Compilers and Formal Languages (4)

Email: christian.urban at kcl.ac.uk
Office: N7.07 (North Wing, Bush House)
Slides: KEATS (also homework is there)

Survey: Thanks!

"…Thanks a million! Thanks without end!"

"Urban is a very talented lecturer: thorough, concise, clear, and cares to make sure that we are learning!"

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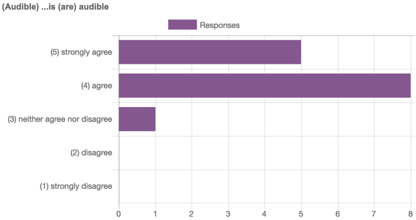

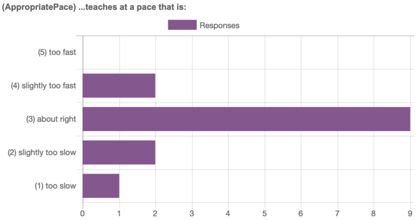

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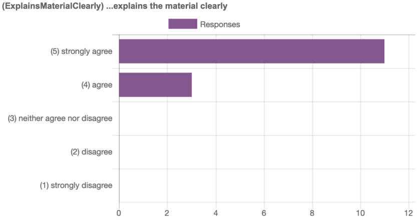

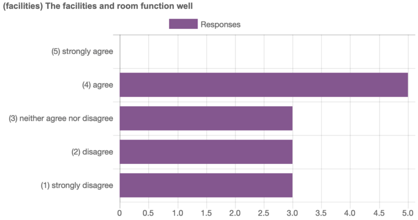

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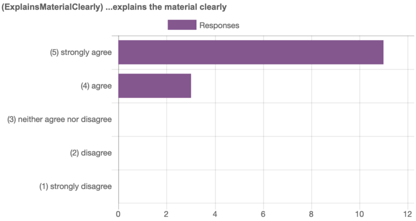
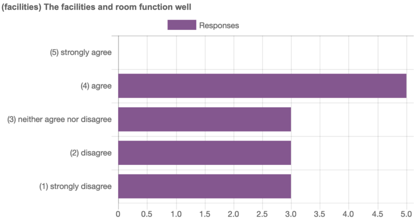
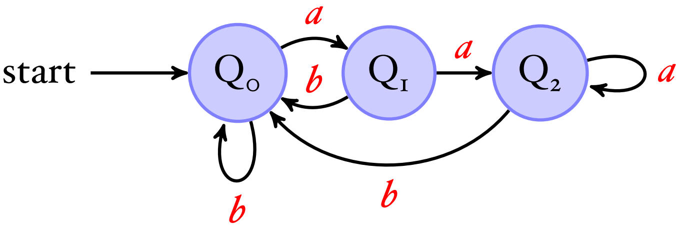




room too hot, 3h lecture



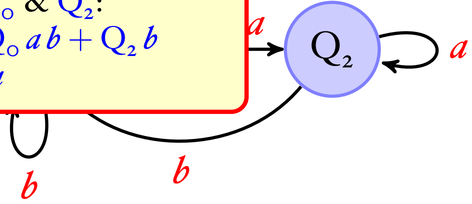
$$
\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\,b + Q_1\,b + Q_2\,b \\
Q_1 &= Q_0\,a \\
Q_2 &= Q_1\,a + Q_2\,a
\end{aligned}
$$

Arden's Lemma:

If $q = q\,r + s$ then $q = s\,r^*$

substitute Q_1 into Q_0 & Q_2:

$$Q_0 = \mathbf{1} + Q_0\, b + Q_0\, a\, b + Q_2\, b$$
$$Q_2 = Q_0\, a\, a + Q_2\, a$$



$$\begin{aligned} Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\ Q_1 &= Q_0\, a \\ Q_2 &= Q_1\, a + Q_2\, a \end{aligned}$$

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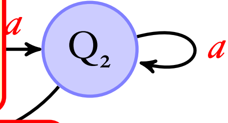

simplifying Q_0:

$$Q_0 = 1 + Q_0\, (b + a\, b) + Q_2\, b$$
$$Q_2 = Q_0\, a\, a + Q_2\, a$$

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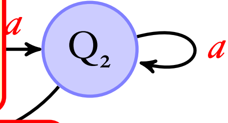

