

Compilers and Formal Languages (2)

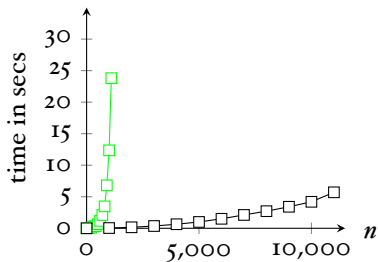
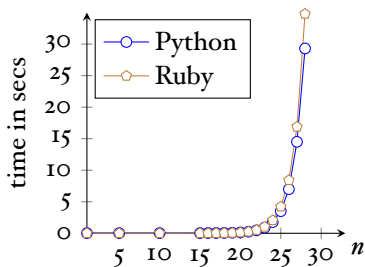
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An Efficient Regular Expression Matcher

Graphs: $a^?{n} \cdot a^{n}$ and strings $\underbrace{a \dots a}_n$



In the handouts is a similar graph with $(a^*)^* \cdot b$ for Java.

Languages

- A **Language** is a set of strings, for example

$$\{\epsilon, \text{hello}, \text{foobar}, a, abc\}$$

- **Concatenation** of strings and languages

$$\text{foo} @ \text{bar} = \text{foobar}$$

$$A @ B \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

For example $A = \{\text{foo}, \text{bar}\}$, $B = \{a, b\}$

$$A @ B = \{\text{fooa}, \text{foob}, \text{bara}, \text{barb}\}$$

The Power Operation

- The **Power** of a language:

$$A^0 \stackrel{\text{def}}{=} \{\epsilon\}$$
$$A^{n+1} \stackrel{\text{def}}{=} A @ A^n$$

For example

$$A^4 = A @ A @ A @ A$$
$$A^1 = A$$
$$A^0 = \{\epsilon\}$$

Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in A^4 ?

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- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in A^4 ?

What if $A = \{[a], [b], [c], []\}$;
how many strings are then in A^4 ?

The Star Operation

- The **Star** of a language:

$$A^* \stackrel{\text{def}}{=} \bigcup_{0 \leq n} A^n$$

This expands to

$$A^0 \cup A^1 \cup A^2 \cup A^3 \cup A^4 \cup \dots$$

$$\{\epsilon\} \cup A \cup A @ A \cup A @ A @ A \cup A @ A @ A @ A \cup \dots$$

The Meaning of a Regular Expression

$$L(\mathbf{0}) \stackrel{\text{def}}{=} \{\}$$

$$L(\mathbf{1}) \stackrel{\text{def}}{=} \{\emptyset\}$$

$$L(c) \stackrel{\text{def}}{=} \{[c]\}$$

$$L(r_1 + r_2) \stackrel{\text{def}}{=} L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in L(r_1) \wedge s_2 \in L(r_2)\}$$

$$L(r^*) \stackrel{\text{def}}{=} (L(r))^* \stackrel{\text{def}}{=} \bigcup_{0 \leq n} L(r)^n$$

L is a function from
regular expressions to sets
of strings

$$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$$

The Specification of Matching

A regular expression r matches a string s provided

$$s \in L(r)$$

...and the point of the this lecture is to decide this problem as fast as possible (unlike Python, Ruby, Java etc)

Semantic Derivative

- The **Semantic Derivative** of a language wrt to a character c :

$$\text{Der } c A \stackrel{\text{def}}{=} \{s \mid c :: s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\text{Der } f A = \{oo, rak\}$$

$$\text{Der } b A = \{ar\}$$

$$\text{Der } a A = \{\}$$

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$$Der f A = \{oo, rak\}$$

$$Der b A = \{ar\}$$

$$Der a A = \{\}$$

We can extend this definition to strings

$$Der s A = \{s' \mid s @ s' \in A\}$$

Regular Expressions

Their inductive definition:

$r ::=$	0	null
	1	empty string / "" / []
	c	character
	$r_1 \cdot r_2$	sequence
	$r_1 + r_2$	alternative / choice
	r^*	star (zero or more)

