

Compilers and Formal Languages

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Slides: KEATS (also home work is there)

There are more problems, than there are programs.

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There must be a problem for which there is no program.

Subsets

If $A \subseteq B$ then A has fewer elements than B

$A \subseteq B$ and $B \subseteq A$

then $A = B$

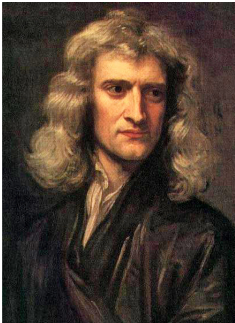


5 elements

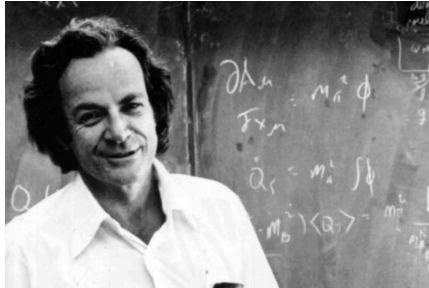


3 elements

Newton vs Feynman



classical physics



quantum physics

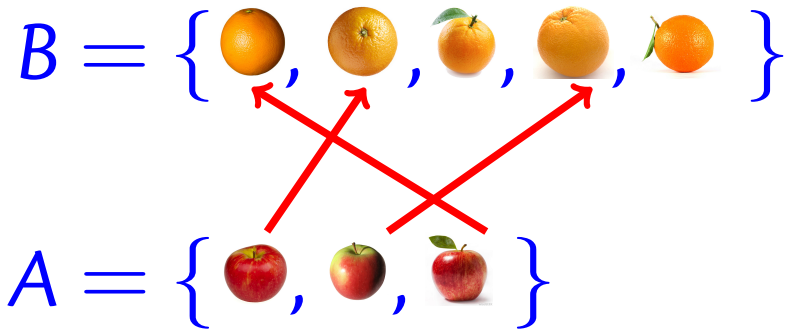
The Goal of the Talk

- show you that something very unintuitive happens with very large sets
- convince you that there are more **problems** than **programs**

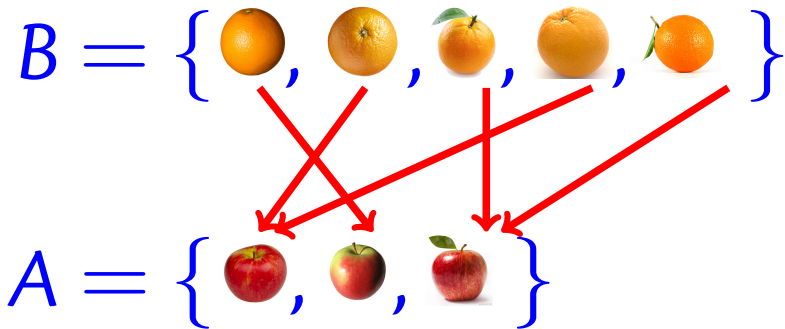
$$B = \{ \text{orange}, \text{orange}, \text{orange}, \text{orange}, \text{orange} \}$$

$$A = \{ \text{apple}, \text{apple}, \text{apple} \}$$

$$|A| = 5, |B| = 3$$



then $|A| \leq |B|$



for $=$ has to be a **one-to-one** mapping

Cardinality

$|A| \stackrel{\text{def}}{=} \text{“how many elements”}$

$$A \subseteq B \Rightarrow |A| \leq |B|$$

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if there is an injective function

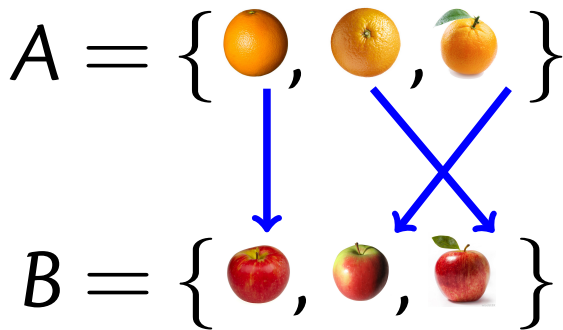
$$f: A \rightarrow B \text{ then } |A| \leq |B|$$

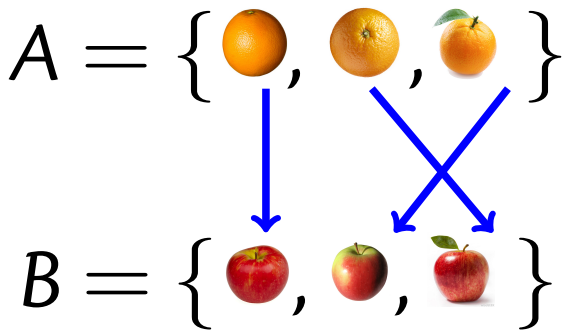
$$\forall xy. f(x) = f(y) \Rightarrow x = y$$

$$A = \{ \text{orange}, \text{orange}, \text{orange} \}$$

$$B = \{ \text{apple}, \text{apple}, \text{apple} \}$$







then $|A| = |B|$

Natural Numbers

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \dots\dots\}$$

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$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \dots\}$$

A is **countable** iff $|A| \leq |\mathbb{N}|$

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

\geq or \leq or $=$?

