# Compilers and Formal Languages (3)

| | |
|---|---|
| Email: | christian.urban at kcl.ac.uk |
| Office: | N7.07 (North Wing, Bush House) |
| Slides: | KEATS (also homework and coursework is there) |

# Scala Book, Exams

- https://nms.kcl.ac.uk/christian.urban/ProgInScala2ed.pdf
- homeworks (written exam 80%)
- coursework (20%)

- short survey at KEATS; to be answered until Sunday

# Regular Expressions

In programming languages they are often used to recognise:

- symbols, digits
- identifiers
- numbers (non-leading zeros)
- keywords
- comments

`http://www.regexper.com`

# Last Week

Last week I showed you a regular expression matcher that works provably correct in all cases (we only started with the proving part though)

$$\mathit{matches}\, s\, r \quad \text{if and only if} \quad s \in L(r)$$

by Janusz Brzozowski (1964)

# The Derivative of a Rexp

$$\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else } (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)\\
 & & \\
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}$$

# Example

Given $r \overset{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$
\begin{array}{rcl}
der\,a\,((a \cdot b) + b)^* & \Rightarrow & der\,a\,\underline{((a \cdot b) + b)^*} \\
 & = & (der\,a\,(\underline{(a \cdot b) + b})) \cdot r \\
 & = & ((der\,a\,(\underline{a \cdot b})) + (der\,a\,b)) \cdot r \\
 & = & (((der\,a\,\underline{a}) \cdot b) + (der\,a\,b)) \cdot r \\
 & = & ((\mathbf{1} \cdot b) + (der\,a\,\underline{b})) \cdot r \\
 & = & ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r
\end{array}
$$

Input: string $abc$ and regular expression $r$

1. $der\,a\,r$
2. $der\,b\,(der\,a\,r)$
3. $der\,c\,(der\,b\,(der\,a\,r))$

Input: string $abc$ and regular expression $r$

1. $der\, a\, r$
2. $der\, b\, (der\, a\, r)$
3. $der\, c\, (der\, b\, (der\, a\, r))$

4. finally check whether the last regular expression can match the empty string

# Simplification

Given $r \overset{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$\begin{aligned}
((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r \quad &\Rightarrow \quad ((\underline{\mathbf{1} \cdot b}) + \mathbf{0}) \cdot r \\
&= \quad (\underline{b + \mathbf{0}}) \cdot r \\
&= \quad b \cdot r
\end{aligned}$$

We proved partially

$$\textit{nullable}(r) \;\text{ if and only if }\; [] \in L(r)$$

by induction on the regular expression $r$.

We proved partially

$$\textit{nullable}(r) \;\text{ if and only if }\; [] \in L(r)$$

by induction on the regular expression $r$.

**Any Questions?**

We need to prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

also by induction on the regular expression $r$.

# Proofs about Rexps

- $P$ holds for $\mathbf{0}$, $\mathbf{1}$ and $c$

- $P$ holds for $r_1 + r_2$ under the assumption that $P$ already holds for $r_1$ and $r_2$.

- $P$ holds for $r_1 \cdot r_2$ under the assumption that $P$ already holds for $r_1$ and $r_2$.

- $P$ holds for $r^*$ under the assumption that $P$ already holds for $r$.

# Proofs about Natural Numbers and Strings

- $P$ holds for $0$ and
- $P$ holds for $n + 1$ under the assumption that $P$ already holds for $n$

- $P$ holds for $[]$ and
- $P$ holds for $c :: s$ under the assumption that $P$ already holds for $s$

# Regular Expressions

| | | | |
|---|---|---|---|
| $r$ | $::=$ | $\mathbf{0}$ | nothing |
| | $\mid$ | $\mathbf{1}$ | empty string / "" / [] |
| | $\mid$ | $c$ | character |
| | $\mid$ | $r_1 \cdot r_2$ | sequence |
| | $\mid$ | $r_1 + r_2$ | alternative / choice |
| | $\mid$ | $r^*$ | star (zero or more) |

How about ranges $[a\text{-}z]$, $r^+$ and $\sim r$? Do they increase the set of languages we can recognise?

# Negation of Regular Expr’s

- $\sim r$ (everything that $r$ cannot recognise)
- $L(\sim r) \stackrel{\text{def}}{=} UNIV - L(r)$
- $nullable(\sim r) \stackrel{\text{def}}{=} not\,(nullable(r))$
- $der\,c\,(\sim r) \stackrel{\text{def}}{=} \sim (der\,c\,r)$

# Negation of Regular Expr's

- $\sim r$ (everything that $r$ cannot recognise)
- $L(\sim r) \overset{\text{def}}{=} UNIV - L(r)$
- $nullable(\sim r) \overset{\text{def}}{=} not\,(nullable(r))$
- $der\,c\,(\sim r) \overset{\text{def}}{=} \sim (der\,c\,r)$

Used often for recognising comments:

$$/ \cdot * \cdot (\sim ([a\text{-}z]^* \cdot * \cdot / \cdot [a\text{-}z]^*)) \cdot * \cdot /$$

# Negation

Assume you have an alphabet consisting of the letters $a$, $b$ and $c$ only. Find a (basic!) regular expression that matches all strings *except* $ab$ and $ac$!

