Compilers and Formal Languages (2)

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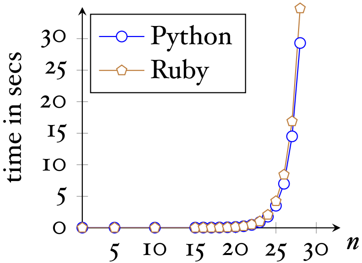
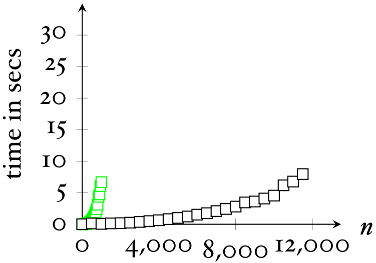
An Efficient Regular Expression Matcher

Graphs: $a^{?\{n\}} \cdot a^{\{n\}}$ and strings $\underbrace{a \ldots a}_{n}$





In the handouts is a similar graph with $(a^*)^* \cdot b$ for Java.

Languages

- A **Language** is a set of strings, for example

$$\{[], hello, foobar, a, abc\}$$

- **Concatenation** of strings and languages

$$foo @ bar \quad = \quad foobar$$
$$A @ B \quad \stackrel{\text{def}}{=} \quad \{s_1 @ s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

For example $A = \{foo, bar\}, B = \{a, b\}$

$$A @ B = \{fooa, foob, bara, barb\}$$

The Power Operation

- The **Power** of a language:

$$A^0 \stackrel{\text{def}}{=} \{[]\}$$
$$A^{n+1} \stackrel{\text{def}}{=} A \,@\, A^n$$

For example

$$A^4 = A \,@\, A \,@\, A \,@\, A$$
$$A^1 = A$$
$$A^0 = \{[]\}$$

Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in A^4?

Homework Question

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How many strings are in A^4?

What if $A = \{[a], [b], [c], []\}$;
how many strings are then in A^4?

The Star Operation

- The **Star** of a language:

$$A\star \stackrel{\text{def}}{=} \bigcup_{0 \leq n} A^n$$

This expands to

$$A^0 \cup A^1 \cup A^2 \cup A^3 \cup A^4 \cup \ldots$$

$$\{[]\} \;\cup\; A \;\cup\; A @ A \;\cup\; A @ A @ A \;\cup\; A @ A @ A @ A \cup \ldots$$

Semantic Derivative

- The **Semantic Derivative** of a language wrt to a character c:

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \{\} \end{aligned}$$

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We can extend this definition to strings

$$Ders\,s\,A = \{s' \mid s @ s' \in A\}$$

Regular Expressions

Their inductive definition:

r	$::=$	$\mathbf{0}$	null
	$\mid$	$\mathbf{1}$	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