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Arden for Q_2:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\,b) + Q_2\, b$$
$$Q_2 = Q_0\, a\, a\,(a^*)$$

$$Q_2 = Q_1\, a + Q_2\, a$$

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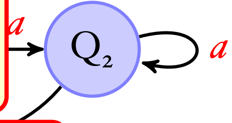

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Substitute Q_2 and simplify:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\, b + a\, a\,(a^*)\, b)$$

Arden's Le

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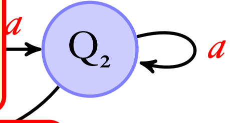

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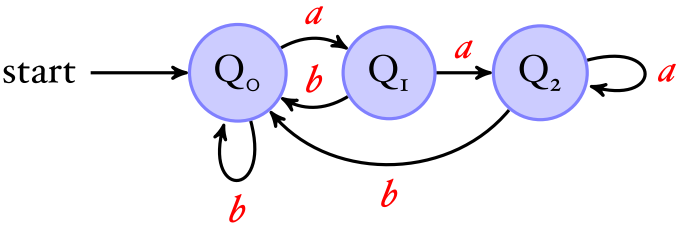
If

Arden again for Q_0:

$$Q_0 = (b + a\, b + a\, a\,(a^*)\, b)^*$$



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\begin{aligned}
Q_0 &= (b + a\, b + a\, a\, (a^*)\, b)^*\\
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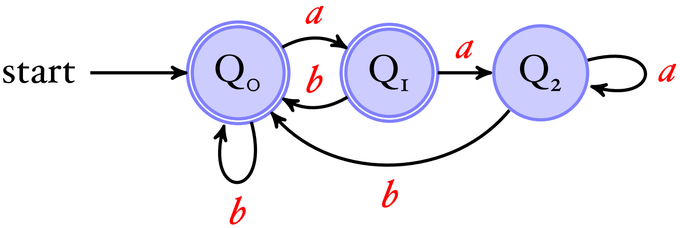

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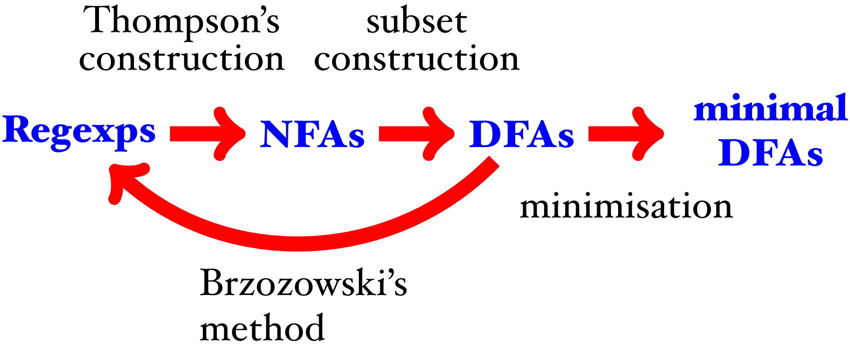
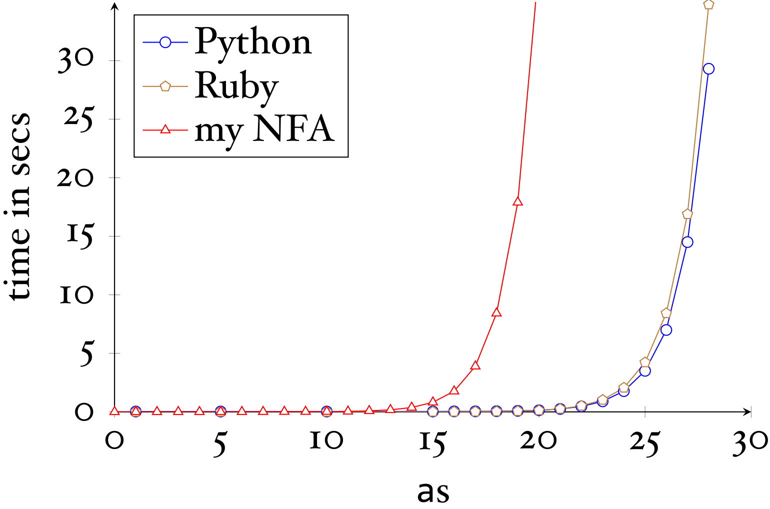
Regexps and Automata



$$a^{?\{n\}} \cdot a^{\{n\}}$$



The punchline is that many existing libraries do depth-first search in NFAs (backtracking).

