

Their inductive definition:

r	$::=$	$\mathbf{0}$	null
	$\mid$	$\mathbf{1}$	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

Th

```
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

r	$::=$	$\mathbf{0}$	null
	$\mid$	$\mathbf{1}$	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

When Are Two Regular Expressions Equivalent?

$$r_1 \equiv r_2 \quad \stackrel{\text{def}}{=} \quad L(r_1) = L(r_2)$$

Concrete Equivalences

$$\begin{array}{rcl}
(a + b) + c & \equiv & a + (b + c) \\
a + a & \equiv & a \\
a + b & \equiv & b + a \\
(a \cdot b) \cdot c & \equiv & a \cdot (b \cdot c) \\
c \cdot (a + b) & \equiv & (c \cdot a) + (c \cdot b)
\end{array}$$

Concrete Equivalences

$$
\begin{array}{rcl}
(a+b)+c & \equiv & a+(b+c) \\
a+a & \equiv & a \\
a+b & \equiv & b+a \\
(a\cdot b)\cdot c & \equiv & a\cdot(b\cdot c) \\
c\cdot(a+b) & \equiv & (c\cdot a)+(c\cdot b)
\end{array}
$$

$$
\begin{array}{rcl}
a\cdot a & \not\equiv & a \\
a+(b\cdot c) & \not\equiv & (a+b)\cdot(a+c)
\end{array}
$$

Corner Cases

$$\begin{array}{rcl} a \cdot \mathbf{0} & \not\equiv & a \\ a + \mathbf{1} & \not\equiv & a \\ \mathbf{1} & \equiv & \mathbf{0}^* \\ \mathbf{1}^* & \equiv & \mathbf{1} \\ \mathbf{0}^* & \not\equiv & \mathbf{0} \end{array}$$

Simplification Rules

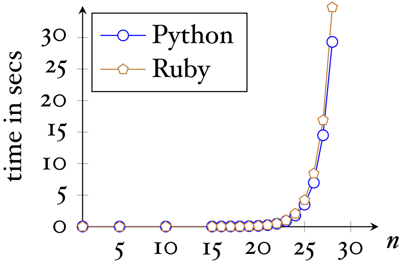
$$
\begin{aligned}
r + \mathbf{0} &\equiv r \\
\mathbf{0} + r &\equiv r \\
r \cdot \mathbf{1} &\equiv r \\
\mathbf{1} \cdot r &\equiv r \\
r \cdot \mathbf{0} &\equiv \mathbf{0} \\
\mathbf{0} \cdot r &\equiv \mathbf{0} \\
r + r &\equiv r
\end{aligned}
$$

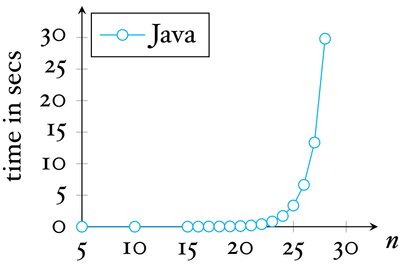
The Specification for Matching

A regular expression r matches a string s if and only if

$$s \in L(r)$$

$(a^{?\{n\}}) \cdot a^{\{n\}}$ **and** $(a^*)^* \cdot b$





Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
  - $(a^{?\{n\}}) \cdot a^{\{n\}}$
  - $(a^*)^*$
  - $([a\text{-}z]^+)^*$
  - $(a + a \cdot a)^*$
  - $(a + a?)^*$
- sometimes also called catastrophic backtracking

A Matching Algorithm

...whether a regular expression can match the empty string:

$$
\begin{array}{lcl}
nullable(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathit{false} \\
nullable(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathit{true} \\
nullable(c) & \stackrel{\text{def}}{=} & \mathit{false} \\
nullable(r_1 + r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \vee nullable(r_2) \\
nullable(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \wedge nullable(r_2) \\
nullable(r^*) & \stackrel{\text{def}}{=} & \mathit{true}
\end{array}
$$

The Derivative of a Rexp

If r matches the string $c :: s$, what is a regular expression that matches just s?

$der\, c\, r$ gives the answer, Brzozowski 1964

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else } (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}
$$

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else } (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}
$$

$$
\begin{array}{lcl}
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}
$$

Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$\begin{array}{l} der\, a\, r = ?\\ der\, b\, r = ?\\ der\, c\, r = ? \end{array}$$

The Algorithm

$$matches\, r\, s \stackrel{\text{def}}{=} nullable(ders\, r\, s)$$

An Example

Does r_1 match abc?