Th

```scala
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

r	$::=$	$\mathbf{0}$	null
	$\mid$	$\mathbf{1}$	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

The Meaning of a Regular Expression

$$
\begin{array}{rcl}
L(\mathbf{0}) & \stackrel{\text{def}}{=} & \{\} \\
L(\mathbf{1}) & \stackrel{\text{def}}{=} & \{[]\} \\
L(c) & \stackrel{\text{def}}{=} & \{[c]\} \\
L(r_1 + r_2) & \stackrel{\text{def}}{=} & L(r_1) \cup L(r_2) \\
L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & L(r_1) \,@\, L(r_2) \\
L(r^*) & \stackrel{\text{def}}{=} & (L(r))\star
\end{array}
$$

L is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

What is $L(a^*)$?

When Are Two Regular Expressions Equivalent?

$$r_1 \equiv r_2 \quad \stackrel{\text{def}}{=} \quad L(r_1) = L(r_2)$$

Concrete Equivalences

$$
\begin{aligned}
(a+b)+c &\equiv a+(b+c)\\
a+a &\equiv a\\
a+b &\equiv b+a\\
(a\cdot b)\cdot c &\equiv a\cdot(b\cdot c)\\
c\cdot(a+b) &\equiv (c\cdot a)+(c\cdot b)
\end{aligned}
$$

Concrete Equivalences

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(a\cdot b)\cdot c & \equiv & a\cdot(b\cdot c) \\
c\cdot(a+b) & \equiv & (c\cdot a)+(c\cdot b)
\end{array}
$$

$$
\begin{array}{rcl}
a\cdot a & \not\equiv & a \\
a+(b\cdot c) & \not\equiv & (a+b)\cdot(a+c)
\end{array}
$$

Corner Cases

$$
\begin{array}{rcl}
a \cdot \mathbf{0} & \not\equiv & a \\
a + \mathbf{1} & \not\equiv & a \\
\mathbf{1} & \equiv & \mathbf{0}^* \\
\mathbf{1}^* & \equiv & \mathbf{1} \\
\mathbf{0}^* & \not\equiv & \mathbf{0}
\end{array}
$$

Simplification Rules

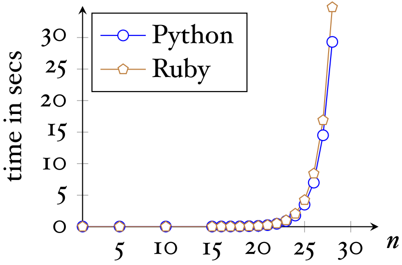
$$
\begin{aligned}
r + \mathbf{0} &\equiv r \\
\mathbf{0} + r &\equiv r \\
r \cdot \mathbf{1} &\equiv r \\
\mathbf{1} \cdot r &\equiv r \\
r \cdot \mathbf{0} &\equiv \mathbf{0} \\
\mathbf{0} \cdot r &\equiv \mathbf{0} \\
r + r &\equiv r
\end{aligned}
$$

The Specification for Matching

A regular expression r matches a string s if and only if

$$s \in L(r)$$

$(a^{?\{n\}}) \cdot a^{\{n\}}$ **and** $(a^*)^* \cdot b$



Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
  - $(a^{?\{n\}}) \cdot a^{\{n\}}$
  - $(a^*)^*$
  - $([a\text{-}z]^+)^*$
  - $(a + a \cdot a)^*$
  - $(a + a?)^*$

A Matching Algorithm

...whether a regular expression can match the empty string:

$$
\begin{array}{lcl}
nullable(\mathbf{0}) & \stackrel{\text{def}}{=} & false \\
nullable(\mathbf{1}) & \stackrel{\text{def}}{=} & true \\
nullable(c) & \stackrel{\text{def}}{=} & false \\
nullable(r_1 + r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \vee nullable(r_2) \\
nullable(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \wedge nullable(r_2) \\
nullable(r^*) & \stackrel{\text{def}}{=} & true
\end{array}
$$

The Derivative of a Rexp

If r matches the string $c :: s$, what is a regular expression that matches just s?

