Automata and Formal Languages (6)

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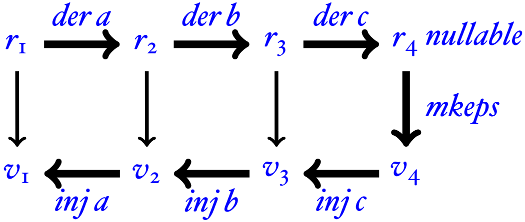
Slides:	KEATS (also home work is there)

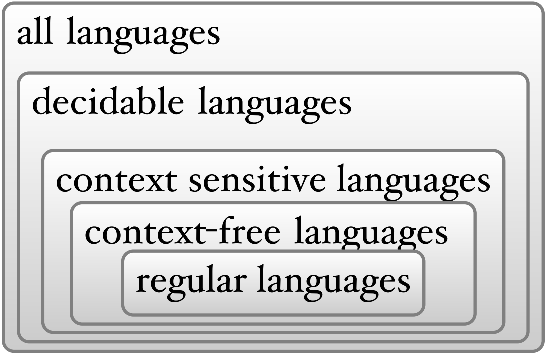
```scala
def concat(A: Set[String], B: Set[String]) : Set[String] =
  for (x <-A ; y <- B) yield x ++ y

def pow(A: Set[String], n: Int) : Set[String] = n match {
  case 0 => Set("")
  case n => concat(A, pow(A, n- 1))
}

val A = Set("a", "b", "c", "d")
pow(A, 4).size                          // -> 256

val B = Set("a", "b", "c", "")
pow(B, 4).size                          // -> 121

val C = Set("a", "b", "")
pow(C, 2)
pow(C, 2).size                          // -> 7

pow(C, 3)
pow(C, 3).size                          // -> 15
```



$$
\begin{array}{lcl}
inj\ (c)\ c\ Empty & \stackrel{\text{def}}{=} & Char\ c\\
inj\ (r_1 + r_2)\ c\ Left(v) & \stackrel{\text{def}}{=} & Left(inj\ r_1\ c\ v)\\
inj\ (r_1 + r_2)\ c\ Right(v) & \stackrel{\text{def}}{=} & Right(inj\ r_2\ c\ v)\\
inj\ (r_1 \cdot r_2)\ c\ Seq(v_1, v_2) & \stackrel{\text{def}}{=} & Seq(inj\ r_1\ c\ v_1, v_2)\\
inj\ (r_1 \cdot r_2)\ c\ Left(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & Seq(inj\ r_1\ c\ v_1, v_2)\\
inj\ (r_1 \cdot r_2)\ c\ Right(v) & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), inj\ r_2\ c\ v)\\
inj\ (r^*)\ c\ Seq(v, vs) & \stackrel{\text{def}}{=} & inj\ r\ c\ v\ ::\ vs
\end{array}
$$

Hierarchy of Languages

Recall that languages are sets of strings.



Two Grammars

Which languages are recognised by the following two grammars?

$$
\begin{array}{lcl}
S & \rightarrow & 1 \cdot S \cdot S \\
  & | & \epsilon
\end{array}
$$

$$
\begin{array}{lcl}
U & \rightarrow & 1 \cdot U \\
  & | & \epsilon
\end{array}
$$

Atomic parsers, for example

$$1 :: rest \;\Rightarrow\; \{(1, rest)\}$$

- you consume one or more token from the input (stream)
- also works for characters and strings

Alternative parser (code $p \mid\mid q$)

- apply p and also q; then combine the outputs

$$p(\text{input}) \cup q(\text{input})$$

Sequence parser (code $p \sim q$)

- apply first p producing a set of pairs
- then apply q to the unparsed parts
- then combine the results:

((output$_1$, output$_2$), unparsed part)

