Compilers and Formal Languages

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Slides: KEATS (also home work is there)

There are more problems, than there are programs.

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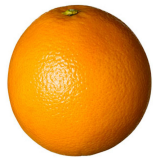
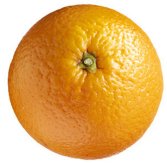
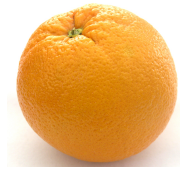
There must be a problem for which there is no program.

Subsets

If $A \subseteq B$ then A has
fewer elements than B

$A \subseteq B$ and $B \subseteq A$

then $A = B$

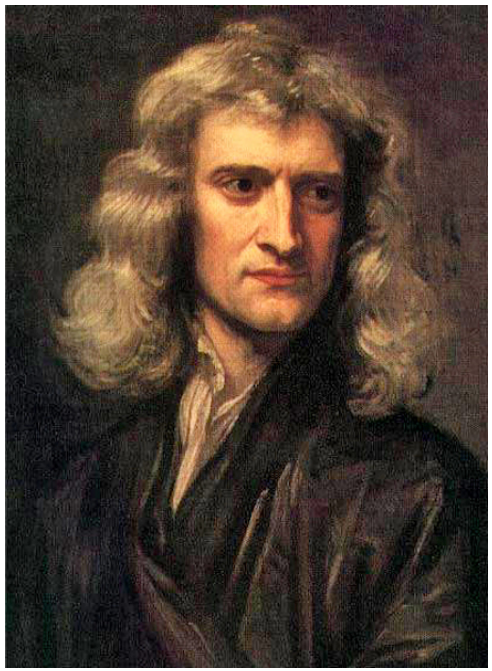
{  ,  ,  ,  ,  }

5 elements

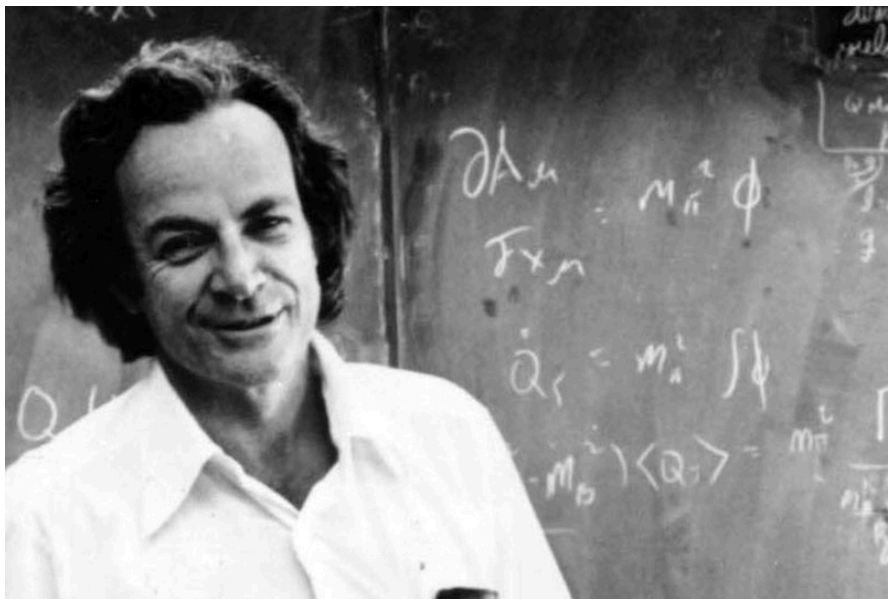
{  ,  ,  }

3 elements

Newton vs Feynman



classical physics

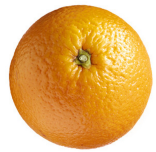


quantum physics

The Goal of the Talk

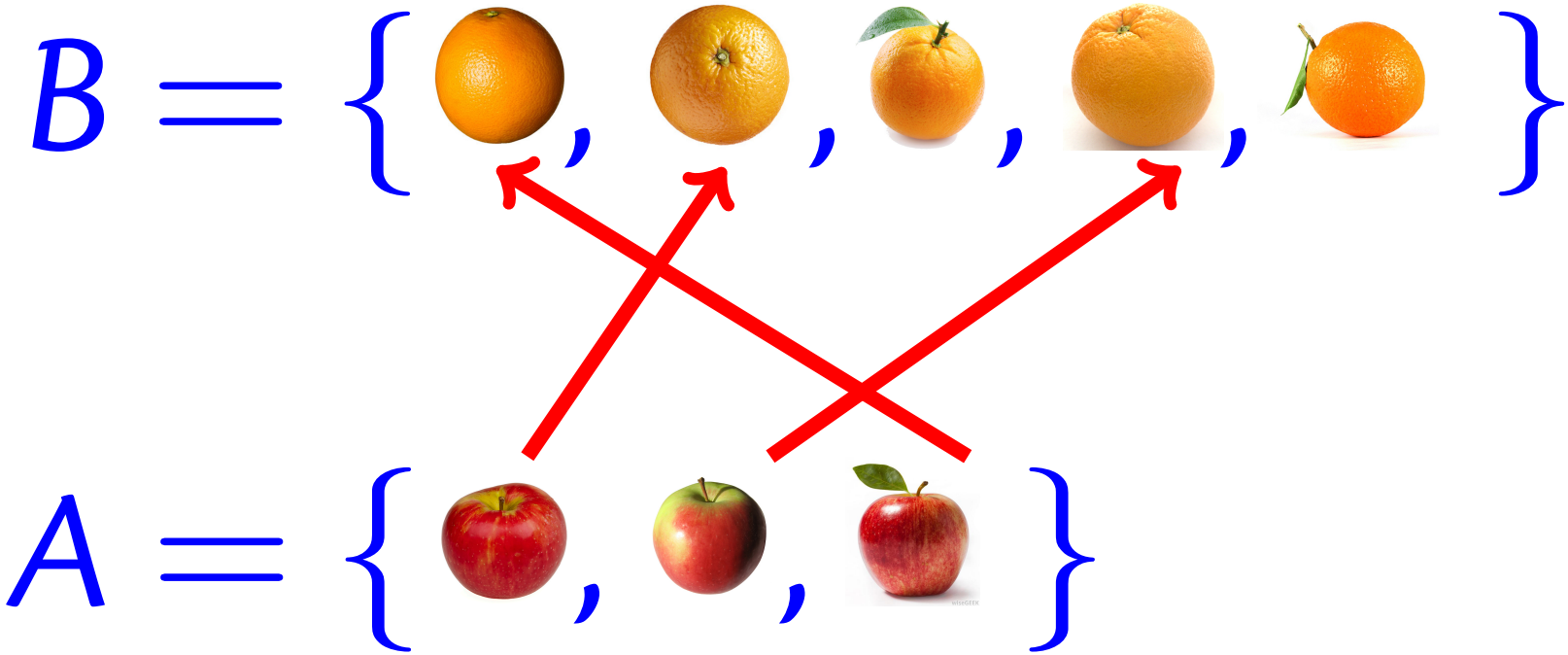
- show you that something very unintuitive happens with very large sets