Step 1:	build derivative of a and r_1	$(r_2 = der\, a\, r_1)$
Step 2:	build derivative of b and r_2	$(r_3 = der\, b\, r_2)$
Step 3:	build derivative of c and r_3	$(r_4 = der\, c\, r_3)$
Step 4:	the string is exhausted: test whether r_4 can recognise the empty string	$(nullable(r_4))$
Output:	result of the test $\Rightarrow$ $true$ or $false$	

The Idea of the Algorithm

If we want to recognise the string *abc* with regular expression r_1 then

1. $Der\, a\, (L(r_1))$

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\, a\, (L(r_1))$
2. $Der\, b\, (Der\, a\, (L(r_1)))$

The Idea of the Algorithm

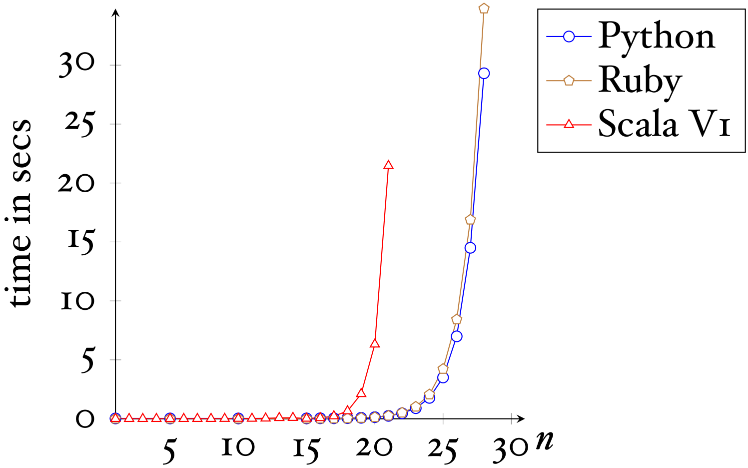
If we want to recognise the string abc with regular expression r_1 then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$
3. $Der\,c\,(Der\,b\,(Der\,a\,(L(r_1))))$

4. finally we test whether the empty string is in this set; same for $Ders\,abc\,(L(r_1))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

Oops... $(a^{?\{n\}}) \cdot a^{\{n\}}$



A Problem

We represented the "n-times" $a^{\{n\}}$ as a sequence regular expression:

$$
\begin{array}{rl}
1: & a \\
2: & a \cdot a \\
3: & a \cdot a \cdot a \\
 & \ldots \\
13: & a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \\
 & \ldots \\
20: &
\end{array}
$$

This problem is aggravated with $a^?$ being represented as $a + \mathbf{1}$.

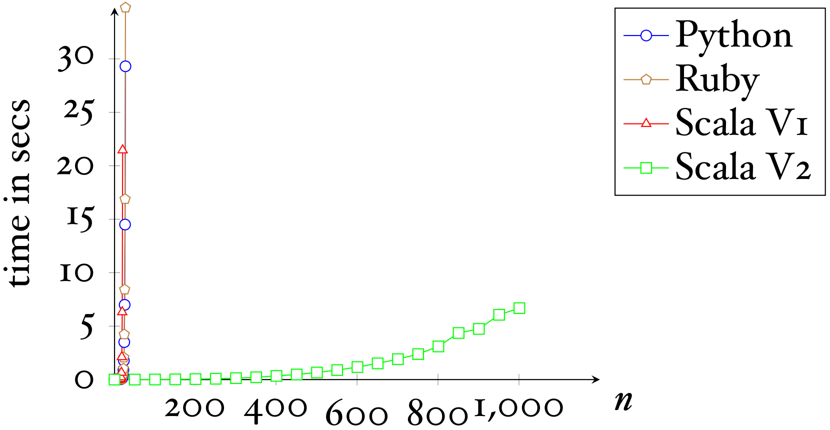
Solving the Problem

What happens if we extend our regular expressions

$$
\begin{array}{rcl}
r & ::= & \ldots \\
  & | & r^{\{n\}} \\
  & | & r^{?}
\end{array}
$$

What is their meaning?
What are the cases for *nullable* and *der*?