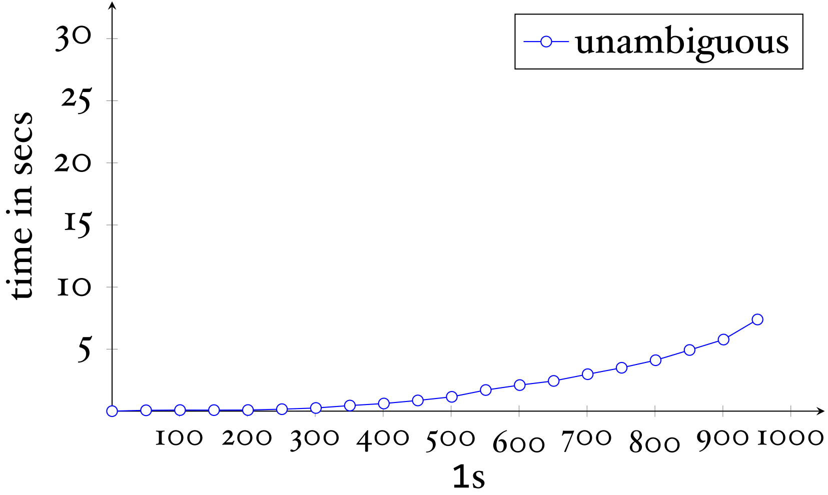
$$\{((o_1, o_2), u_2) \mid (o_1, u_1) \in p(\text{input}) \wedge (o_2, u_2) \in q(u_1)\}$$

Function parser (code $p \Rightarrow f$)

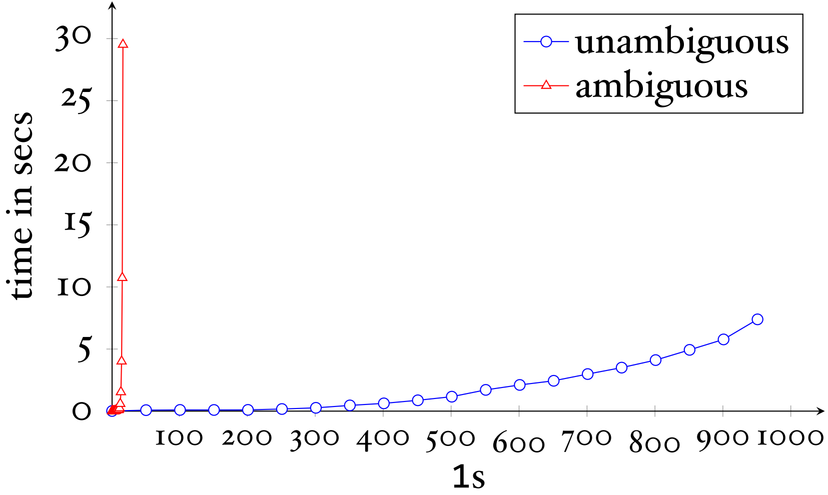
- apply p producing a set of pairs
- then apply the function f to each first component

$$\{(f(o_1), u_1) \mid (o_1, u_1) \in p(\text{input})\}$$

Ambiguous Grammars



Ambiguous Grammars



Arithmetic Expressions

A grammar for arithmetic expressions and numbers:

$$\begin{array}{lcl} E & \rightarrow & E \cdot + \cdot E \,|\, E \cdot * \cdot E \,|\, (\cdot E \cdot) \,|\, N \\ N & \rightarrow & N \cdot N \,|\, 0 \,|\, 1 \,|\, \ldots \,|\, 9 \end{array}$$

Unfortunately it is left-recursive (and ambiguous).

A problem for recursive descent parsers (e.g. parser combinators).

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Numbers

$$N \rightarrow N \cdot N \mid 0 \mid 1 \mid \ldots \mid 9$$

A non-left-recursive, non-ambiguous grammar for numbers:

$$N \rightarrow 0 \cdot N \mid 1 \cdot N \mid \ldots \mid 0 \mid 1 \mid \ldots \mid 9$$

Removing Left-Recursion

The rule for numbers is directly left-recursive:

$$N \rightarrow N \cdot N \mid 0 \mid 1 \quad (\ldots)$$

Translate

$$\begin{array}{rcl} N & \rightarrow & N \cdot \alpha \\ & \mid & \beta \end{array} \quad \Rightarrow \quad \begin{array}{rcl} N & \rightarrow & \beta \cdot N' \\ N' & \rightarrow & \alpha \cdot N' \\ & \mid & \epsilon \end{array}$$

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Which means

$$\begin{array}{lll} N & \rightarrow & 0 \cdot N' \mid 1 \cdot N' \\ N' & \rightarrow & N \cdot N' \mid \epsilon \end{array}$$

Operator Precedences

To disambiguate

$$E \;\rightarrow\; E \cdot + \cdot E \mid E \cdot * \cdot E \mid (\cdot E \cdot) \mid N$$

Decide on how many precedence levels, say

highest for $()$, medium for $*$, lowest for $+$

$$\begin{array}{lcl} E_{low} & \rightarrow & E_{med} \cdot + \cdot E_{low} \mid E_{med} \\ E_{med} & \rightarrow & E_{hi} \cdot * \cdot E_{med} \mid E_{hi} \\ E_{hi} & \rightarrow & (\cdot E_{low} \cdot) \mid N \end{array}$$

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What happens with $1 + 3 + 4$?