- convince you that there are more **problems** than **programs**

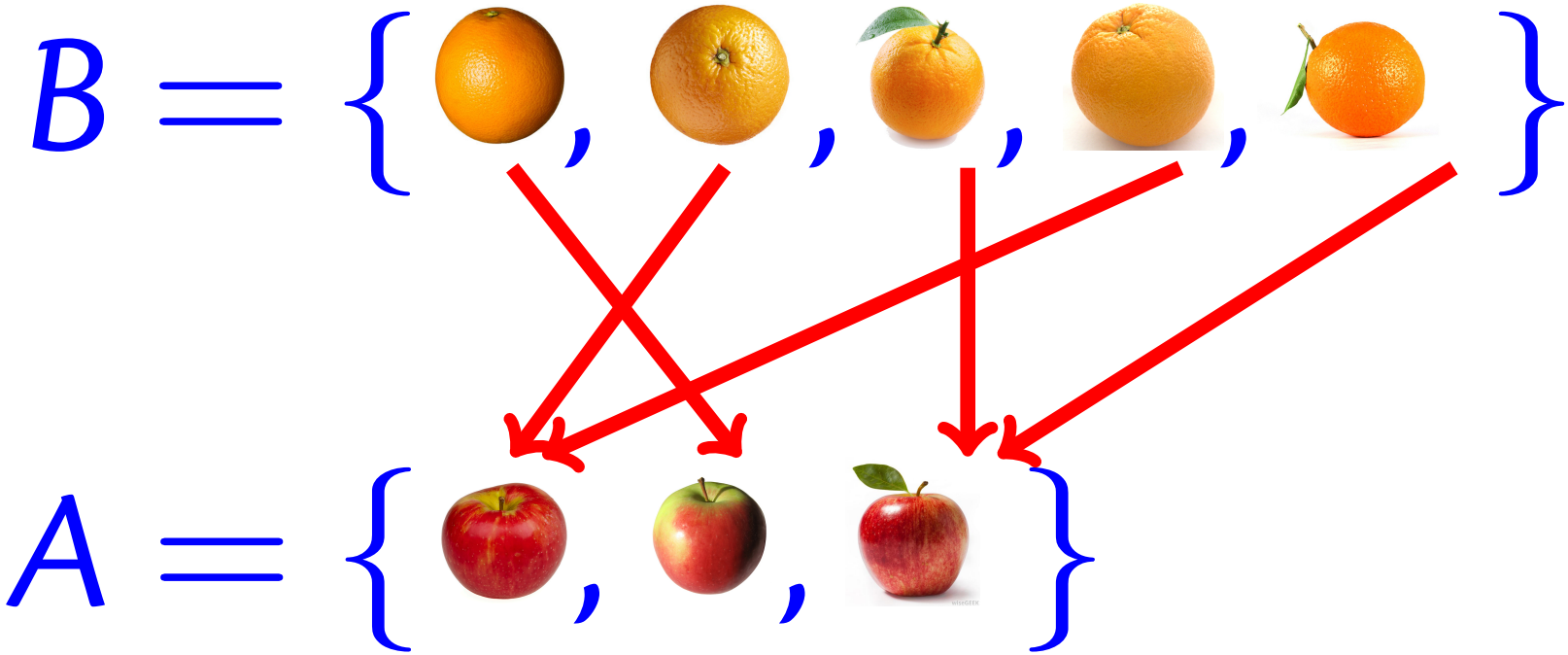
$B = \{$  ,  ,  ,  ,  $\}$

$A = \{$  ,  ,  $\}$

$$|A| = 5, |B| = 3$$



then $|A| \leq |B|$



for $=$ has to be a **one-to-one** mapping

Cardinality

$|A| \stackrel{\text{def}}{=}$ "how many elements"

