Compilers and Formal Languages

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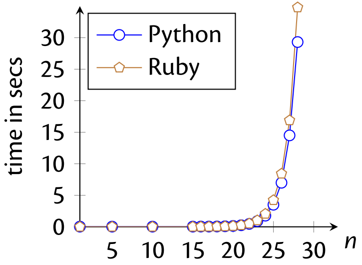
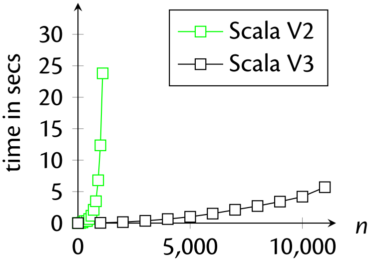
Slides & Progs: KEATS (also homework is there)

Let’s Implement an Efficient Regular Expression Matcher

Graphs: $a^{?\{n\}} \cdot a^{\{n\}}$ and strings $\underbrace{a\ldots a}_{n}$





In the handouts is a similar graph for $(a^*)^* \cdot b$ and Java 8, JavaScript and Python.

(Basic) Regular Expressions

Their inductive definition:

$$
\begin{array}{rcll}
r & ::= & \mathbf{0} & \text{nothing}\\
  & \mid & \mathbf{1} & \text{empty string / "" / []}\\
  & \mid & c & \text{character}\\
  & \mid & r_1 + r_2 & \text{alternative / choice}\\
  & \mid & r_1 \cdot r_2 & \text{sequence}\\
  & \mid & r^* & \text{star (zero or more)}
\end{array}
$$

When Are Two Regular Expressions Equivalent?

Two regular expressions r_1 and r_2 are **equivalent** provided:

$$r_1 \equiv r_2 \quad \stackrel{\text{def}}{=} \quad L(r_1) = L(r_2)$$

Some Concrete Equivalences

$$
\begin{array}{rcl}
(a + b) + c & \equiv & a + (b + c) \\
a + a & \equiv & a \\
a + b & \equiv & b + a \\
(a \cdot b) \cdot c & \equiv & a \cdot (b \cdot c) \\
c \cdot (a + b) & \equiv & (c \cdot a) + (c \cdot b)
\end{array}
$$

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c \cdot (a + b) & \equiv & (c \cdot a) + (c \cdot b)
\end{array}
$$

$$
\begin{array}{rcl}
a \cdot a & \not\equiv & a \\
a + (b \cdot c) & \not\equiv & (a + b) \cdot (a + c)
\end{array}
$$

Some Corner Cases

$$\begin{array}{rcl}
a \cdot \mathbf{0} & \not\equiv & a \\
a + \mathbf{1} & \not\equiv & a \\
\mathbf{1} & \equiv & \mathbf{0}^* \\
\mathbf{1}^* & \equiv & \mathbf{1} \\
\mathbf{0}^* & \not\equiv & \mathbf{0}
\end{array}$$

Some Simplification Rules

$$
\begin{array}{rcl}
r + \mathbf{0} & \equiv & r \\
\mathbf{0} + r & \equiv & r \\
r \cdot \mathbf{1} & \equiv & r \\
\mathbf{1} \cdot r & \equiv & r \\
r \cdot \mathbf{0} & \equiv & \mathbf{0} \\
\mathbf{0} \cdot r & \equiv & \mathbf{0} \\
r + r & \equiv & r
\end{array}
$$

Simplification Example

$$
\begin{array}{rcl}
((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r & \Rightarrow & ((\underline{\mathbf{1} \cdot b}) + \mathbf{0}) \cdot r \\
& = & (\underline{b + \mathbf{0}}) \cdot r \\
& = & b \cdot r
\end{array}
$$

Simplification Example

$$
\begin{aligned}
((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r \quad \Rightarrow & \quad ((\underline{\mathbf{0} \cdot b}) + \mathbf{0}) \cdot r \\
= & \quad (\underline{\mathbf{0} + \mathbf{0}}) \cdot r \\
= & \quad \mathbf{0} \cdot r \\
= & \quad \mathbf{0}
\end{aligned}
$$

Semantic Derivative

- The **Semantic Derivative** of a language w.r.t. to a character c:

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c :: s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \{\} \end{aligned}$$

Semantic Derivative

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$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \{\} \end{aligned}$$