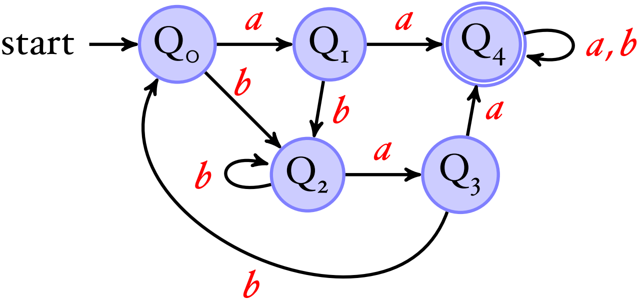
# Automata

A **deterministic finite automaton**, DFA, consists of:

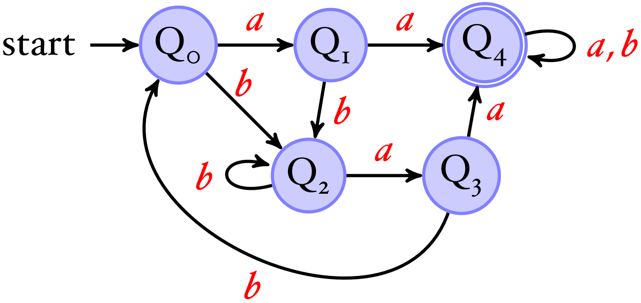
- an alphabet $\Sigma$
- a set of states $\mathcal{Q}$
- one of these states is the start state $Q_0$
- some states are accepting states $F$, and
- there is transition function $\delta$

which takes a state as argument and a character and produces a new state; this function might not be everywhere defined ⇒ partial function

$$A(\Sigma, Q, Q_0, F, \delta)$$



- the start state can be an accepting state
- it is possible that there is no accepting state
- all states might be accepting (but this does not necessarily mean all strings are accepted)



for this automaton $\delta$ is the function

$$\begin{array}{lll} (Q_0, a) \rightarrow Q_1 & (Q_1, a) \rightarrow Q_4 & (Q_4, a) \rightarrow Q_4 \\ (Q_0, b) \rightarrow Q_2 & (Q_1, b) \rightarrow Q_2 & (Q_4, b) \rightarrow Q_4 \end{array} \quad \ldots$$

# Accepting a String

Given

$$A(\Sigma, Q, Q_0, F, \delta)$$

you can define

$$\begin{aligned}\widehat{\delta}(q, []) &\stackrel{\text{def}}{=} q\\ \widehat{\delta}(q, c :: s) &\stackrel{\text{def}}{=} \widehat{\delta}(\delta(q, c), s)\end{aligned}$$

# Accepting a String

Given

$$A(\Sigma, Q, Q_0, F, \delta)$$

you can define

$$\begin{aligned}\widehat{\delta}(q, []) &\stackrel{\text{def}}{=} q\\ \widehat{\delta}(q, c :: s) &\stackrel{\text{def}}{=} \widehat{\delta}(\delta(q, c), s)\end{aligned}$$

Whether a string $s$ is accepted by $A$?

$$\widehat{\delta}(Q_0, s) \in F$$

# Regular Languages

A **language** is a set of strings.

A **regular expression** specifies a language.

A language is **regular** iff there exists a regular expression that recognises all its strings.

# Regular Languages

A **language** is a set of strings.

A **regular expression** specifies a language.

A language is **regular** iff there exists a regular expression that recognises all its strings.

not all languages are regular, e.g. $a^n b^n$ is not

# Regular Languages (2)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

# Non-Deterministic Finite Automata

A non-deterministic finite automaton (NFA) consists again of:

- a finite set of states
- some these states are the start states
- some states are accepting states, and
- there is transition relation

$$\begin{array}{l}(Q_1, a) \rightarrow Q_2 \\ (Q_1, a) \rightarrow Q_3 \end{array} \quad \ldots$$

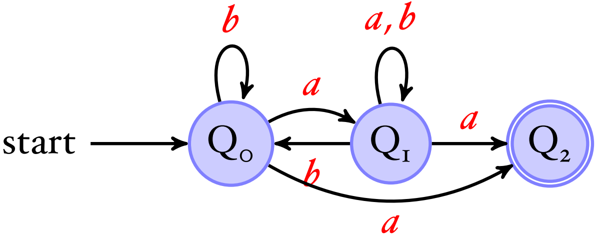
# Non-Deterministic Finite Automata

A non-deterministic finite automaton (NFA) consists again of:

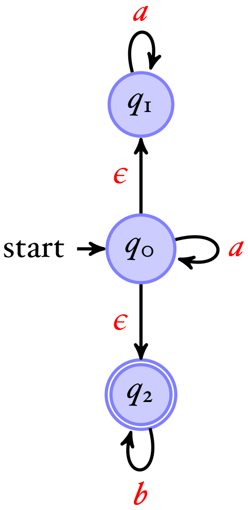
- a finite set of states
- some these states are the start states
- some states are accepting states, and
- there is transition relation