Chomsky Normal Form

All rules must be of the form

$$A \rightarrow a$$

or

$$A \rightarrow B \cdot C$$

No rule can contain ϵ.

ϵ-Removal

1. If $A \rightarrow \alpha \cdot B \cdot \beta$ and $B \rightarrow \epsilon$ are in the grammar, then add $A \rightarrow \alpha \cdot \beta$ (iterate if necessary).
2. Throw out all $B \rightarrow \epsilon$.

$$\begin{array}{lcl} N & \rightarrow & 0 \cdot N' \mid 1 \cdot N' \\ N' & \rightarrow & N \cdot N' \mid \epsilon \end{array}$$

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$$\begin{array}{lcl} N & \rightarrow & 0 \cdot N' \mid 1 \cdot N' \mid 0 \mid 1 \\ N' & \rightarrow & N \cdot N' \mid N \end{array}$$

$$N \rightarrow 0 \cdot N \mid 1 \cdot N \mid 0 \mid 1$$

CYK Algorithm

If grammar is in Chomsky normalform ...

$$\begin{array}{lcl} S & \rightarrow & N \cdot P \\ P & \rightarrow & V \cdot N \\ N & \rightarrow & N \cdot N \\ N & \rightarrow & \texttt{students} \mid \texttt{Jeff} \mid \texttt{geometry} \mid \texttt{trains} \\ V & \rightarrow & \texttt{trains} \end{array}$$

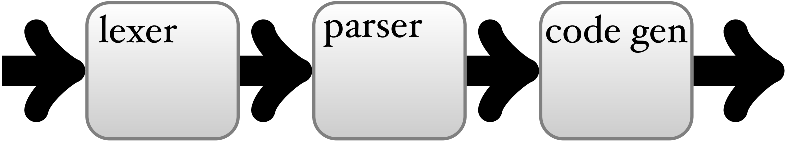
`Jeff trains geometry students`

CYK Algorithm

- fastest possible algorithm for recognition problem
- runtime is $O(n^3)$
- grammars need to be transferred into CNF

The Goal of this Course

Write a Compiler



We have lexer and parser.

$$
\begin{array}{lcl}
\mathit{Stmt} & \rightarrow & \texttt{skip} \\
 & | & \mathit{Id}\ \texttt{:=}\ \mathit{AExp} \\
 & | & \texttt{if}\ \mathit{BExp}\ \texttt{then}\ \mathit{Block}\ \texttt{else}\ \mathit{Block} \\
 & | & \texttt{while}\ \mathit{BExp}\ \texttt{do}\ \mathit{Block} \\
 & | & \texttt{read}\ \mathit{Id} \\
 & | & \texttt{write}\ \mathit{Id} \\
 & | & \texttt{write}\ \mathit{String} \\
\mathit{Stmts} & \rightarrow & \mathit{Stmt}\ \texttt{;}\ \mathit{Stmts} \\
 & | & \mathit{Stmt} \\
\mathit{Block} & \rightarrow & \{\ \mathit{Stmts}\ \} \\
 & | & \mathit{Stmt} \\
\mathit{AExp} & \rightarrow & \ldots \\
\mathit{BExp} & \rightarrow & \ldots
\end{array}
$$

```
write "Fib";
read n;
minus1 := 0;
minus2 := 1;
while n > 0 do {
       temp := minus2;
       minus2 := minus1 + minus2;
       minus1 := temp;
       n := n - 1
};
write "Result";
write minus2
```

An Interpreter

$$
\begin{array}{l}
\{ \\
\quad x := 5; \\
\quad y := x * 3; \\
\quad y := x * 4; \\
\quad x := u * 3 \\
\}
\end{array}
$$

- the interpreter has to record the value of x before assigning a value to y

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\}
\end{array}
$$

- the interpreter has to record the value of x before assigning a value to y
- eval(stmt, env)