Regular Languages

Two equivalent definitions:

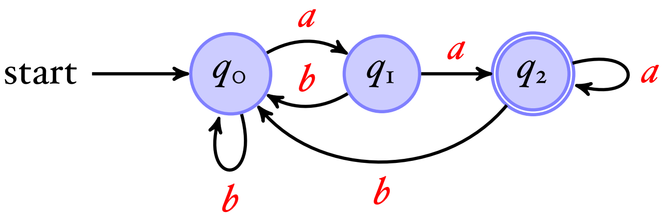
A language is regular iff there exists a regular expression that recognises all its strings.

A language is regular iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

Negation

Regular languages are closed under negation:



But requires that the automaton is completed!

Negation

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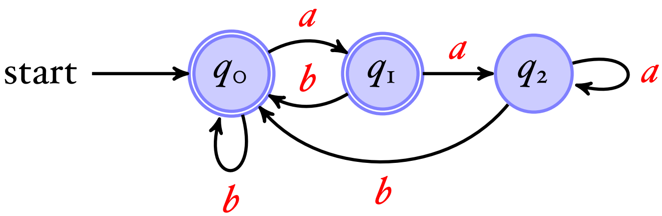

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The Goal of this Course

Write a compiler

Today a lexer.

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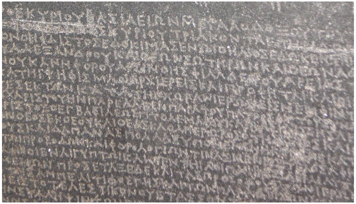

lexing ⇒ recognising words (Stone of Rosetta)

Regular Expressions

In programming languages they are often used to recognise:

- symbols, digits
- identifiers
- numbers (non-leading zeros)
- keywords
- comments

`http://www.regexper.com`

Lexing: Test Case

```
write "Fib";
read n;
minus1 := 0;
minus2 := 1;
while n > 0 do {
       temp := minus2;
       minus2 := minus1 + minus2;
       minus1 := temp;
       n := n - 1
};
write "Result";
write minus2
```

"if true then then 42 else +"

KEYWORD:

  if, then, else,

WHITESPACE:

  " ", \n,

IDENTIFIER:

  $\texttt{LETTER} \cdot (\texttt{LETTER} + \texttt{DIGIT} + \_)^*$

NUM:

  $(\texttt{NONZERODIGIT} \cdot \texttt{DIGIT}^*) + 0$

OP:

  +, -, *, %, <, <=

COMMENT:

  $/* \cdot \sim(\texttt{ALL}^* \cdot (*/) \cdot \texttt{ALL}^*) \cdot */$

”if true then then 42 else +”

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

”if true then then 42 else +”

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

There is one small problem with the tokenizer. How should we tokenize...?

"x-3"

```
ID: ...
OP:
  "+", "-"
NUM:
  (NONZERODIGIT · DIGIT*) + "0"
NUMBER:
  NUM + ("-" · NUM)
```

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string abc.

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Or, keywords are `if` and identifiers are letters followed by "letters + numbers + _"*

$$if \qquad iffoo$$

POSIX: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as the next token.

- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

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traditional lexers are fast, but hairy

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\ a} r_2$$

Sulzmann & Lu Matcher

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We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4$$

Sulzmann & Lu Matcher

We want to match the string *abc* using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4 \; nullable?$$

Sulzmann & Lu Matcher

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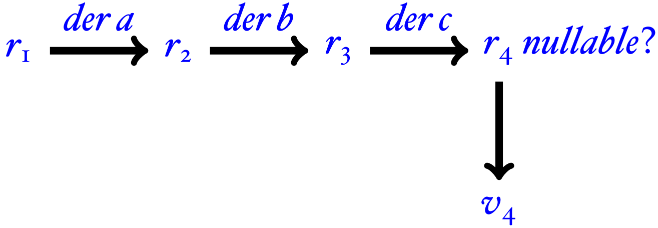

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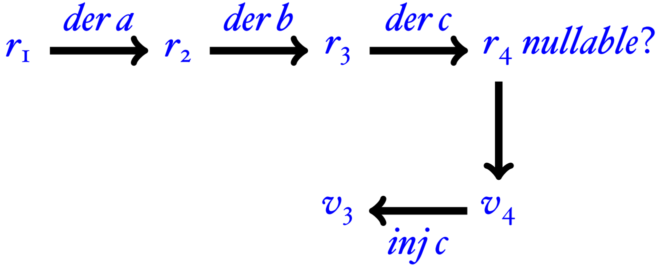

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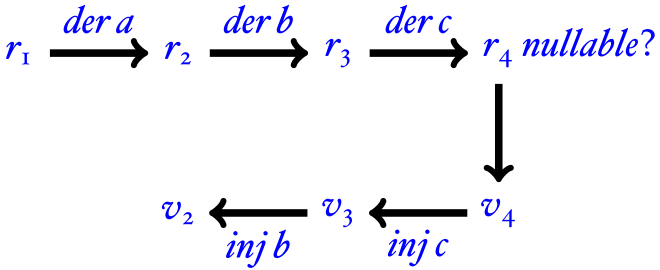

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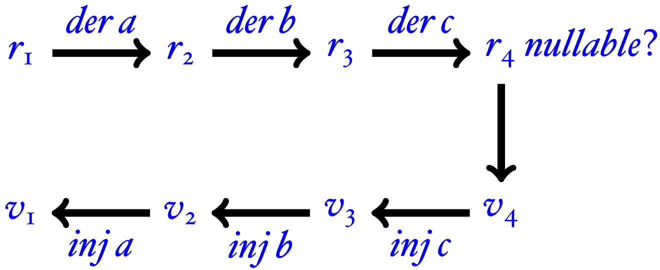

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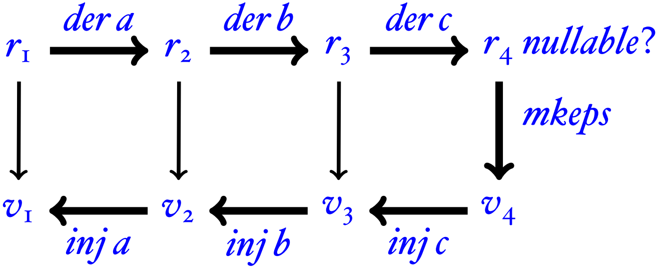


Regexes and Values

Regular expressions and their corresponding values:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
& | & \mathbf{1} \\
& | & c \\
& | & r_1 \cdot r_2 \\
& | & r_1 + r_2 \\
& & \\
& | & r^*
\end{array}
\qquad
\begin{array}{rcl}
v & ::= & \\
& & \mathit{Empty} \\
& | & \mathit{Char}(c) \\
& | & \mathit{Seq}(v_1, v_2) \\
& | & \mathit{Left}(v) \\
& | & \mathit{Right}(v) \\
& | & \mathit{Stars}\,[] \\
& | & \mathit{Stars}\,[v_1, \ldots\, v_n]
\end{array}
$$

```scala
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp

abstract class Val
case object Empty extends Val
case class Chr(c: Char) extends Val
case class Sequ(v1: Val, v2: Val) extends Val
case class Left(v: Val) extends Val
case class Right(v: Val) extends Val
case class Stars(vs: List[Val]) extends Val
```

Mkeps

Finding a (posix) value for recognising the empty string:

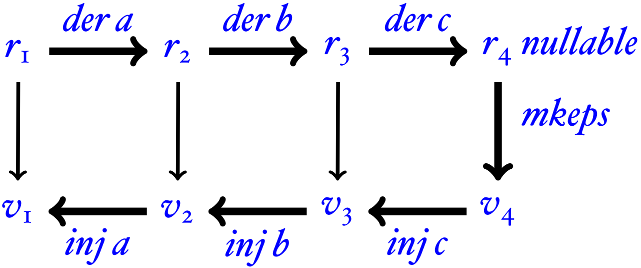
$$
\begin{aligned}
mkeps\,(\mathbf{1}) &\stackrel{\text{def}}{=} Empty \\
mkeps\,(r_1 + r_2) &\stackrel{\text{def}}{=} \text{if } nullable(r_1) \\
&\qquad \text{then } Left(mkeps(r_1)) \\
&\qquad \text{else } Right(mkeps(r_2)) \\
mkeps\,(r_1 \cdot r_2) &\stackrel{\text{def}}{=} Seq(mkeps(r_1), mkeps(r_2)) \\
mkeps\,(r^*) &\stackrel{\text{def}}{=} Stars\,[]
\end{aligned}
$$

Inject

Injecting ("Adding") a character to a value

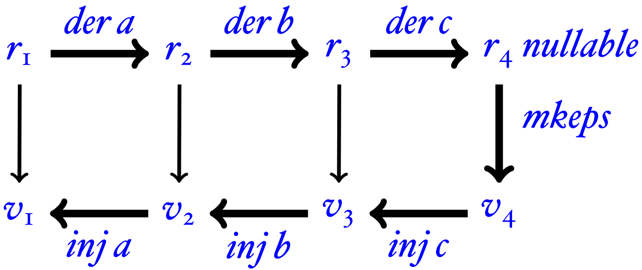
$$
\begin{array}{lcl}
inj\,(c)\,c\,(Empty) & \stackrel{\text{def}}{=} & Char\,c\\
inj\,(r_1 + r_2)\,c\,(Left(v)) & \stackrel{\text{def}}{=} & Left(inj\,r_1\,c\,v)\\
inj\,(r_1 + r_2)\,c\,(Right(v)) & \stackrel{\text{def}}{=} & Right(inj\,r_2\,c\,v)\\
inj\,(r_1 \cdot r_2)\,c\,(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & Seq(inj\,r_1\,c\,v_1, v_2)\\
inj\,(r_1 \cdot r_2)\,c\,(Left(Seq(v_1, v_2))) & \stackrel{\text{def}}{=} & Seq(inj\,r_1\,c\,v_1, v_2)\\
inj\,(r_1 \cdot r_2)\,c\,(Right(v)) & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), inj\,r_2\,c\,v)\\
inj\,(r^*)\,c\,(Seq(v, Stars\,vs)) & \stackrel{\text{def}}{=} & Stars\,(inj\,r\,c\,v\ ::\ vs)
\end{array}
$$

inj: 1st arg $\mapsto$ a rexp; 2nd arg $\mapsto$ a character; 3rd arg $\mapsto$ a value



der a
der b
der c
r1
r2
r3
r4 nullable
mkeps
v1
v2
v3
v4
inj a
inj b
inj c

r_1: $a \cdot (b \cdot c)$
r_2: $\mathbf{1} \cdot (b \cdot c)$
r_3: $(\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c)$
r_4: $(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$



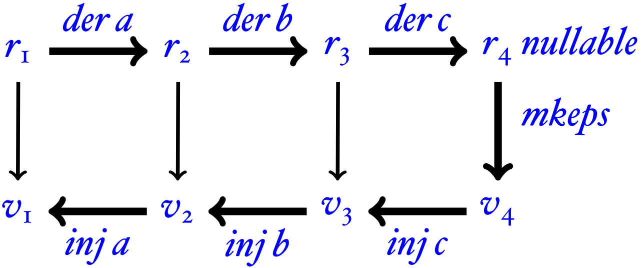
v_1: $Seq(Char(a), Seq(Char(b), Char(c)))$
v_2: $Seq(Empty, Seq(Char(b), Char(c)))$
v_3: $Right(Seq(Empty, Char(c)))$
v_4: $Right(Right(Empty))$