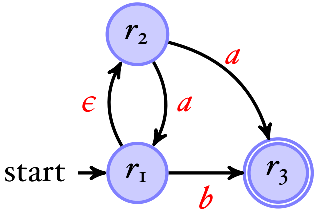
$$\begin{array}{l}(Q_1, a) \rightarrow Q_2 \\ (Q_1, a) \rightarrow Q_3\end{array} \quad \ldots \qquad (Q_1, a) \rightarrow \{Q_2, Q_3\}$$

# An NFA Example



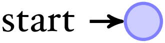
# Two Epsilon NFA Examples



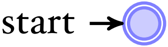


# Rexp to NFA

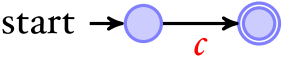
$\mathbf{0}$



$\mathbf{1}$

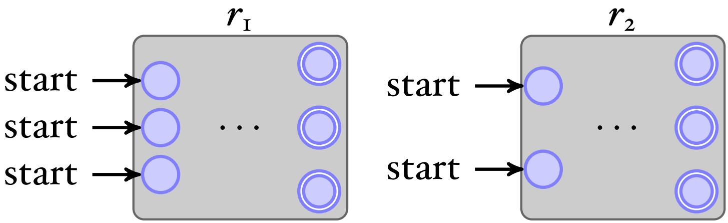


$c$



# Case $r_1 \cdot r_2$

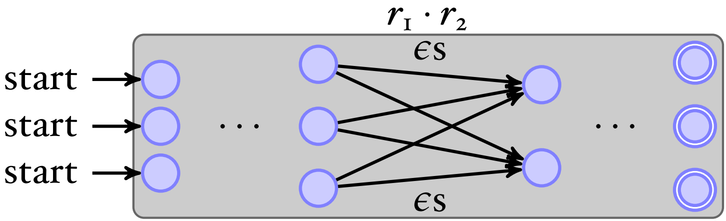
By recursion we are given two automata:



We need to (1) change the accepting nodes of the first automaton into non-accepting nodes, and (2) connect them via $\epsilon$-transitions to the starting state of the second automaton.

# Case $r_1 \cdot r_2$

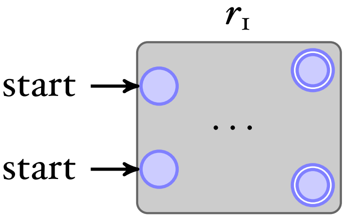
By recursion we are given two automata:

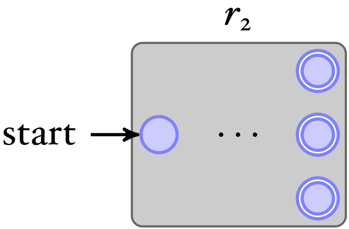


We need to (1) change the accepting nodes of the first automaton into non-accepting nodes, and (2) connect them via $\epsilon$-transitions to the starting state of the second automaton.

# Case $r_1 + r_2$

By recursion we are given two automata:

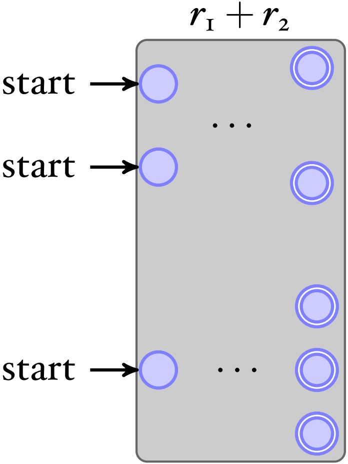




We can just put both automata together.

# Case $r_1 + r_2$

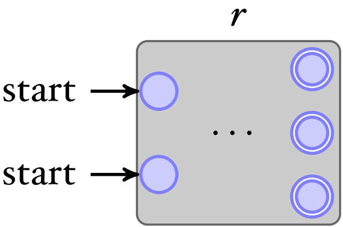
By recursion we are given two automata:



We can just put both automata together.

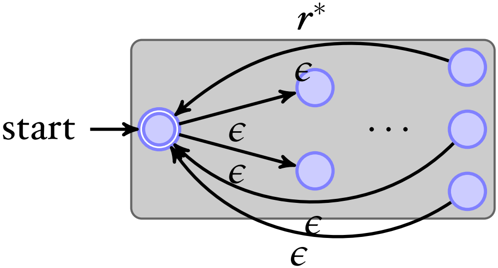
# Case $r^*$

By recursion we are given an automaton for $r$:



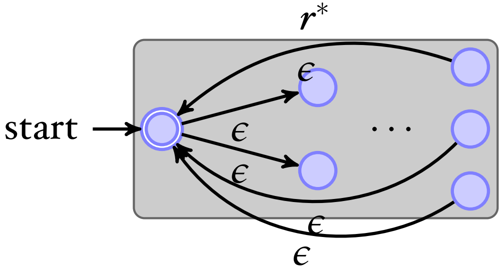
# Case $r^*$

By recursion we are given an automaton for $r$:



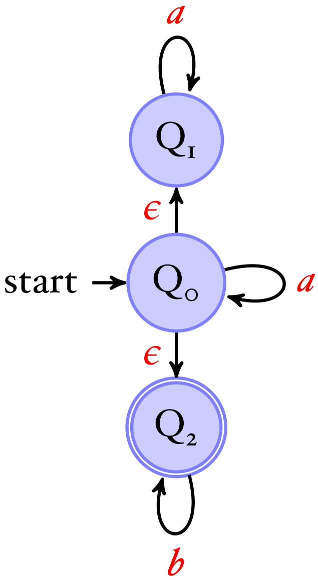
# Case $r^*$

By recursion we are given an automaton for $r$:



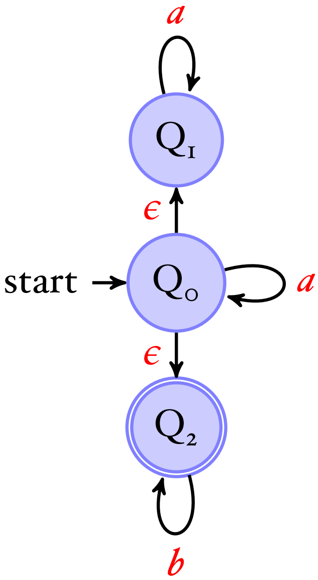
Why can't we just have an epsilon transition from the accepting states to the starting state?

# Subset Construction



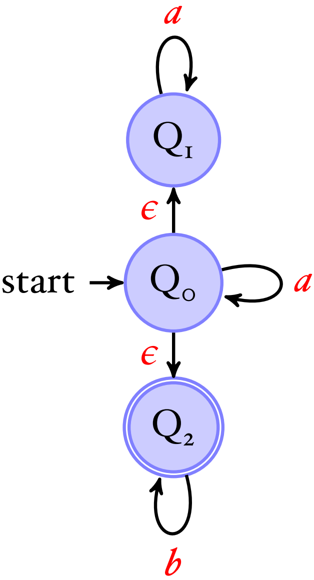
| nodes | $a$ | $b$ |
|---:|---|---|
| $\{\}$ | | |
| $\{0\}$ | | |
| $\{1\}$ | | |
| $\{2\}$ | | |
| $\{0, 1\}$ | | |
| $\{0, 2\}$ | | |
| $\{1, 2\}$ | | |
| $\{0, 1, 2\}$ | | |

# Subset Construction



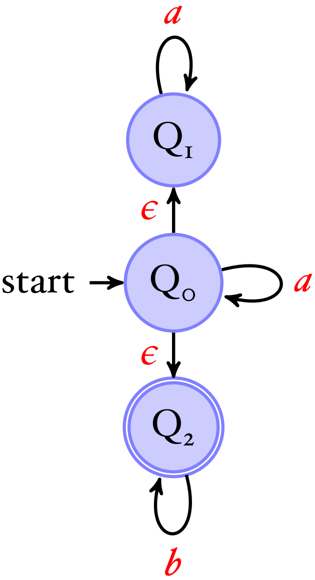
| nodes | $a$ | $b$ |
|---:|---|---|
| $\{\}$ | $\{\}$ | $\{\}$ |
| $\{0\}$ | | |
| $\{1\}$ | | |
| $\{2\}$ | | |
| $\{0,1\}$ | | |
| $\{0,2\}$ | | |
| $\{1,2\}$ | | |
| $\{0,1,2\}$ | | |

# Subset Construction



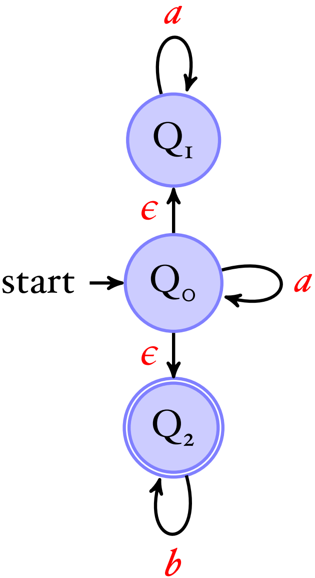
| nodes | $a$ | $b$ |
|---:|:---:|:---:|
| $\{\}$ | $\{\}$ | $\{\}$ |
| $\{0\}$ | $\{0,1,2\}$ | $\{2\}$ |
| $\{1\}$ | $\{1\}$ | $\{\}$ |
| $\{2\}$ | $\{\}$ | $\{2\}$ |
| $\{0,1\}$ | | |
| $\{0,2\}$ | | |
| $\{1,2\}$ | | |
| $\{0,1,2\}$ | | |

