Compilers and
Formal Languages (10)

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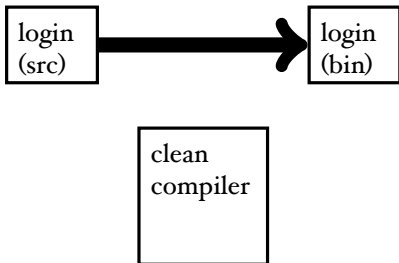Office: N7.07 (North Wing, Bush House)
Slides: KEATS (also home work is there)
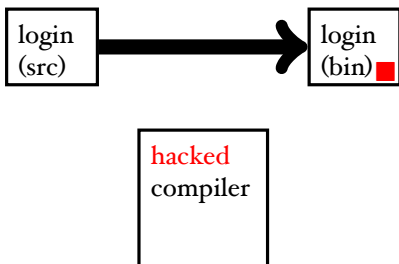
**Using a compiler,
how can you mount the
perfect attack against a system?**

What is a **perfect** attack?

1. you can potentially completely take over a target system
2. your attack is (nearly) undetectable
3. the victim has (almost) no chance to recover

```
clean
compiler
```

my compiler (src)

V0.01

Scala

host language

my compiler (src)

V0.01   V0.02   ...   V1.00

Scala   Scala   Scala

host language

V1.01   V1.02   ...

my compiler (src)

V0.01   V0.02   ...   V1.00   V1.00   V1.01   ...
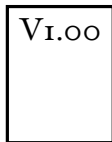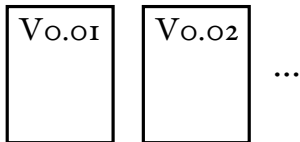
Scala   Scala   Scala
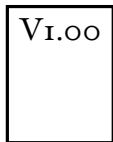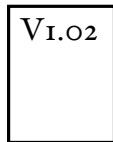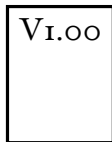
host language   no host language needed

Hacking Compilers

Ken Thompson
Turing Award, 1983

Ken Thompson showed how to hide a Trojan Horse in a compiler without leaving any traces in the source code.

No amount of source level verification will protect you from such Thompson-hacks.

Hacking Compilers

Ken Thompson
Turing Award, 198

1) *Assume you ship the compiler as binary and also with sources.*
2) *Make the compiler aware when it compiles itself.*
3) *Add the Trojan horse.*
4) *Compile.*
5) *Delete Trojan horse from the sources of the compiler.*
6) *Go on holiday for the rest of your life. ;o)*

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Compilers & Boeings 777

First flight in 1994. They want to achieve triple redundancy in hardware faults.

They compile 1 Ada program to

- Intel 80486
- Motorola 68040 (old Macintosh's)
- AMD 29050 (RISC chips used often in laser printers)

using 3 independent compilers.

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Airbus uses C and static analysers. Recently started using CompCert.

How many strings are in $L(a^*)$?

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$$[]\quad a\quad aa\quad aaa\quad aaaa\quad ...$$
$$0\quad 1\quad 2\quad 3\quad 4\quad\quad ...$$

There are more problems, than there are programs.

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than there are programs.

There must be a problem for
which there is no program.

Subsets

If $A \subseteq B$ then A has fewer or equal elements than B

$A \subseteq B$ and $B \subseteq A$

then $A = B$
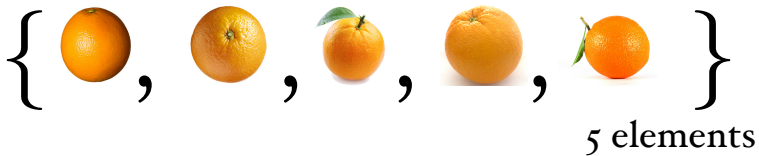
5 elements

3 elements

Newton vs Feynman
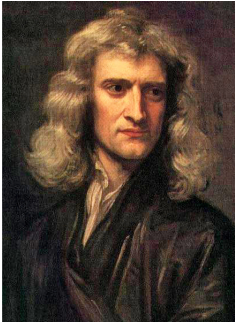
classical physics
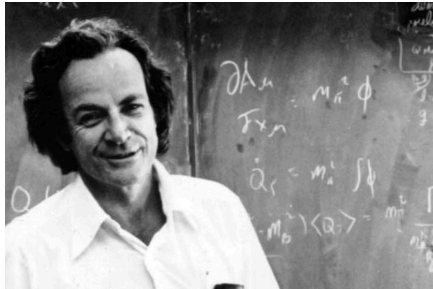
quantum physics