Th

```
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

$r ::=$	O	null
	I	empty string / "" / []
	c	character
	$r_1 \cdot r_2$	sequence
	$r_1 + r_2$	alternative / choice
	r^*	star (zero or more)

When Are Two Regular Expressions Equivalent?

$$r_1 \equiv r_2 \stackrel{\text{def}}{=} L(r_1) = L(r_2)$$

Concrete Equivalences

$$(a + b) + c \equiv a + (b + c)$$

$$a + a \equiv a$$

$$a + b \equiv b + a$$

$$(a \cdot b) \cdot c \equiv a \cdot (b \cdot c)$$

$$c \cdot (a + b) \equiv (c \cdot a) + (c \cdot b)$$

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$$c \cdot (a + b) \equiv (c \cdot a) + (c \cdot b)$$

$$a \cdot a \not\equiv a$$

$$a + (b \cdot c) \not\equiv (a + b) \cdot (a + c)$$

Corner Cases

$$\begin{aligned} a \cdot \mathbf{0} &\neq a \\ a + \mathbf{1} &\neq a \\ \mathbf{1} &\equiv \mathbf{0}^* \\ \mathbf{1}^* &\equiv \mathbf{1} \\ \mathbf{0}^* &\neq \mathbf{0} \end{aligned}$$

Simplification Rules

$$r + \mathbf{0} \equiv r$$

$$\mathbf{0} + r \equiv r$$

$$r \cdot \mathbf{1} \equiv r$$

$$\mathbf{1} \cdot r \equiv r$$

$$r \cdot \mathbf{0} \equiv \mathbf{0}$$

$$\mathbf{0} \cdot r \equiv \mathbf{0}$$

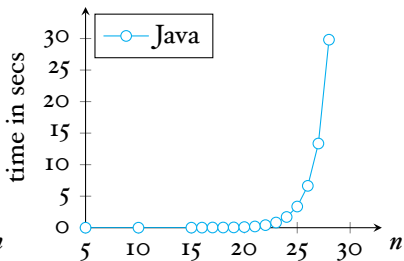
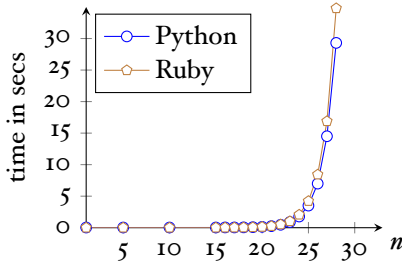
$$r + r \equiv r$$

The Specification for Matching

A regular expression r matches a string s
if and only if

$$s \in L(r)$$

$$(a^{\{n\}}) \cdot a^{\{n\}} \text{ and } (a^*)^* \cdot b$$



Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
 - $(a^?\{n\}) \cdot a^{\{n\}}$
 - $(a^*)^*$
 - $([a-z]^+)^*$
 - $(a + a \cdot a)^*$
 - $(a + a?)^*$
- sometimes also called catastrophic backtracking

A Matching Algorithm

...whether a regular expression can match the empty string:

$$\text{nullable}(\mathbf{0}) \stackrel{\text{def}}{=} \text{false}$$

$$\text{nullable}(\mathbf{1}) \stackrel{\text{def}}{=} \text{true}$$

$$\text{nullable}(c) \stackrel{\text{def}}{=} \text{false}$$

$$\text{nullable}(r_1 + r_2) \stackrel{\text{def}}{=} \text{nullable}(r_1) \vee \text{nullable}(r_2)$$

$$\text{nullable}(r_1 \cdot r_2) \stackrel{\text{def}}{=} \text{nullable}(r_1) \wedge \text{nullable}(r_2)$$

$$\text{nullable}(r^*) \stackrel{\text{def}}{=} \text{true}$$

The Derivative of a Rexp

If r matches the string $c::s$, what is a regular expression that matches just s ?

$der\ c\ r$ gives the answer, Brzozowski 1964

The Derivative of a Rexp

$$\mathit{der} c (\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0}$$

$$\mathit{der} c (\mathbf{1}) \stackrel{\text{def}}{=} \mathbf{0}$$

$$\mathit{der} c (d) \stackrel{\text{def}}{=} \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}$$

$$\mathit{der} c (r_1 + r_2) \stackrel{\text{def}}{=} \mathit{der} c r_1 + \mathit{der} c r_2$$

$$\mathit{der} c (r_1 \cdot r_2) \stackrel{\text{def}}{=} \text{if } \mathit{nullable}(r_1) \\ \text{then } (\mathit{der} c r_1) \cdot r_2 + \mathit{der} c r_2 \\ \text{else } (\mathit{der} c r_1) \cdot r_2$$