$der\,c\,r$ gives the answer, Brzozowski 1964

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\, c\, (\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0} \\
der\, c\, (\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0} \\
der\, c\, (d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0} \\
der\, c\, (r_1 + r_2) & \stackrel{\text{def}}{=} & der\, c\, r_1 + der\, c\, r_2 \\
der\, c\, (r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } (der\, c\, r_1) \cdot r_2 + der\, c\, r_2 \\
 & & \text{else } (der\, c\, r_1) \cdot r_2 \\
der\, c\, (r^*) & \stackrel{\text{def}}{=} & (der\, c\, r) \cdot (r^*)
\end{array}
$$

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if}\;c = d\;\text{then}\;\mathbf{1}\;\text{else}\;\mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if}\;nullable(r_1)\\
 & & \text{then}\;(der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else}\;(der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)\\
\\
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}
$$

Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$
\begin{aligned}
der\,a\,r &= ?\\
der\,b\,r &= ?\\
der\,c\,r &= ?
\end{aligned}
$$

The Algorithm

$$matches\ r\ s \stackrel{\text{def}}{=} nullable(ders\ r\ s)$$

An Example

Does r_1 match *abc*?

Step 1:	build derivative of a and r_1	$(r_2 = der\,a\,r_1)$
Step 2:	build derivative of b and r_2	$(r_3 = der\,b\,r_2)$
Step 3:	build derivative of c and r_3	$(r_4 = der\,c\,r_3)$
Step 4:	the string is exhausted: test whether r_4 can recognise the empty string	$(nullable(r_4))$
Output:	result of the test $\Rightarrow$ *true* or *false*	

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\, a\, (L(r_1))$

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2. $Der\, b\, (Der\, a\, (L(r_1)))$

The Idea of the Algorithm

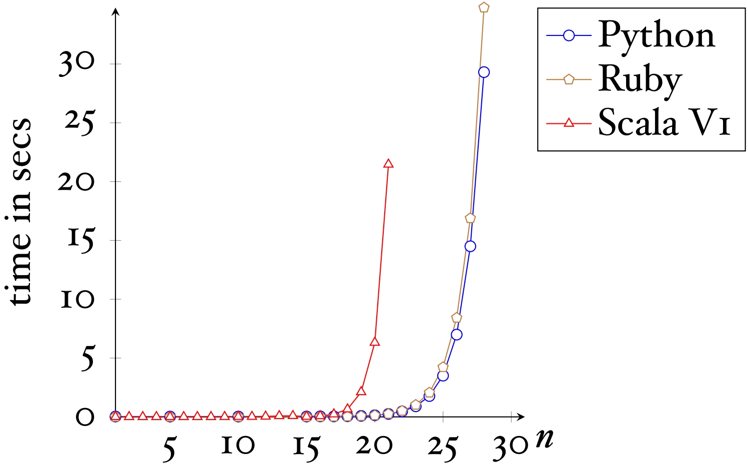
If we want to recognise the string abc with regular expression r_1 then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$
3. $Der\,c\,(Der\,b\,(Der\,a\,(L(r_1))))$

4. finally we test whether the empty string is in this set; same for $Ders\,abc\,(L(r_1))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

Oops... $(a^{?\{n\}}) \cdot a^{\{n\}}$



A Problem

We represented the "n-times" $a^{\{n\}}$ as a sequence regular expression:

$$
\begin{array}{rl}
1\text{:} & a \\
2\text{:} & a \cdot a \\
3\text{:} & a \cdot a \cdot a \\
 & \ldots \\
13\text{:} & a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \\
 & \ldots \\
20\text{:} &
\end{array}
$$

This problem is aggravated with $a^?$ being represented as $a + \mathbf{1}$.

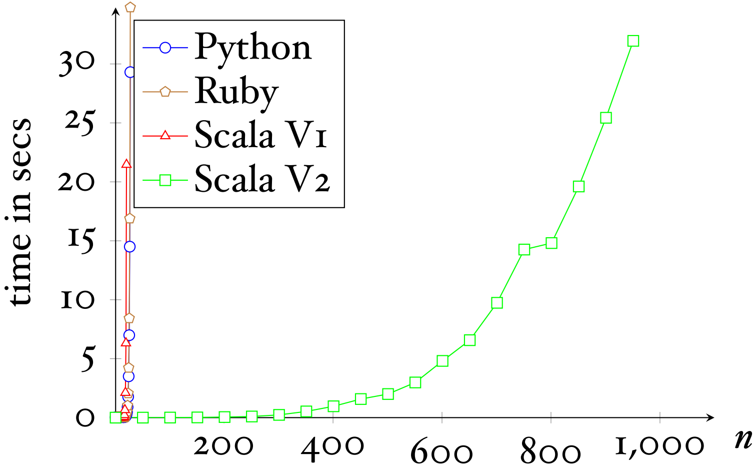
Solving the Problem

What happens if we extend our regular expressions

$$\begin{array}{rcl} r & ::= & \ldots \\ & | & r^{\{n\}} \\ & | & r^{?} \end{array}$$