$$(a^{?\{n\}}) \cdot a^{\{n\}}$$


time in secs
0
5
10
15
20
25
30
200
400
600
800
1,000
n
Python
Ruby
Scala V1
Scala V2

Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$\begin{aligned} der\, a\, r &= ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r \\ der\, b\, r &= ((\mathbf{0} \cdot b) + \mathbf{1}) \cdot r \\ der\, c\, r &= ((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r \end{aligned}$$

What are these regular expressions equivalent to?

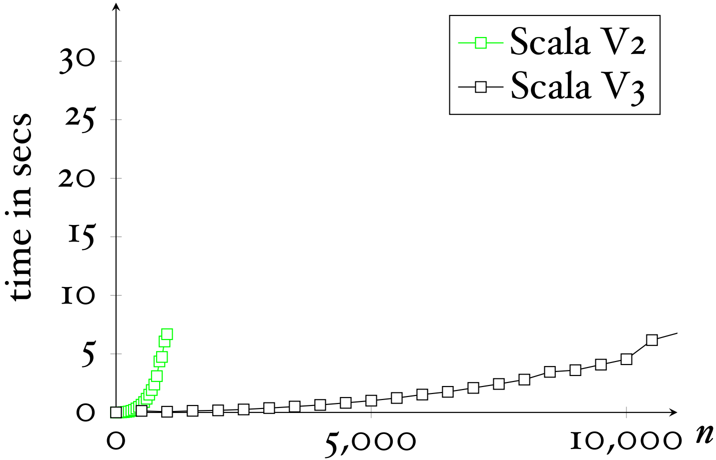
Simplifiaction

$$
\begin{array}{rcl}
r + \mathbf{0} & \Rightarrow & r \\
\mathbf{0} + r & \Rightarrow & r \\
r \cdot \mathbf{1} & \Rightarrow & r \\
\mathbf{1} \cdot r & \Rightarrow & r \\
r \cdot \mathbf{0} & \Rightarrow & \mathbf{0} \\
\mathbf{0} \cdot r & \Rightarrow & \mathbf{0} \\
r + r & \Rightarrow & r
\end{array}
$$

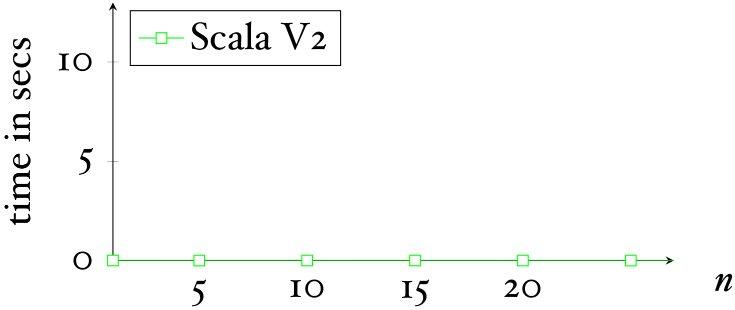
```
def ders(s: List[Char], r: Rexp) : Rexp = s match {
  case Nil => r
  case c::s => ders(s, simp(der(c, r)))
}
```

```scala
def simp(r: Rexp) : Rexp = r match {
  case ALT(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, r2s) => r2s
      case (r1s, ZERO) => r1s
      case (r1s, r2s) =>
        if (r1s == r2s) r1s else ALT(r1s, r2s)
    }
  }
  case SEQ(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, _) => ZERO
      case (_, ZERO) => ZERO
      case (ONE, r2s) => r2s
      case (r1s, ONE) => r1s
      case (r1s, r2s) => SEQ(r1s, r2s)
    }
  }
  case NTIMES(r, n) => NTIMES(simp(r), n)
  case r => r
}
```

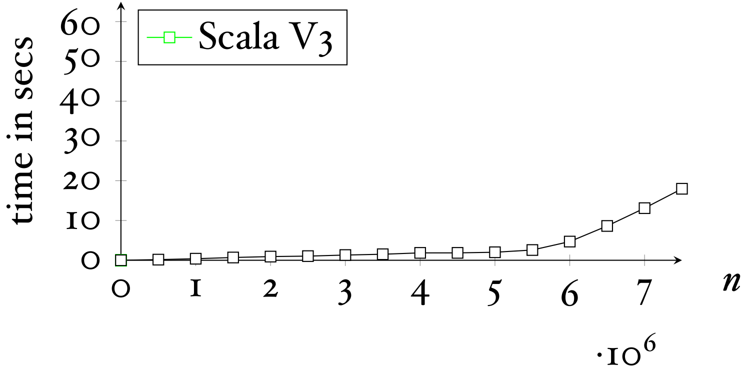
$$(a^{?\{n\}}) \cdot a^{\{n\}}$$



$$(a^*)^* \cdot b$$



$$(a^*)^* \cdot b$$



What is good about this Alg.