The Goal of the Talk

- show you that something very unintuitive happens with very large sets

- convince you that there are more **problems** than **programs**

$$B = \{ \text{🍊}, \text{🍊}, \text{🍊}, \text{🍊}, \text{🍊} \}$$

$$A = \{ \text{🍎}, \text{🍎}, \text{🍎} \}$$

$$|A| = 5, |B| = 3$$

$B = \{$ $,$ $,$ $,$ $,$ $\}$

$A = \{$ $,$ $,$ $\}$
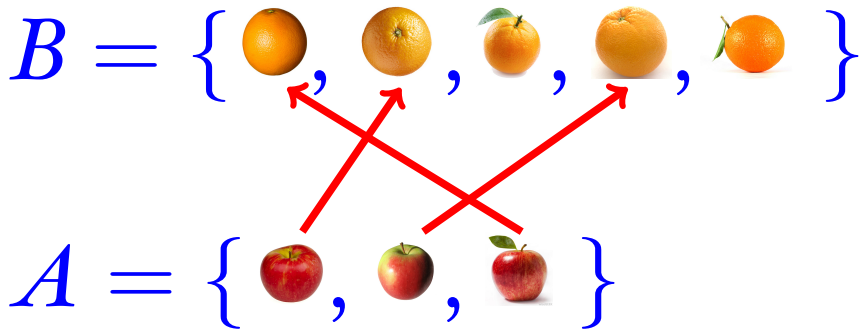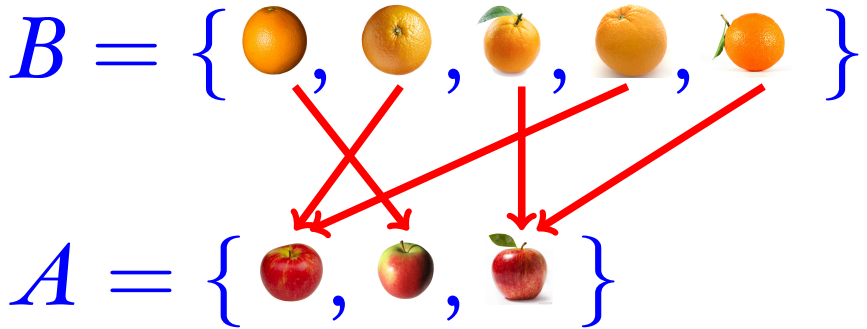
then $|A| \leq |B|$

for $=$ has to be a **one-to-one** mapping

Cardinality

$|A| \stackrel{\text{def}}{=}$ "how many elements"

$A \subseteq B \Rightarrow |A| \leq |B|$

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$A \subseteq B \Rightarrow |A| \leq |B|$

if there is an injective function
$f : A \to B$ then $|A| \leq |B|$

$$\forall xy.\, f(x) = f(y) \Rightarrow x = y$$
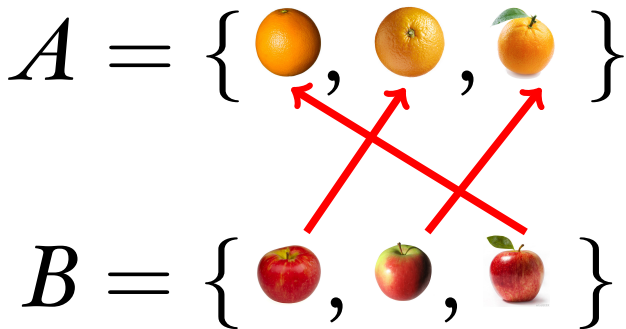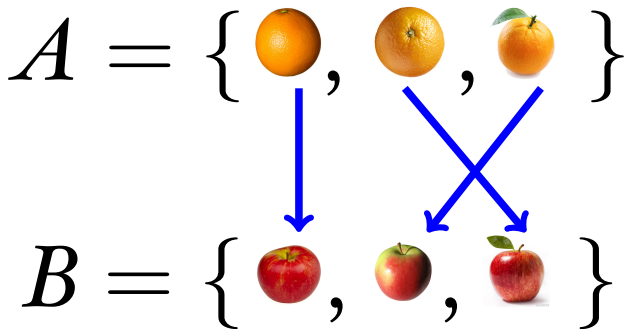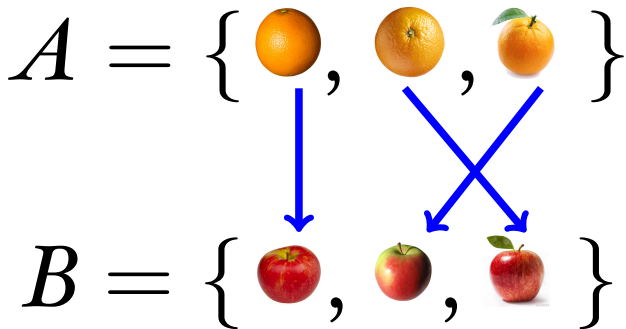
then $|A| = |B|$

Natural Numbers

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\ldots\}$$

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A is **countable** iff $|A| \leq |\mathbb{N}|$

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

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$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

$x \mapsto x + 1,$

$$|\mathbb{N} - \{0\}| = |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\quad \{1, 3, 5 \ldots\ldots\}$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\quad \{1, 3, 5......\}$