What is their meaning?
What are the cases for *nullable* and *der*?

$$(a^{?\{n\}}) \cdot a^{\{n\}}$$



Python
Ruby
Scala V1
Scala V2
time in secs
0
5
10
15
20
25
30
200
400
600
800
1,000
n

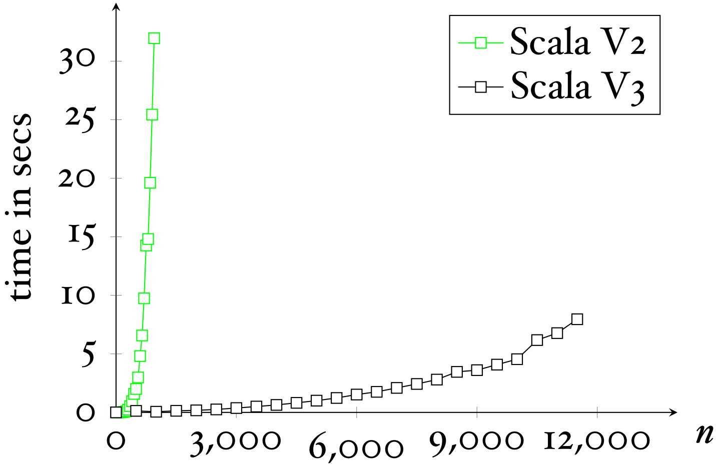
Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$
\begin{aligned}
der\, a\, r &= ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r \\
der\, b\, r &= ((\mathbf{0} \cdot b) + \mathbf{1}) \cdot r \\
der\, c\, r &= ((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r
\end{aligned}
$$

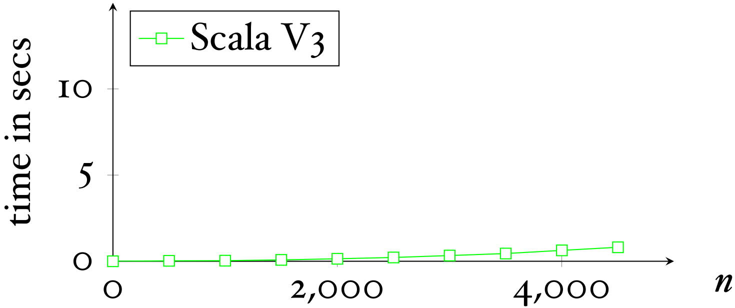
What are these regular expressions equivalent to?

$$(a^{?\{n\}}) \cdot a^{\{n\}}$$



Scala V2
Scala V3
time in secs
0
5
10
15
20
25
30
0
3,000
6,000
9,000
12,000
n

$$(a^*)^* \cdot b$$



What is good about this Alg.

- extends to most regular expressions, for example $\sim r$
- is easy to implement in a functional language
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is brand new work)
- we can prove its correctness...

Proofs about Rexps

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
  & \mid & \mathbf{1} \\
  & \mid & c \\
  & \mid & r_1 \cdot r_2 \\
  & \mid & r_1 + r_2 \\
  & \mid & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{c}$
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$nullable(r) \text{ if and only if } [] \in L(r)$$

Proofs about Rexp (4)

$$\begin{aligned}
rev(\mathbf{0}) &\stackrel{\text{def}}{=} \mathbf{0}\\
rev(\mathbf{1}) &\stackrel{\text{def}}{=} \mathbf{1}\\
rev(c) &\stackrel{\text{def}}{=} c\\
rev(r_1 + r_2) &\stackrel{\text{def}}{=} rev(r_1) + rev(r_2)\\
rev(r_1 \cdot r_2) &\stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1)\\
rev(r^*) &\stackrel{\text{def}}{=} rev(r)^*
\end{aligned}$$

We can prove

$$L(rev(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on r.

Correctness Proof for our Matcher

- We started from

$$\begin{array}{rl} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\, s\, (L(r)) \end{array}$$

Correctness Proof for our Matcher

- We started from

$$\begin{aligned} & s \in L(r) \\ \Leftrightarrow \quad & [] \in Ders\, s\, (L(r)) \end{aligned}$$

- if we can show $Ders\, s\, (L(r)) = L(ders\, s\, r)$ we have

$$\begin{aligned} \Leftrightarrow \quad & [] \in L(ders\, s\, r) \\ \Leftrightarrow \quad & nullable(ders\, s\, r) \\ \stackrel{\text{def}}{=} \quad & matches\, s\, r \end{aligned}$$

Proofs about Rexp (5)

Let $Der\,c\,A$ be the set defined as

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

We can prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

by induction on r.

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

We can then prove

$$Ders\,s\,(L(r)) = L(ders\,s\,r)$$

We can finally prove

$matches\,s\,r$ if and only if $s \in L(r)$