# Subset Construction



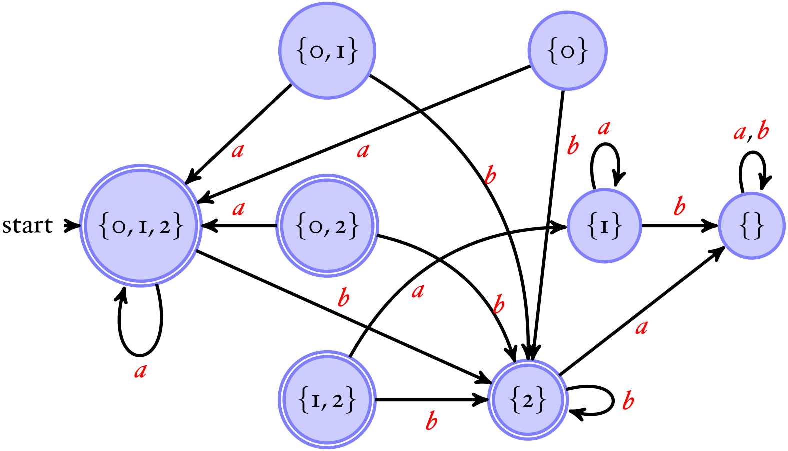
| nodes | $a$ | $b$ |
|---:|:---:|:---:|
| $\{\}$ | $\{\}$ | $\{\}$ |
| $\{0\}$ | $\{0,1,2\}$ | $\{2\}$ |
| $\{1\}$ | $\{1\}$ | $\{\}$ |
| $\{2\}$ | $\{\}$ | $\{2\}$ |
| $\{0,1\}$ | $\{0,1,2\}$ | $\{2\}$ |
| $\{0,2\}$ | $\{0,1,2\}$ | $\{2\}$ |
| $\{1,2\}$ | $\{1\}$ | $\{2\}$ |
| $\{0,1,2\}$ | $\{0,1,2\}$ | $\{2\}$ |

# Subset Construction



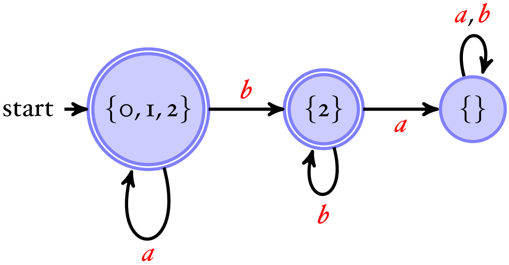
| nodes |  | $a$ | $b$ |
|---|---|---|---|
| $\{\}$ |  | $\{\}$ | $\{\}$ |
| $\{0\}$ |  | $\{0, 1, 2\}$ | $\{2\}$ |
| $\{1\}$ |  | $\{1\}$ | $\{\}$ |
| $\{2\}$ | * | $\{\}$ | $\{2\}$ |
| $\{0, 1\}$ |  | $\{0, 1, 2\}$ | $\{2\}$ |
| $\{0, 2\}$ | * | $\{0, 1, 2\}$ | $\{2\}$ |
| $\{1, 2\}$ | * | $\{1\}$ | $\{2\}$ |
| s: $\{0, 1, 2\}$ | * | $\{0, 1, 2\}$ | $\{2\}$ |

# The Result

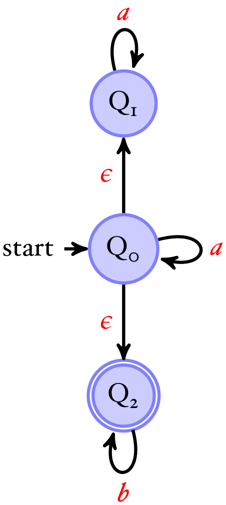


# Removing Dead States

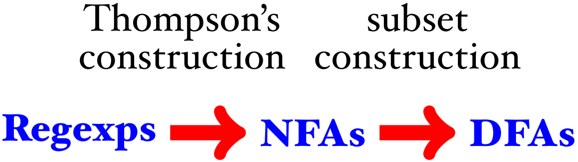
DFA:



(original) NFA:



# Regexps and Automata



Thompson's construction

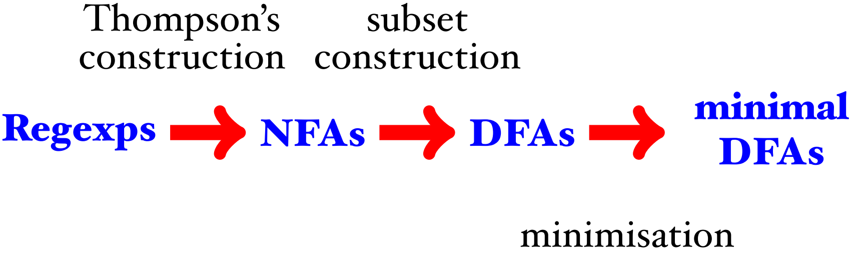
subset construction

**Regexps**

**NFAs**

**DFAs**

# Regexps and Automata



Thompson's construction

subset construction

Regexps → NFAs → DFAs → minimal DFAs

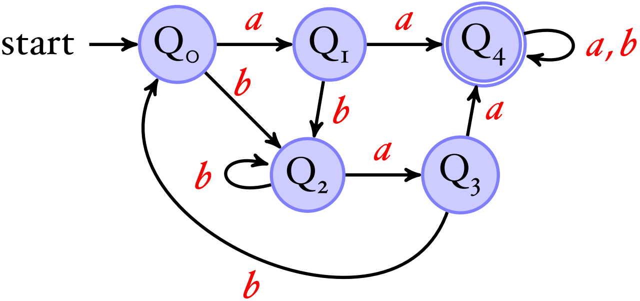
minimisation

# DFA Minimisation

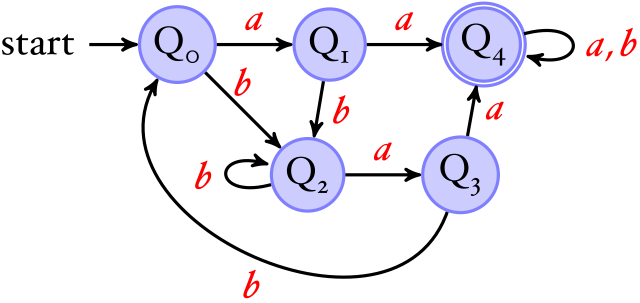
1. Take all pairs $(q,p)$ with $q \neq p$
2. Mark all pairs that accepting and non-accepting states
3. For all unmarked pairs $(q,p)$ and all characters $c$ test whether

   $$(\delta(q,c),\delta(p,c))$$

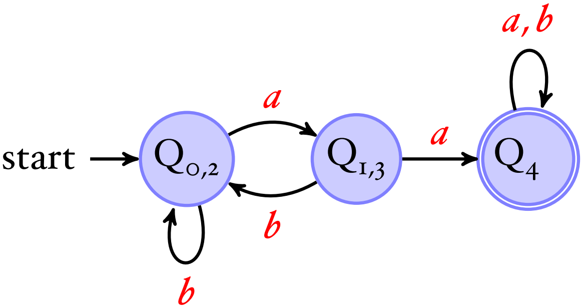
   are marked. If yes in at least one case, then also mark $(q,p)$.
4. Repeat last step until no change.
5. All unmarked pairs can be merged.



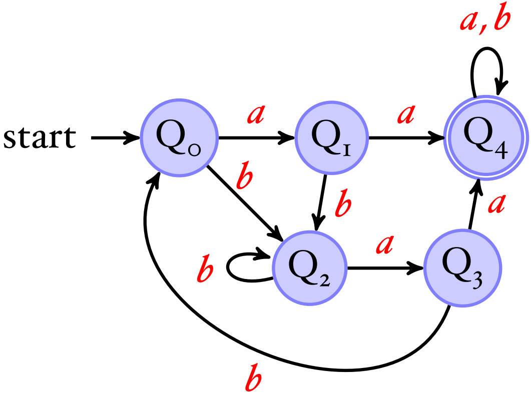
| | $Q_0$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $Q_1$ | | | | |
| $Q_2$ | | | | |
| $Q_3$ | | | | |
| $Q_4$ | ⋆ | ⋆ | ⋆ | ⋆ |



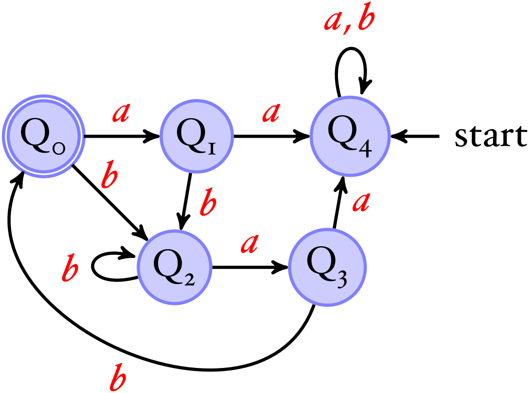
|  | $Q_0$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $Q_1$ | ★ |  |  |  |
| $Q_2$ |  | ★ |  |  |
| $Q_3$ | ★ |  | ★ |  |
| $Q_4$ | ★ | ★ | ★ | ★ |