Interpreter

$$
\begin{array}{lcl}
\text{eval}(n, E) & \stackrel{\text{def}}{=} & n \\
\text{eval}(x, E) & \stackrel{\text{def}}{=} & E(x) \quad \text{lookup } x \text{ in } E \\
\text{eval}(a_1 + a_2, E) & \stackrel{\text{def}}{=} & \text{eval}(a_1, E) + \text{eval}(a_2, E) \\
\text{eval}(a_1 - a_2, E) & \stackrel{\text{def}}{=} & \text{eval}(a_1, E) - \text{eval}(a_2, E) \\
\text{eval}(a_1 * a_2, E) & \stackrel{\text{def}}{=} & \text{eval}(a_1, E) * \text{eval}(a_2, E) \\
\\
\text{eval}(a_1 = a_2, E) & \stackrel{\text{def}}{=} & \text{eval}(a_1, E) = \text{eval}(a_2, E) \\
\text{eval}(a_1 \mathrel{!}= a_2, E) & \stackrel{\text{def}}{=} & \neg(\text{eval}(a_1, E) = \text{eval}(a_2, E)) \\
\text{eval}(a_1 < a_2, E) & \stackrel{\text{def}}{=} & \text{eval}(a_1, E) < \text{eval}(a_2, E)
\end{array}
$$

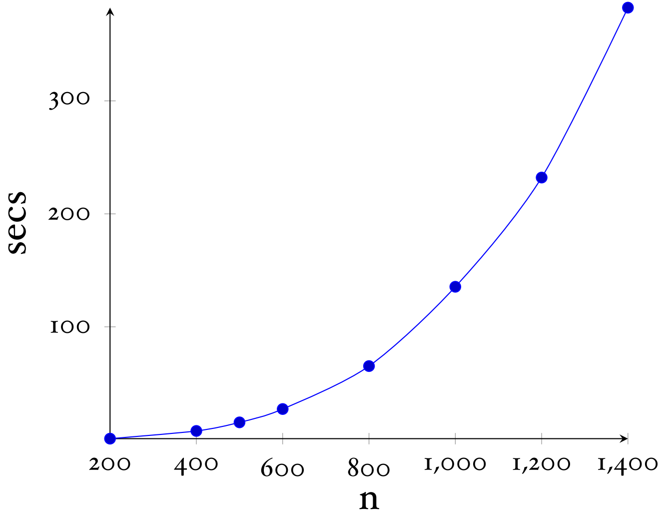
Interpreter (2)

$$
\begin{array}{lcl}
\text{eval}(\text{skip}, E) & \stackrel{\text{def}}{=} & E \\
\text{eval}(x := a, E) & \stackrel{\text{def}}{=} & E(x \mapsto \text{eval}(a, E)) \\
\text{eval}(\text{if } b \text{ then } cs_1 \text{ else } cs_2, E) & \stackrel{\text{def}}{=} & \\
\multicolumn{3}{l}{\qquad\qquad \text{if eval}(b, E) \text{ then eval}(cs_1, E)} \\
\multicolumn{3}{l}{\qquad\qquad\qquad\qquad\qquad\quad \text{else eval}(cs_2, E)} \\
\text{eval}(\text{while } b \text{ do } cs, E) & \stackrel{\text{def}}{=} & \\
\multicolumn{3}{l}{\qquad\qquad \text{if eval}(b, E)} \\
\multicolumn{3}{l}{\qquad\qquad \text{then eval}(\text{while } b \text{ do } cs, \text{eval}(cs, E))} \\
\multicolumn{3}{l}{\qquad\qquad \text{else } E} \\
\text{eval}(\text{write } x, E) & \stackrel{\text{def}}{=} & \{\ \text{println}(E(x))\ ;\ E\ \}
\end{array}
$$

Test Program

```
start := 1000;
x := start;
y := start;
z := start;
while 0 < x do {
 while 0 < y do {
  while 0 < z do { z := z - 1 };
  z := start;
  y := y - 1
 };
 y := start;
 x := x - 1
}
```

Interpreted Code



Java Virtual Machine

- introduced in 1995
- is a stack-based VM (like Postscript, CLR of .Net)
- contains a JIT compiler
- many languages take advantage of JVM's infrastructure (JRE)
- is garbage collected $\Rightarrow$ no buffer overflows
- some languages compile to the JVM: Scala, Clojure...