$$A \subseteq B \Rightarrow |A| \leq |B|$$

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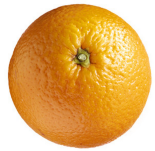
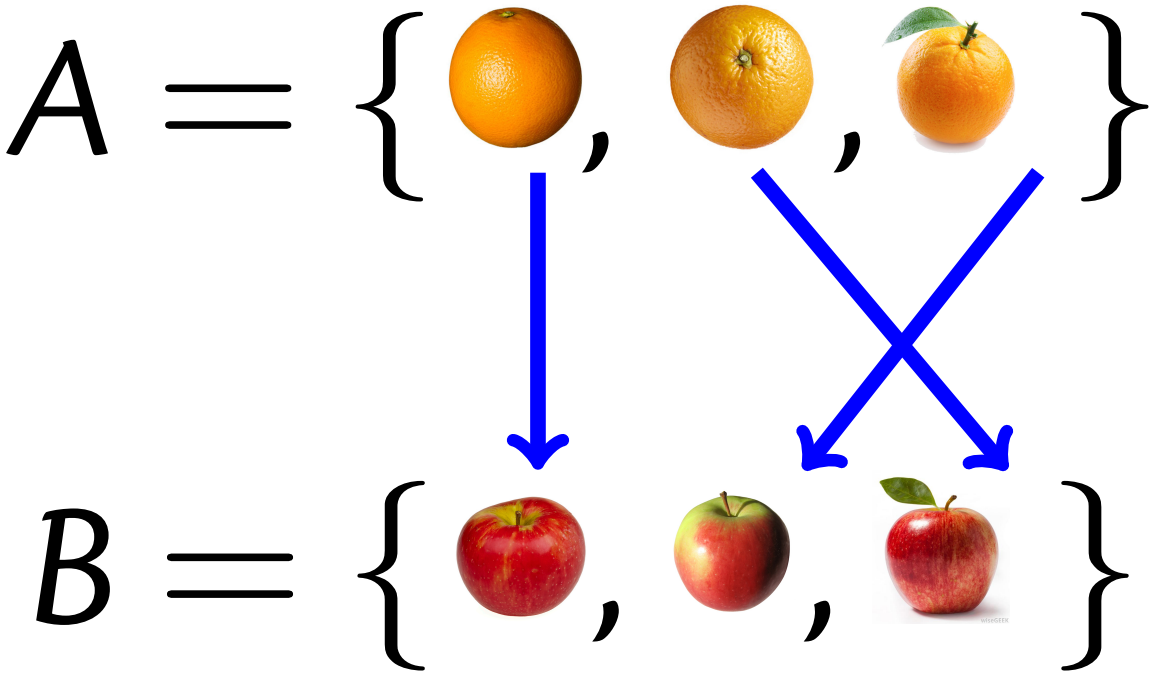
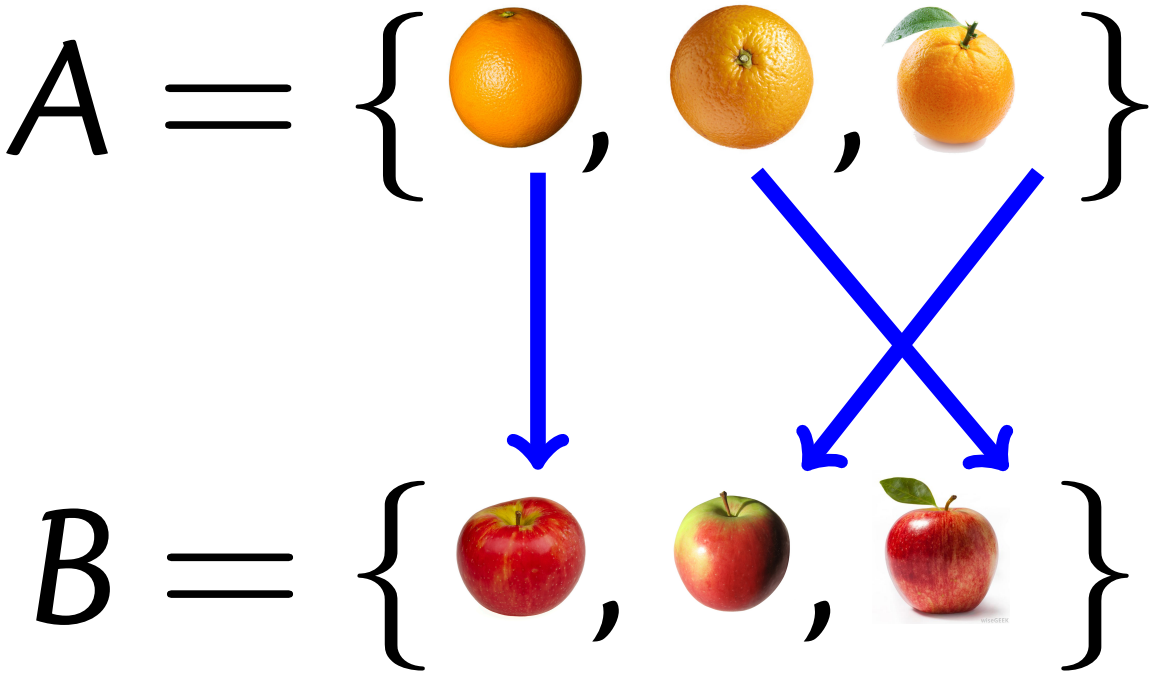
if there is an injective function $f : A \longrightarrow B$ then $|A| \leq |B|$