Flatten

Obtaining the string underlying a value:

$$
\begin{array}{lcl}
|Empty| & \stackrel{\text{def}}{=} & [] \\
|Char(c)| & \stackrel{\text{def}}{=} & [c] \\
|Left(v)| & \stackrel{\text{def}}{=} & |v| \\
|Right(v)| & \stackrel{\text{def}}{=} & |v| \\
|Seq(v_1, v_2)| & \stackrel{\text{def}}{=} & |v_1| \,@\, |v_2| \\
|[v_1, \ldots, v_n]| & \stackrel{\text{def}}{=} & |v_1| \,@\ldots @\, |v_n|
\end{array}
$$

r_1: $a \cdot (b \cdot c)$
r_2: $\mathbf{1} \cdot (b \cdot c)$
r_3: $(\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c)$
r_4: $(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$



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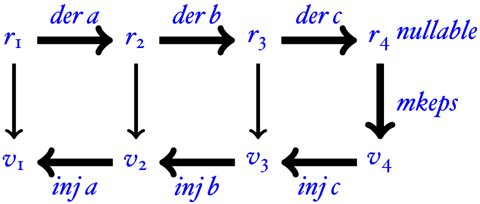
v_3: $Right(Seq(Empty, Char(c)))$
v_4: $Right(Right(Empty))$

$|v_1|$: abc
$|v_2|$: bc
$|v_3|$: c
$|v_4|$: $[]$

Lexing

$$\begin{aligned} lex\ r\ [] &\stackrel{\text{def}}{=} \text{if } nullable(r) \text{ then } mkeps(r) \text{ else } error \\ lex\ r\ c :: s &\stackrel{\text{def}}{=} inj\ r\ c\ lex(der(c,r), s) \end{aligned}$$

lex: returns a value



Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

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- new regex: $(x : r)$ new value: $Rec(x, v)$
- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\,c\,(x : r) \stackrel{\text{def}}{=} (x : der\,c\,r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\,(x : r)\,c\,v \stackrel{\text{def}}{=} Rec(x, inj\,r\,c\,v)$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$
- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\,c\,(x : r) \stackrel{\text{def}}{=} (x : der\,c\,r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\,(x : r)\,c\,v \stackrel{\text{def}}{=} Rec(x, inj\,r\,c\,v)$

for extracting subpatterns $(z : ((x : ab) + (y : ba))$

- A regular expression for email addresses

$$(\text{name}: [a\text{-}z0\text{-}9\_.-]^{+}) \cdot @ \cdot (\text{domain}: [a\text{-}z0\text{-}9.-]^{+}) \cdot . \cdot (\text{top\_level}: [a\text{-}z.]^{\{2,6\}})$$

`christian.urban@kcl.ac.uk`

- the result environment:

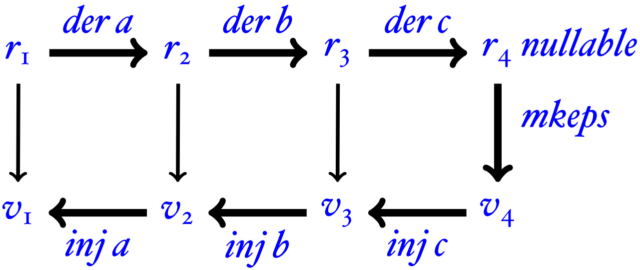
$$[(\mathit{name} : \texttt{christian.urban}), (\mathit{domain} : \texttt{kcl}), (\mathit{top\_level} : \texttt{ac.uk})]$$

While Tokens

$$
\texttt{WHILE\_REGS} \stackrel{\text{def}}{=} ((\text{"k"} : \texttt{KEYWORD}) + (\text{"i"} : \texttt{ID}) + (\text{"o"} : \texttt{OP}) + (\text{"n"} : \texttt{NUM}) + (\text{"s"} : \texttt{SEMI}) + (\text{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + (\text{"b"} : (\texttt{BEGIN} + \texttt{END})) + (\text{"w"} : \texttt{WHITESPACE}))^*
$$

Simplification

- If we simplify after the derivative, then we are building the value for the simplified regular expression, but **not** for the original regular expression.



$$(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1}) \mapsto \mathbf{1}$$

Normally we would have

$$(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$$

and answer how this regular expression matches the empty string with the value

$$Right(Right(Empty))$$

But now we simplify this to $\mathbf{1}$ and would produce *Empty* (see *mkeps*).