- extends to most regular expressions, for example $\sim r$
- is easy to implement in a functional language
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is brand new work)
- we can prove its correctness...

Proofs about Rexps

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
  & \mid & \mathbf{1} \\
  & \mid & c \\
  & \mid & r_1 \cdot r_2 \\
  & \mid & r_1 + r_2 \\
  & \mid & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\mathbf{0}$, $\mathbf{1}$ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$nullable(r) \text{ if and only if } [] \in L(r)$$

Proofs about Rexp (4)

$$
\begin{aligned}
rev(\mathbf{0}) &\stackrel{\text{def}}{=} \mathbf{0}\\
rev(\mathbf{1}) &\stackrel{\text{def}}{=} \mathbf{1}\\
rev(c) &\stackrel{\text{def}}{=} c\\
rev(r_1 + r_2) &\stackrel{\text{def}}{=} rev(r_1) + rev(r_2)\\
rev(r_1 \cdot r_2) &\stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1)\\
rev(r^*) &\stackrel{\text{def}}{=} rev(r)^*
\end{aligned}
$$

We can prove

$$L(rev(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on r.

Correctness Proof for our Matcher

- We started from

$$\begin{array}{ll} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\, s\, (L(r)) \end{array}$$

Correctness Proof for our Matcher

- We started from

$$
\begin{aligned}
& s \in L(r) \\
\Leftrightarrow \quad & [] \in \mathit{Ders}\, s\, (L(r))
\end{aligned}
$$

- if we can show $\mathit{Ders}\, s\, (L(r)) = L(\mathit{ders}\, s\, r)$ we have

$$
\begin{aligned}
\Leftrightarrow \quad & [] \in L(\mathit{ders}\, s\, r) \\
\Leftrightarrow \quad & \mathit{nullable}(\mathit{ders}\, s\, r) \\
\stackrel{\text{def}}{=} \quad & \mathit{matches}\, s\, r
\end{aligned}
$$

Proofs about Rexp (5)

Let $Der\,c\,A$ be the set defined as

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

We can prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

by induction on r.

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

We can then prove

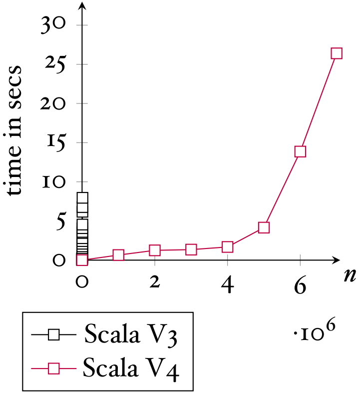
$$Ders\, s\, (L(r)) = L(ders\, s\, r)$$

We can finally prove

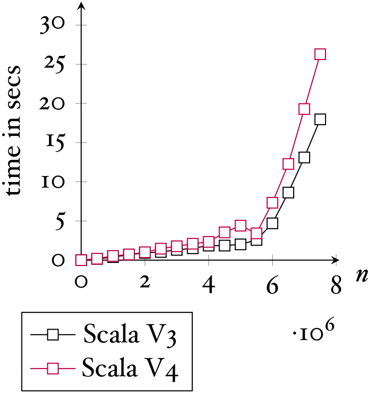
$$matches\, s\, r \text{ if and only if } s \in L(r)$$

Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$



Graph: $(a^*)^* \cdot b$



Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$

Graph: $(a^*)^* \cdot b$


```
def ders2(s: List[Char], r: Rexp) : Rexp = (s, r) match {
  case (Nil, r) => r
  case (s, ZERO) => ZERO
  case (s, ONE) => if (s == Nil) ONE else ZERO
  case (s, CHAR(c)) => if (s == List(c)) ONE else
                       if (s == Nil) CHAR(c) else ZERO
  case (s, ALT(r1, r2)) => ALT(ders2(s, r2), ders2(s, r2))
  case (c::s, r) => ders2(s, simp(der(c, r)))
}
```