$$\forall xy.\, f(x) = f(y) \Rightarrow x = y$$

$A = \{$  ,  ,  $\}$

$B = \{$  ,  ,  $\}$


A = { , , }
B = { , , }



then $|A| = |B|$

Natural Numbers

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, ...\}$$

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A is countable iff

$$|A| \leq |\mathbb{N}|$$

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

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$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

$x \mapsto x + 1$,

$$|\mathbb{N} - \{0\}| = |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\quad \{1, 3, 5.....\}$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\{1, 3, 5.....\}$

$\mathbb{E} \stackrel{\text{def}}{=}$ even numbers $\{0, 2, 4.....\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{N} \stackrel{\text{def}}{=}$ positive numbers

$\{0, 1, 2, 3, ...\}$

$-\mathbb{N} \stackrel{\text{def}}{=}$ negative numbers

$\{0, -1, -2, -3, ...\}$

A is countable if there exists an injective $f : A \to \mathbb{N}$

A is uncountable if there does not exist an injective $f : A \to \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$

uncountable: $|A| > |\mathbb{N}|$

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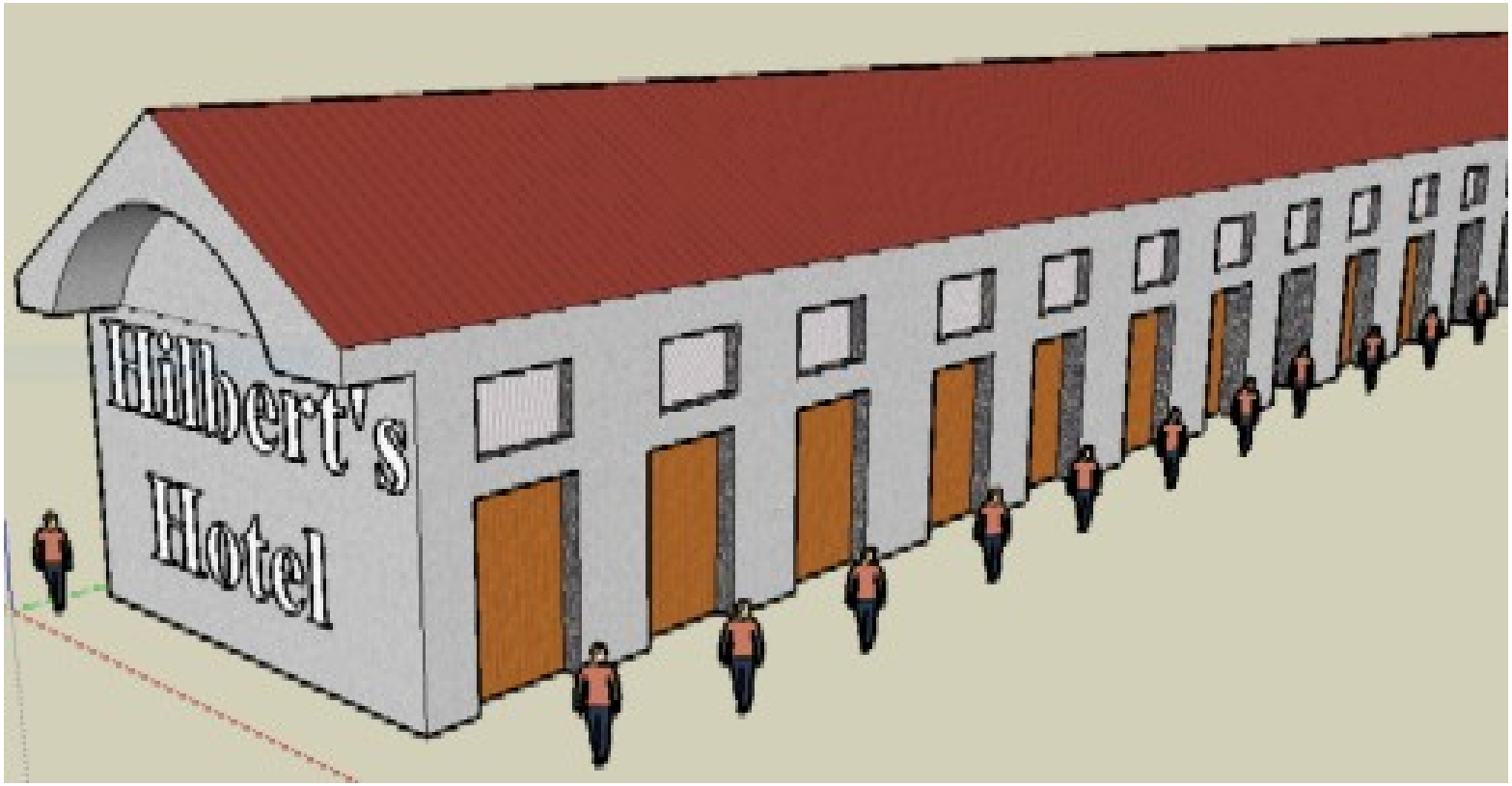
Hilbert's Hotel