Rectification

rectification functions:

$$
\begin{array}{lcll}
r \cdot \mathbf{0} & \mapsto & \mathbf{0} & \\
\mathbf{0} \cdot r & \mapsto & \mathbf{0} & \\
r \cdot \mathbf{1} & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\mathbf{1} \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \mathbf{0} & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\mathbf{0} + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}
$$

Rectification

rectification functions:

$$\begin{array}{lcll}
r \cdot \mathbf{0} & \mapsto & \mathbf{0} & \\
\mathbf{0} \cdot r & \mapsto & \mathbf{0} & \\
r \cdot \mathbf{1} & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\mathbf{1} \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \mathbf{0} & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\mathbf{0} + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}$$

old *simp* returns a rexp;
new *simp* returns a rexp and a rectification function.

Rectification

$simp(r)$:

case $r = r_1 + r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(r_{2s}, \lambda v.\, Right(f_{2s}(v)))$

case $r_{2s} = \mathbf{0}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$

case $r_{1s} = r_{2s}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$

otherwise: return $(r_{1s} + r_{2s}, f_{alt}(f_{1s}, f_{2s}))$

$$f_{alt}(f_1, f_2) \stackrel{\text{def}}{=}$$

$$\lambda v.\ \text{case } v = Left(v'): \text{ return } Left(f_1(v'))$$

$$\text{case } v = Right(v'): \text{ return } Right(f_2(v'))$$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case ALT(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (r2s, F_RIGHT(f2s))
      case (_, ZERO) => (r1s, F_LEFT(f1s))
      case _ =>
         if (r1s == r2s) (r1s, F_LEFT(f1s))
         else (ALT (r1s, r2s), F_ALT(f1s, f2s))
    }
  }
  ...
}

def F_RIGHT(f: Val => Val) = (v:Val) => Right(f(v))
def F_LEFT(f: Val => Val) = (v:Val) => Left(f(v))
def F_ALT(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Right(v) => Right(f2(v))
    case Left(v) => Left(f1(v)) }
```

Rectification

$simp(r)$:...

case $r = r_1 \cdot r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{2s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{1s} = \mathbf{1}$: return $(r_{2s}, \lambda v.\, Seq(f_{1s}(Empty), f_{2s}(v)))$

case $r_{2s} = \mathbf{1}$: return $(r_{1s}, \lambda v.\, Seq(f_{1s}(v), f_{2s}(Empty)))$

otherwise: return $(r_{1s} \cdot r_{2s}, f_{seq}(f_{1s}, f_{2s}))$

$f_{seq}(f_1, f_2) \stackrel{\text{def}}{=}$

$\lambda v.$ case $v = Seq(v_1, v_2)$: return $Seq(f_1(v_1), f_2(v_2))$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case SEQ(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (ZERO, F_ERROR)
      case (_, ZERO) => (ZERO, F_ERROR)
      case (ONE, _) => (r2s, F_SEQ_Void1(f1s, f2s))
      case (_, ONE) => (r1s, F_SEQ_Void2(f1s, f2s))
      case _ => (SEQ(r1s,r2s), F_SEQ(f1s, f2s))
    }
  }
  ...

def F_SEQ_Void1(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(Void), f2(v))
def F_SEQ_Void2(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(v), f2(Void))
def F_SEQ(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Sequ(v1, v2) => Sequ(f1(v1), f2(v2)) }
```

Rectification Example

$$(b \cdot c) + (\mathbf{0} + \mathbf{1}) \mapsto (b \cdot c) + \mathbf{1}$$

Rectification Example

$$(\underline{b \cdot c}) + (\underline{\mathbf{0} + \mathbf{1}}) \mapsto (b \cdot c) + \mathbf{1}$$

