

Automata and Formal Languages (2)

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Languages

A **language** is a set of strings.

A **regular expression** specifies a set of strings or language.

Regular Expressions

Their inductive definition:

$r ::=$	\emptyset	null
	ϵ	empty string / "" / []
	c	character
	$r_1 \cdot r_2$	sequence
	$r_1 + r_2$	alternative / choice
	r^*	star (zero or more)

Regular Expressions

Their implementation in Scala:

```
1 abstract class Rexp
2
3 case object NULL extends Rexp
4 case object EMPTY extends Rexp
5 case class CHAR(c: Char) extends Rexp
6 case class ALT(r1: Rexp, r2: Rexp) extends Rexp
7 case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
8 case class STAR(r: Rexp) extends Rexp
```

The Meaning of a Regular Expression

$$L(\emptyset) \stackrel{\text{def}}{=} \emptyset$$

$$L(\epsilon) \stackrel{\text{def}}{=} \{\epsilon\}$$

$$L(c) \stackrel{\text{def}}{=} \{c\}$$

$$L(r_1 + r_2) \stackrel{\text{def}}{=} L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) \stackrel{\text{def}}{=} L(r_1) @ L(r_2)$$

$$L(r^*) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} L(r)^n$$

$$L(r)^0 \stackrel{\text{def}}{=} \{\epsilon\}$$

$$L(r)^{n+1} \stackrel{\text{def}}{=} L(r) @ L(r)^n$$

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$A @ B$

...you take out every string from A and concatenate it with every string in B

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L is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

What is $L(a^*)$?

Reg Exp Equivalences

$$(a + b) + c \equiv? a + (b + c)$$

$$a + a \equiv? a$$

$$(a \cdot b) \cdot c \equiv? a \cdot (b \cdot c)$$

$$a \cdot a \equiv? a$$

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The Meaning of Matching

a regular expression r matches a string s
is defined as

$$s \in L(r)$$

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if $r_1 \equiv r_2$, then $s \in L(r_1)$ iff $s \in L(r_2)$

A Matching Algorithm

- given a regular expression r and a string s , say yes or no for whether

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- Identifiers (strings of letters or digits, starting with a letter)
- Integers (a non-empty sequence of digits)
- Keywords (else, if, while, ...)
- White space (a non-empty sequence of blanks, newlines and tabs)

A Matching Algorithm

whether a regular expression matches the empty string:

```
1 def nullable (r: Rexp) : Boolean = r match {
2   case NULL => false
3   case EMPTY => true
4   case CHAR(_) => false
5   case ALT(r1, r2) => nullable(r1) || nullable(r2)
6   case SEQ(r1, r2) => nullable(r1) && nullable(r2)
7   case STAR(_) => true
8 }
```

The Derivative of a Rexp

If r matches the string $c::s$, what is a regular expression that matches s ?

$\text{der } c \ r$ gives the answer

The Derivative of a Rexp (2)

$$\text{der } c (\emptyset) \stackrel{\text{def}}{=} \emptyset$$

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$$\text{der } c (d) \stackrel{\text{def}}{=} \text{if } c = d \text{ then } \epsilon \text{ else } \emptyset$$

$$\text{der } c (r_1 + r_2) \stackrel{\text{def}}{=} (\text{der } c r_1) + (\text{der } c r_2)$$

$$\text{der } c (r_1 \cdot r_2) \stackrel{\text{def}}{=} \begin{array}{l} \text{if nullable } r_1 \\ \text{then } ((\text{der } c r_1) \cdot r_2) + (\text{der } c r_2) \\ \text{else } (\text{der } c r_1) \cdot r_2 \end{array}$$

$$\text{der } c (r^*) \stackrel{\text{def}}{=} (\text{der } c r) \cdot (r^*)$$

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$$\text{der } c (r^*) \stackrel{\text{def}}{=} (\text{der } c r) \cdot (r^*)$$

$$\text{ders } [] r \stackrel{\text{def}}{=} r$$

$$\text{ders } (c::s) r \stackrel{\text{def}}{=} \text{ders } s (\text{der } c r)$$

The Derivative

```
1 def deriv (r: Rexp, c: Char) : Rexp = r match {
2   case NULL => NULL
3   case EMPTY => NULL
4   case CHAR(d) => if (c == d) EMPTY else NULL
5   case ALT(r1, r2) => ALT(deriv(r1, c), deriv(r2, c))
6   case SEQ(r1, r2) =>
7     if (nullable(r1)) ALT(SEQ(deriv(r1, c), r2), deriv(r2, c))
8     else SEQ(deriv(r1, c), r2)
9   case STAR(r) => SEQ(deriv(r, c), STAR(r))
10 }
```

The Rexp Matcher

```
1 def matches(r: Rexp, s: String) : Boolean =
2   nullable(derivs(r, s.toList))
3
4
5  /* Examples */
6
7  println(matches(SEQ(SEQ(CHAR('c'), CHAR('a')), CHAR('b')), "cab"))
8  println(matches(STAR(CHAR('a')), "aaa"))
9
10 /* Convenience using implicits */
11 implicit def string2rexp(s : String) : Rexp = {
12   s.foldRight (EMPTY: Rexp) ( (c, r) => SEQ(CHAR(c), r) )
13 }
14
15 println(matches("cab" , "cab"))
16 println(matches(STAR("a"), "aaa"))
17 println(matches(STAR("a"), "aaab"))
```


Proofs about Rexp

Remember their inductive definition:

$$\begin{array}{l} r ::= \emptyset \\ | \epsilon \\ | c \\ | r_1 \cdot r_2 \\ | r_1 + r_2 \\ | r^* \end{array}$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for \emptyset , ϵ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2 .
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2 .
- P holds for r^* under the assumption that P already holds for r .

Proofs about Rexp (3)

Assume $P(r)$ is the property:

nullable(r) if and only if $\epsilon \in L(r)$

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

Let $\text{Der } c \ A$ be the set defined as

$$\text{Der } c \ A \stackrel{\text{def}}{=} \{ s \mid c::s \in A \}$$

Assume that $L(\text{der } c \ r) = \text{Der } c \ (L(r))$. Prove that

$\text{matcher}(r, s)$ if and only if $s \in L(r)$

Regular Languages

A language (set of strings) is **regular** iff there exists a regular expression that recognises all its strings.

Automata

A deterministic finite automaton consists of:

- a set of states
- one of these states is the start state
- some states are accepting states, and
- there is transition function

which takes a state as argument and a character and produces a new state

this function might not always be defined