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$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

\geq or \leq or $=$?

$$x \mapsto x + 1,$$

$$|\mathbb{N} - \{0\}| = |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=} \text{odd numbers} \quad \{1, 3, 5, \dots\}$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=} \text{odd numbers} \quad \{1, 3, 5, \dots\}$

$\mathbb{E} \stackrel{\text{def}}{=} \text{even numbers} \quad \{0, 2, 4, \dots\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{N} \stackrel{\text{def}}{=} \text{positive numbers} \quad \{0, 1, 2, 3, \dots\}$

$-\mathbb{N} \stackrel{\text{def}}{=} \text{negative numbers} \quad \{0, -1, -2, -3, \dots\}$

A is **countable** if there exists an injective
 $f: A \rightarrow \mathbb{N}$

A is **uncountable** if there does not exist
an injective $f: A \rightarrow \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$

uncountable: $|A| > |\mathbb{N}|$

A is **countable** if there exists an injective
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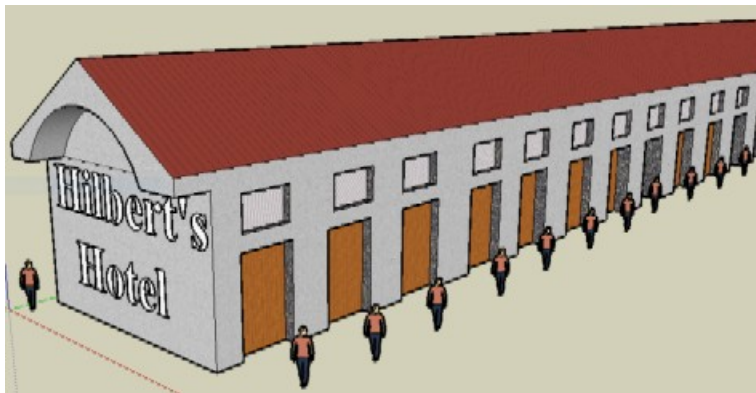
A is **uncountable** if there does not exist
an injective $f: A \rightarrow \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$

uncountable: $|A| > |\mathbb{N}|$

Does there exist such an A ?

Hilbert's Hotel



- ...has as many rooms as there are natural numbers

Real Numbers between 0 and 1

1	3	3	3	3	3	3
2	1	2	3	4	5	6	7	
3	0	1	0	1	0	...		
4	7	8	5	3	9	...		
								...

Real Numbers between 0 and 1

1	4	3	3	3	3	3
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...

$$|\mathbb{N}| < |\mathbb{R}|$$

The Set of Problems

 \mathbb{Z}_0

	0	1	2	3	4	5	...
1	0	1	0	1	0	1	...
2	0	0	0	1	1	0	0
3	0	0	0	0	0	...	
4	1	1	0	1	1	...	
...							

The Set of Problems

 \mathbb{Z}_0

	0	1	2	3	4	5	...
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3	0	0	0	0	0	...	
4	1	1	0	1	1	...	
...							

$$|\text{Progs}| = |\mathbb{N}| < |\text{Probs}|$$

Halting Problem

Assume a program H that decides for all programs A and all input data D whether

- $H(A, D) \stackrel{\text{def}}{=} 1$ iff $A(D)$ terminates
- $H(A, D) \stackrel{\text{def}}{=} 0$ otherwise

Halting Problem (2)

Given such a program H define the following program C : for all programs A

- $C(A) \stackrel{\text{def}}{=} 0$ iff $H(A, A) = 0$
- $C(A) \stackrel{\text{def}}{=} \text{loops}$ otherwise

Contradiction

$H(C, C)$ is either 0 or 1.

- $H(C, C) = 1 \xRightarrow{\text{def } H} C(C) \downarrow \xRightarrow{\text{def } C} H(C, C) = 0$

- $H(C, C) = 0 \xRightarrow{\text{def } H} C(C) \text{ loops} \xRightarrow{\text{def } C}$

$$H(C, C) = 1$$

Contradiction in both cases. So H cannot exist.

Take Home Points

- there are sets that are more infinite than others
- even with the most powerful computer we can imagine, there are problems that cannot be solved by any program
- in CS we actually hit quite often such problems (halting problem)