# Alternatives

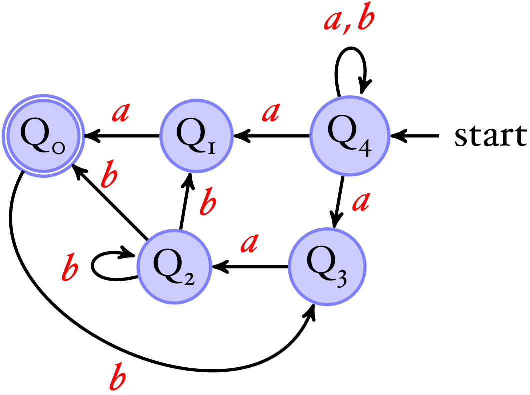


# Alternatives



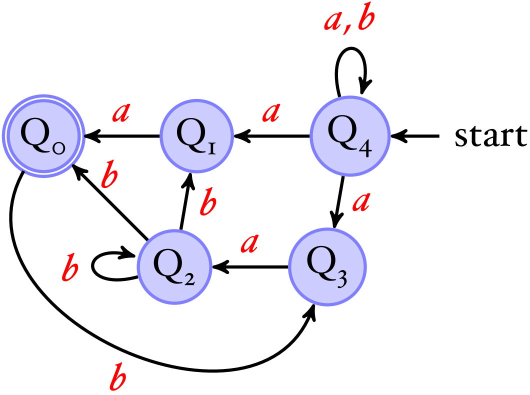
- exchange initial / accepting states

# Alternatives



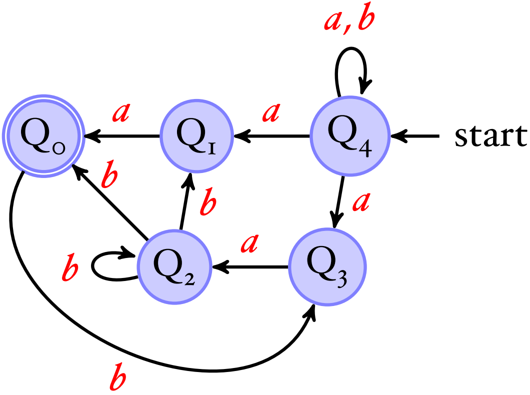
- exchange initial / accepting states
- reverse all edges

# Alternatives



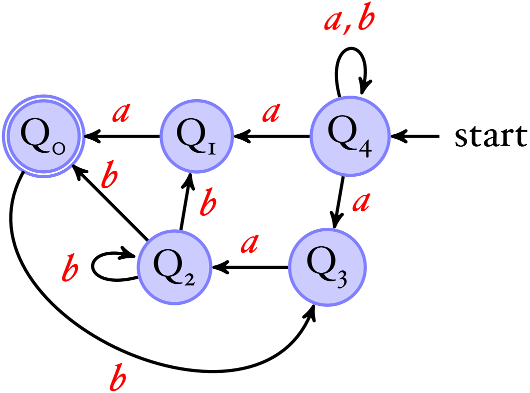
- exchange initial / accepting states
- reverse all edges
- subset construction ⇒ DFA

# Alternatives



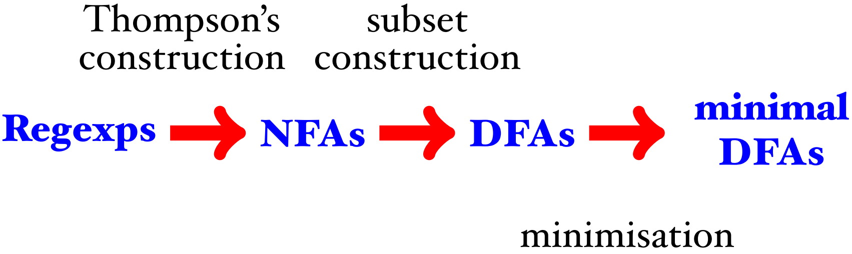
- exchange initial / accepting states
- reverse all edges
- subset construction ⇒ DFA
- remove dead states

# Alternatives



- exchange initial / accepting states
- reverse all edges
- subset construction ⇒ DFA
- remove dead states
- repeat once more ⇒ minimal DFA

# Regexps and Automata



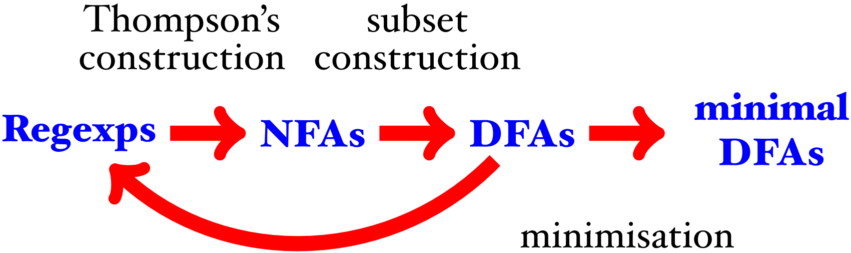
Thompson's construction

subset construction

minimisation

**Regexps** → **NFAs** → **DFAs** → **minimal DFAs**

# Regexps and Automata



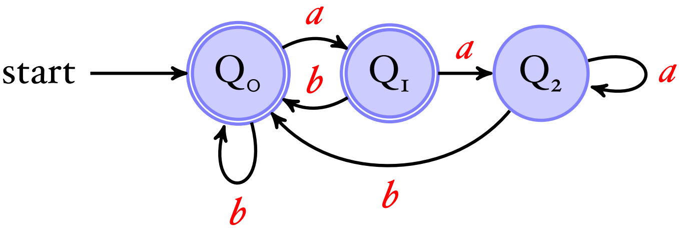
Thompson's construction

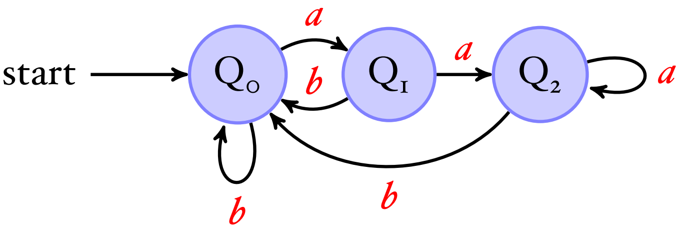
subset construction

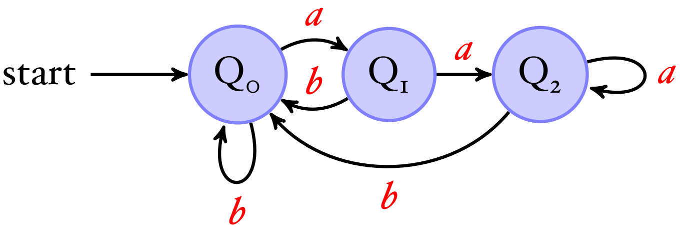
**Regexps** → **NFAs** → **DFAs** → **minimal DFAs**