$$\mathit{der} c (r^*) \stackrel{\text{def}}{=} (\mathit{der} c r) \cdot (r^*)$$

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$$\mathit{der} c (r^*) \stackrel{\text{def}}{=} (\mathit{der} c r) \cdot (r^*)$$

$$\mathit{ders} [] r \stackrel{\text{def}}{=} r$$

$$\mathit{ders} (c :: s) r \stackrel{\text{def}}{=} \mathit{ders} s (\mathit{der} c r)$$

Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$\text{der } a r = ?$$

$$\text{der } b r = ?$$

$$\text{der } c r = ?$$

The Algorithm

matches r s $\stackrel{\text{def}}{=} \text{nullable}(\text{ders } r s)$

An Example

Does r_1 match *abc*?

Step 1: build derivative of *a* and r_1 ($r_2 = \text{der } a r_1$)

Step 2: build derivative of *b* and r_2 ($r_3 = \text{der } b r_2$)

Step 3: build derivative of *c* and r_3 ($r_4 = \text{der } c r_3$)

Step 4: the string is exhausted: ($\text{nullable}(r_4)$)
test whether r_4 can recognise
the empty string

Output: result of the test
 \Rightarrow *true* or *false*

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_I then

① $Der a (L(r_I))$

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_I then

- 1 $Der a (L(r_I))$
- 2 $Der b (Der a (L(r_I)))$

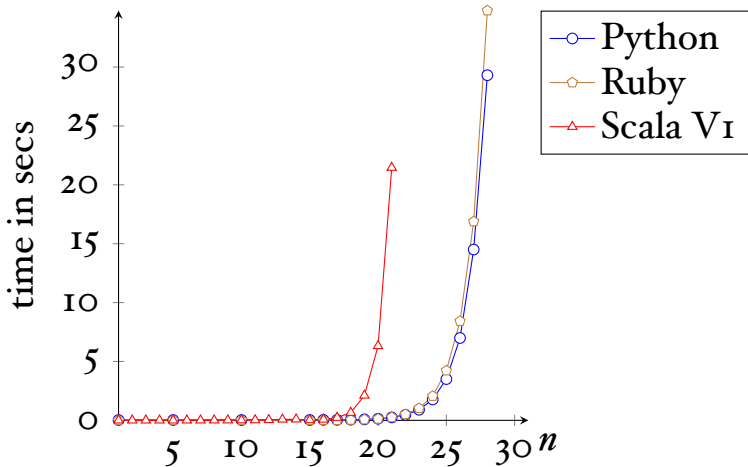
The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_I then

- 1 $Der a (L(r_I))$
- 2 $Der b (Der a (L(r_I)))$
- 3 $Der c (Der b (Der a (L(r_I))))$
- 4 finally we test whether the empty string is in this set; same for $Ders abc (L(r_I))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

Oops... $(a^{\{n\}}) \cdot a^{\{n\}}$



A Problem

We represented the “n-times” $a^{\{n\}}$ as a sequence regular expression:

1: a

2: $a \cdot a$

3: $a \cdot a \cdot a$

...

13: $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

...

20:

This problem is aggravated with $a^?$ being represented as $a + \mathbf{1}$.

Solving the Problem

What happens if we extend our regular expressions

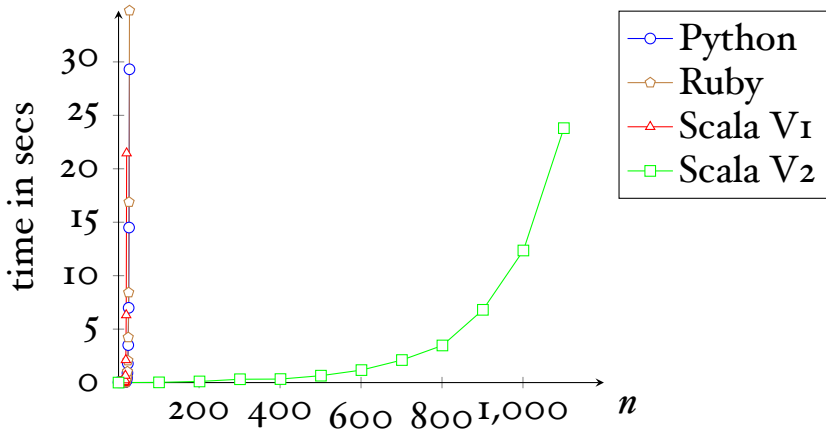
$$r ::= \dots$$

	$r^{\{n\}}$
	$r^?$

What is their meaning?

What are the cases for *nullable* and *der*?

$$(a^{\{n\}}) \cdot a^{\{n\}}$$



Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$\text{der } a r = ((\mathbf{I} \cdot b) + \mathbf{0}) \cdot r$$

$$\text{der } b r = ((\mathbf{0} \cdot b) + \mathbf{I}) \cdot r$$

$$\text{der } c r = ((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r$$

What are these regular expressions equivalent to?

Simplification

$$r + \mathbf{0} \Rightarrow r$$

$$\mathbf{0} + r \Rightarrow r$$

$$r \cdot \mathbf{I} \Rightarrow r$$

$$\mathbf{I} \cdot r \Rightarrow r$$

$$r \cdot \mathbf{0} \Rightarrow \mathbf{0}$$

$$\mathbf{0} \cdot r \Rightarrow \mathbf{0}$$

$$r + r \Rightarrow r$$

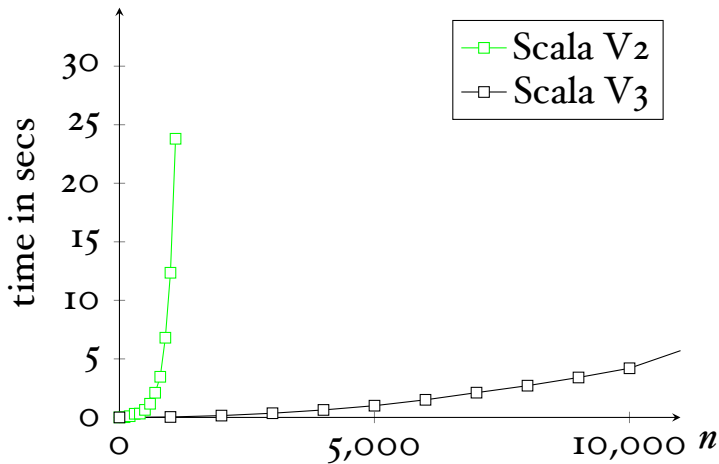
```
def ders(s: List[Char], r: Rexp) : Rexp = s match {  
  case Nil => r  
  case c::s => ders(s, simp(der(c, r)))  
}
```