- …has as many rooms as there are natural numbers

Real Numbers between 0 and 1

1	3	3	3	3	3	3	...	...
2	1	2	3	4	5	6	7	
3	0	1	0	1	0	...		
4	7	8	5	3	9	...		
	...							

Real Numbers between 0 and 1

1	4	3	3	3	3	3	...	...
2	1	2	3	4	5	6	7	
3	0	1	0	1	0	...		
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Real Numbers between 0 and 1

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	...							

Real Numbers between 0 and 1

1	4	3	3	3	3	3	$\cdots$	$\cdots$
2	1	3	3	4	5	6	7	
3	0	1	1	1	0	$\cdots$		
4	7	8	5	4	9	$\cdots$		

$\cdots$

$$|\mathbb{N}| < |R|$$

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	$\cdots$
1	0	1	0	1	0	1	$\cdots$
2	0	0	0	1	1	0	0
3	0	0	0	0	0	$\cdots$	
4	1	1	0	1	1	$\cdots$	

$\cdots$ $\cdots$

$\cdots$

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	$\cdots$
1	0	1	0	1	0	1	$\cdots$
2	0	0	0	1	1	0	0
3	0	0	0	0	0	$\cdots$	
4	1	1	0	1	1	$\cdots$	

$\cdots$ $\cdots$

$$|\text{Progs}| = |\mathbb{N}| < |\text{Probs}|$$

Halting Problem

Assume a program H that decides for all programs A and all input data D whether

- $H(A, D) \stackrel{\text{def}}{=} 1$ iff $A(D)$ terminates
- $H(A, D) \stackrel{\text{def}}{=} 0$ otherwise

Halting Problem (2)

Given such a program H
define the following program
C: for all programs A

- $C(A) \stackrel{\text{def}}{=} 0$ iff $H(A, A) = 0$
- $C(A) \stackrel{\text{def}}{=}$ loops otherwise

Contradiction

$H(C,C)$ is either 0 or 1.

- $H(C,C) = 1 \overset{\text{def } H}{\Rightarrow} C(C) \downarrow \overset{\text{def } C}{\Rightarrow}$
  $H(C,C) = 0$
- $H(C,C) = 0 \overset{\text{def } H}{\Rightarrow} C(C)$ loops $\overset{\text{def } C}{\Rightarrow}$

  $H(C,C) = 1$

Contradiction in both cases. So H cannot exist.

Take Home Points

- there are sets that are more infinite than others
- even with the most powerful computer we can imagine, there are problems that cannot be solved by any program
- in CS we actually hit quite often such problems (halting problem)