minimisation

# DFA to Rexp

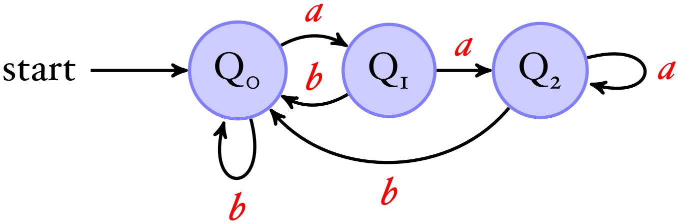


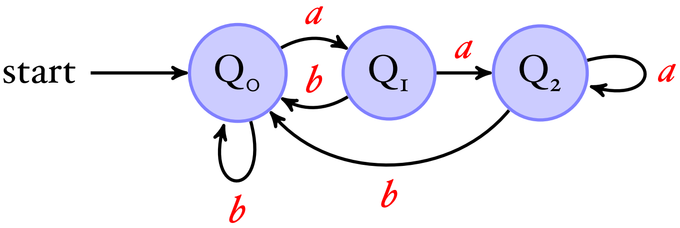

start
Q0
Q1
Q2
a
b
a
a
b
b



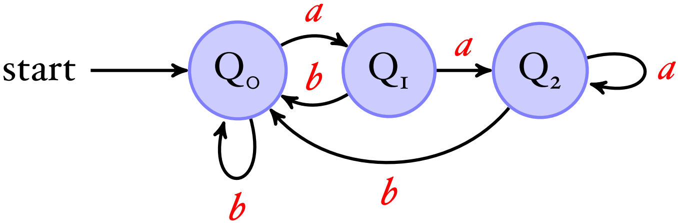
You know how to solve since school days, no?

$$
\begin{aligned}
Q_0 &= 2\,Q_0 + 3\,Q_1 + 4\,Q_2 \\
Q_1 &= 2\,Q_0 + 3\,Q_1 + 1\,Q_2 \\
Q_2 &= 1\,Q_0 + 5\,Q_1 + 2\,Q_2
\end{aligned}
$$


start
$Q_0$
$Q_1$
$Q_2$
$a$
$b$
$a$
$a$
$b$
$b$



$$
\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\,b + Q_1\,b + Q_2\,b \\
Q_1 &= Q_0\,a \\
Q_2 &= Q_1\,a + Q_2\,a
\end{aligned}
$$

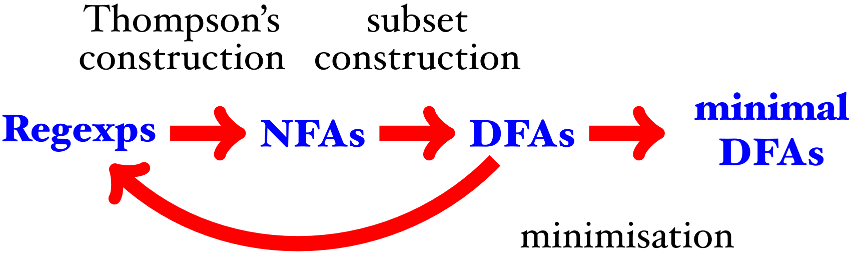


$$\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\,b + Q_1\,b + Q_2\,b \\
Q_1 &= Q_0\,a \\
Q_2 &= Q_1\,a + Q_2\,a
\end{aligned}$$

Arden's Lemma:

If $q = q\,r + s$ then $q = s\,r^*$

# Regexps and Automata



Thompson's construction

subset construction

**Regexps**

**NFAs**

**DFAs**

**minimal DFAs**

minimisation

# Regular Languages (3)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

# Regular Languages (3)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

Why is every finite set of strings a regular language?

Given the function

$$\begin{aligned}
rev(\mathbf{0}) &\overset{\text{def}}{=} \mathbf{0}\\
rev(\mathbf{1}) &\overset{\text{def}}{=} \mathbf{1}\\
rev(c) &\overset{\text{def}}{=} c\\
rev(r_1 + r_2) &\overset{\text{def}}{=} rev(r_1) + rev(r_2)\\
rev(r_1 \cdot r_2) &\overset{\text{def}}{=} rev(r_2) \cdot rev(r_1)\\
rev(r^*) &\overset{\text{def}}{=} rev(r)^*
\end{aligned}$$

and the set

$$Rev\,A \overset{\text{def}}{=} \{s^{-1} \mid s \in A\}$$

prove whether

$$L(rev(r)) = Rev(L(r))$$