```

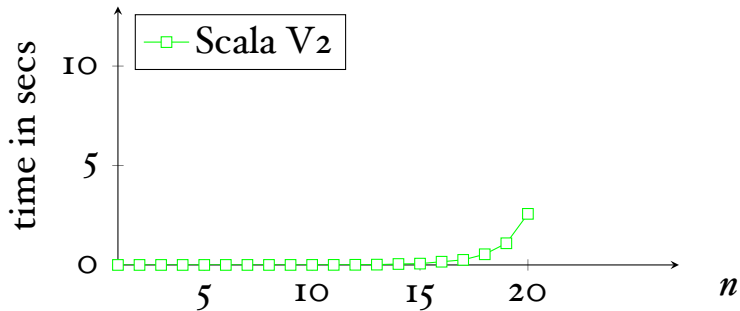
def simp(r: Rexp) : Rexp = r match {
  case ALT(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, r2s) => r2s
      case (r1s, ZERO) => r1s
      case (r1s, r2s) =>
        if (r1s == r2s) r1s else ALT(r1s, r2s)
    }
  }
  case SEQ(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, _) => ZERO
      case (_, ZERO) => ZERO
      case (ONE, r2s) => r2s
      case (r1s, ONE) => r1s
      case (r1s, r2s) => SEQ(r1s, r2s)
    }
  }
  case NTIMES(r, n) => NTIMES(simp(r), n)
  case r => r
}

```

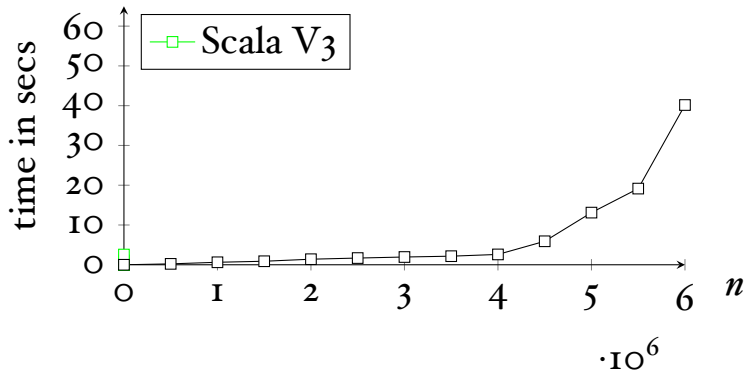
$$(a^{\{n\}}) \cdot a^{\{n\}}$$



$$(a^*)^* \cdot b$$



$$(a^*)^* \cdot b$$



What is good about this Alg.

- extends to most regular expressions, for example
 $\sim r$
- is easy to implement in a functional language
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is brand new work)
- we can prove its correctness...

Proofs about Rexprs

Remember their inductive definition:

$$r ::= \begin{array}{l} \mathbf{0} \\ \mathbf{1} \\ c \\ r_1 \cdot r_2 \\ r_1 + r_2 \\ r^* \end{array}$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\mathbf{0}$, $\mathbf{1}$ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2 .
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2 .
- P holds for r^* under the assumption that P already holds for r .

Proofs about Rexp (3)

Assume $P(r)$ is the property:

nullable(r) if and only if $[\] \in L(r)$

Proofs about Rexp (4)

$$\text{rev}(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0}$$

$$\text{rev}(\mathbf{1}) \stackrel{\text{def}}{=} \mathbf{1}$$

$$\text{rev}(c) \stackrel{\text{def}}{=} c$$

$$\text{rev}(r_1 + r_2) \stackrel{\text{def}}{=} \text{rev}(r_1) + \text{rev}(r_2)$$

$$\text{rev}(r_1 \cdot r_2) \stackrel{\text{def}}{=} \text{rev}(r_2) \cdot \text{rev}(r_1)$$