Rectification Example

$$(\underline{b \cdot c}) + (\underline{\mathbf{0} + \mathbf{1}}) \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$f_{alt}(f_{s1}, f_{s2}) \stackrel{\text{def}}{=}$$
$$\lambda v.\ \text{case } v = Left(v'):\quad \text{return } Left(f_{s1}(v'))$$
$$\text{case } v = Right(v'): \text{return } Right(f_{s2}(v'))$$

Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$\begin{array}{ll} \lambda v. & \text{case } v = Left(v'): \quad \text{return } Left(v') \\ & \text{case } v = Right(v'): \text{return } Right(Right(v')) \end{array}$$

Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

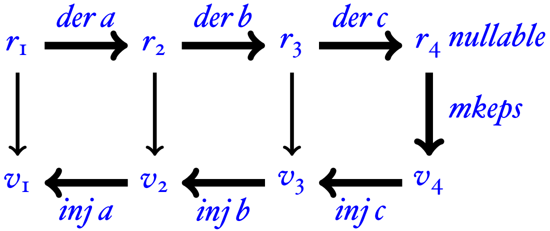
$$\begin{array}{rcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$\lambda v.$ case $v = Left(v')$: return $Left(v')$
case $v = Right(v')$: return $Right(Right(v'))$

mkeps simplified case: $Right(Empty)$
rectified case: $Right(Right(Empty))$

Lexing with Simplification

$$
\begin{array}{lcl}
lex\,r\,[] & \stackrel{\text{def}}{=} & \text{if } nullable(r) \text{ then } mkeps(r) \text{ else } error \\
lex\,r\,c::s & \stackrel{\text{def}}{=} & \text{let } (r', frect) = simp(der(c,r)) \\
 & & inj\,r\,c\,(frect(lex(r',s)))
\end{array}
$$



Environments

Obtaining the "recorded" parts of a value:

$$
\begin{array}{lcl}
env(Empty) & \stackrel{\text{def}}{=} & [] \\
env(Char(c)) & \stackrel{\text{def}}{=} & [] \\
env(Left(v)) & \stackrel{\text{def}}{=} & env(v) \\
env(Right(v)) & \stackrel{\text{def}}{=} & env(v) \\
env(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & env(v_1) \mathbin{@} env(v_2) \\
env(Stars\,[v_1, \ldots, v_n]) & \stackrel{\text{def}}{=} & env(v_1) \mathbin{@} \ldots \mathbin{@} env(v_n) \\
env(Rec(x : v)) & \stackrel{\text{def}}{=} & (x : |v|) :: env(v)
\end{array}
$$

While Tokens

$$
\begin{aligned}
\texttt{WHILE\_REGS} \stackrel{\text{def}}{=} ((&\text{"k"} : \texttt{KEYWORD}) + \\
(&\text{"i"} : \texttt{ID}) + \\
(&\text{"o"} : \texttt{OP}) + \\
(&\text{"n"} : \texttt{NUM}) + \\
(&\text{"s"} : \texttt{SEMI}) + \\
(&\text{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + \\
(&\text{"b"} : (\texttt{BEGIN} + \texttt{END})) + \\
(&\text{"w"} : \texttt{WHITESPACE}))^*
\end{aligned}
$$

”if true then then 42 else +”

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

”if true then then 42 else +”

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

Lexer: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as next token.

- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

$$
\begin{array}{lcl}
\textit{zeroable}(\mathbf{0}) & \stackrel{\text{def}}{=} & \textit{true} \\
\textit{zeroable}(\mathbf{1}) & \stackrel{\text{def}}{=} & \textit{false} \\
\textit{zeroable}(c) & \stackrel{\text{def}}{=} & \textit{false} \\
\textit{zeroable}(r_1 + r_2) & \stackrel{\text{def}}{=} & \textit{zeroable}(r_1) \wedge \textit{zeroable}(r_2) \\
\textit{zeroable}(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \textit{zeroable}(r_1) \vee \textit{zeroable}(r_2) \\
\textit{zeroable}(r^*) & \stackrel{\text{def}}{=} & \textit{false}
\end{array}
$$

$\textit{zeroable}(r)$ if and only if $L(r) = \{\}$