$\mathbb{E} \stackrel{\text{def}}{=}$ even numbers $\quad \{0, 2, 4......\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{N} \overset{\text{def}}{=}$ positive numbers $\quad \{0, 1, 2, 3, \ldots\ldots\}$

$-\mathbb{N} \overset{\text{def}}{=}$ negative numbers $\quad \{0, -1, -2, -3, \ldots\ldots\}$

A is countable if there exists an injective $f : A \rightarrow \mathbb{N}$

A is uncountable if there does not exist an injective $f : A \rightarrow \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$
uncountable: $|A| > |\mathbb{N}|$

A is countable if there exists an injective $f : A \to \mathbb{N}$

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Does there exist such an A ?

Hilbert's Hotel
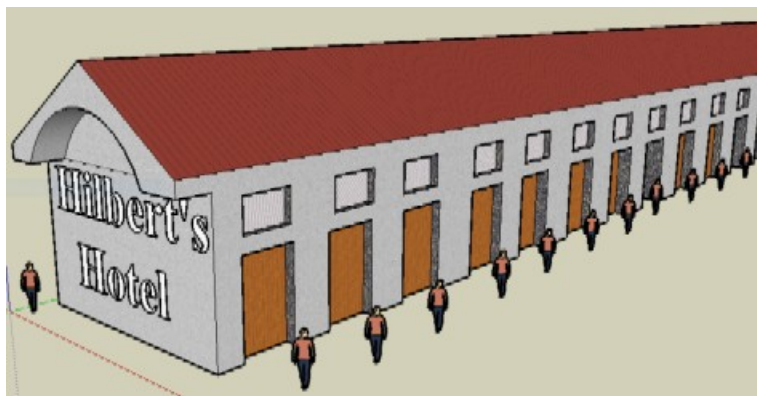
- ...has as many rooms as there are natural numbers

Real Numbers between 0 and 1

1	3	3	3	3	3	3	$\cdots$	$\cdots$
2	1	2	3	4	5	6	7	
3	0	1	0	1	0	$\cdots$		
4	7	8	5	3	9	$\cdots$		

$\cdots$

Real Numbers between 0 and 1

1	4	3	3	3	3	3	$\cdots$	$\cdots$
2	1	2	3	4	5	6	7	
3	0	1	0	1	0	$\cdots$		
4	7	8	5	3	9	$\cdots$		

$\cdots$

Real Numbers between 0 and 1

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. . .

Real Numbers between 0 and 1

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$\cdots$

Real Numbers between 0 and 1

1	4	3	3	3	3	3	...	...
2	1	3	3	4	5	6	7	
3	0	1	1	1	0	...		
4	7	8	5	4	9	...		

...

Real Numbers between 0 and 1

1	4	3	3	3	3	3	$\cdots$	$\cdots$
2	1	3	3	4	5	6	7	
3	0	1	1	1	0	$\cdots$		
4	7	8	5	4	9	$\cdots$		

$\cdots$

$$|\mathbb{N}| < |R|$$

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	$\cdots$
1	0	1	0	1	0	1	$\cdots$
2	0	0	0	1	1	0	0
3	0	0	0	0	0	$\cdots$	
4	1	1	0	1	1	$\cdots$	

$\cdots$

$\cdots$

The Set of Problems

$\aleph_0$

	O	I	2	3	4	5	$\cdots$
I	O	I	O	I	O	I	$\cdots$
2	O	O	O	I	I	O	O
3	O	O	O	O	O	$\cdots$	
4	I	I	O	I	I	$\cdots$	

$\cdots$

$|\text{Progs}| = |\mathbb{N}| < |\text{Probs}|$

Halting Problem

Assume a program H that decides for all programs A and all input data D whether

- $H(A,D) \stackrel{\text{def}}{=} 1$ iff $A(D)$ terminates
- $H(A,D) \stackrel{\text{def}}{=} 0$ otherwise

Halting Problem (2)

Given such a program H define the
following program C: for all programs A

- $C(A) \stackrel{\text{def}}{=} \circ$ iff $H(A, A) = \circ$
- $C(A) \stackrel{\text{def}}{=}$ loops otherwise

Contradiction

$H(C, C)$ is either o or 1.

- $H(C, C) = 1 \overset{\text{def}H}{\Rightarrow} C(C) \downarrow \overset{\text{def}C}{\Rightarrow} H(C, C) = o$

- $H(C, C) = o \overset{\text{def}H}{\Rightarrow} C(C) \text{ loops } \overset{\text{def}C}{\Rightarrow}$
$$H(C, C) = 1$$

Contradiction in both cases. So H cannot exist.

Take Home Points

- there are sets that are more infinite than others

- even with the most powerful computer we can imagine, there are problems that cannot be solved by any program

- in CS we actually hit quite often such problems (halting problem)