$$\text{rev}(r^*) \stackrel{\text{def}}{=} \text{rev}(r)^*$$

We can prove

$$L(\text{rev}(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on r .

Correctness Proof for our Matcher

- We started from

$$s \in L(r)$$

$$\Leftrightarrow \square \in \text{Ders } s(L(r))$$

Correctness Proof for our Matcher

- We started from

$$s \in L(r)$$

$$\Leftrightarrow [] \in Ders\ s\ (L(r))$$

- if we can show $Ders\ s\ (L(r)) = L(ders\ s\ r)$ we have

$$\Leftrightarrow [] \in L(ders\ s\ r)$$

$$\Leftrightarrow \text{nullable}(ders\ s\ r)$$

$$\stackrel{\text{def}}{=} \text{matches}\ s\ r$$

Proofs about Rexp (5)

Let $Der\ c\ A$ be the set defined as

$$Der\ c\ A \stackrel{\text{def}}{=} \{s \mid c :: s \in A\}$$

We can prove

$$L(\text{der}\ c\ r) = Der\ c\ (L(r))$$

by induction on r .

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

We can then prove

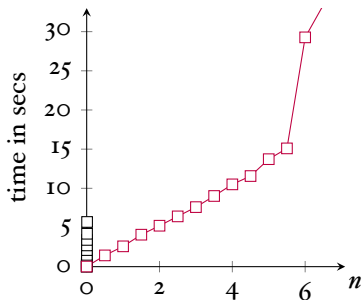
$$\text{Ders } s (L(r)) = L(\text{ders } s r)$$

We can finally prove

$$\text{matches } s r \text{ if and only if } s \in L(r)$$

Epilogue

Graph: $a^{\{n\}} \cdot a^{\{n\}}$

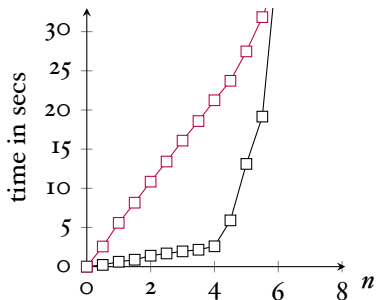


—□— Scala V3

—□— Scala V4

$\cdot 10^6$

Graph: $(a^*)^* \cdot b$



—□— Scala V3

—□— Scala V4

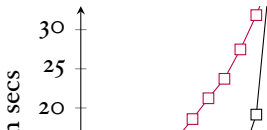
$\cdot 10^6$

Epilogue

Graph: $a^{\{n\}} \cdot a^{\{n\}}$



Graph: $(a^*)^* \cdot b$



```
def ders2(s: List[Char], r: Rexp) : Rexp = (s, r) match {  
  case (Nil, r) => r  
  case (s, ZERO) => ZERO  
  case (s, ONE) => if (s == Nil) ONE else ZERO  
  case (s, CHAR(c)) => if (s == List(c)) ONE else  
                        if (s == Nil) CHAR(c) else ZERO  
  case (s, ALT(r1, r2)) => ALT(ders2(s, r1), ders2(s, r2))  
  case (c::s, r) => ders2(s, simp(der(c, r)))  
}
```