We can extend this definition to strings

$$Ders\,s\,A = \{s' \mid s\,@\,s' \in A\}$$

The Specification for Matching

A regular expression r matches a string s provided:

$$s \in L(r)$$

...and the point of the this lecture is to decide this problem as fast as possible (unlike Python, Ruby, Java etc)

Brzozowski's Algorithm (1)

…whether a regular expression can match the empty string:

$$
\begin{aligned}
nullable(\mathbf{0}) &\stackrel{\text{def}}{=} false\\
nullable(\mathbf{1}) &\stackrel{\text{def}}{=} true\\
nullable(c) &\stackrel{\text{def}}{=} false\\
nullable(r_1 + r_2) &\stackrel{\text{def}}{=} nullable(r_1) \vee nullable(r_2)\\
nullable(r_1 \cdot r_2) &\stackrel{\text{def}}{=} nullable(r_1) \wedge nullable(r_2)\\
nullable(r^*) &\stackrel{\text{def}}{=} true
\end{aligned}
$$

The Derivative of a Rexp

If r matches the string $c :: s$, what is a regular expression that matches just s?

$der\, c\, r$ gives the answer, Brzozowski 1964

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0} \\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0} \\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0} \\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2 \\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2 \\
 & & \text{else } (der\,c\,r_1) \cdot r_2 \\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}
$$

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else } (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}
$$

$$
\begin{array}{lcl}
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}
$$

Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$
\begin{aligned}
der\, a\, r &= ?\\
der\, b\, r &= ?\\
der\, c\, r &= ?
\end{aligned}
$$

Derivative Example

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$
\begin{array}{rcl}
der\,a\,((a \cdot b) + b)^* & \Rightarrow & der\,a\,\underline{((a \cdot b) + b)^*} \\
 & = & (der\,a\,(\underline{(a \cdot b) + b})) \cdot r \\
 & = & ((der\,a\,(\underline{a \cdot b})) + (der\,a\,b)) \cdot r \\
 & = & (((der\,a\,\underline{a}) \cdot b) + (der\,a\,b)) \cdot r \\
 & = & ((\mathbf{1} \cdot b) + (der\,a\,\underline{b})) \cdot r \\
 & = & ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r
\end{array}
$$

The Brzozowski Algorithm

$$matcher\, r\, s \stackrel{\text{def}}{=} nullable(ders\ s\ r)$$

Brzozowski: An Example

Does r_1 match abc?

Step 1:	build derivative of a and r_1	$(r_2 = der\, a\, r_1)$
Step 2:	build derivative of b and r_2	$(r_3 = der\, b\, r_2)$
Step 3:	build derivative of c and r_3	$(r_4 = der\, c\, r_3)$
Step 4:	the string is exhausted: test whether r_4 can recognise the empty string	$(nullable(r_4))$
Output:	result of the test $\Rightarrow$ *true* or *false*	

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\, a\, (L(r_1))$

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$
3. $Der\,c\,(Der\,b\,(Der\,a\,(L(r_1))))$

4. finally we test whether the empty string is in this set; same for $Ders\,abc\,(L(r_1))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

The Idea with Derivatives

Input: string abc and regular expression r

1. $der\, a\, r$
2. $der\, b\, (der\, a\, r)$
3. $der\, c\, (der\, b\, (der\, a\, r))$

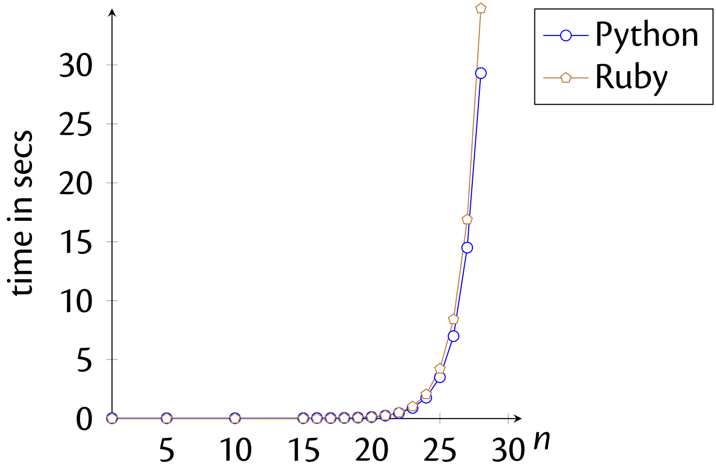
The Idea with Derivatives

Input: string abc and regular expression r

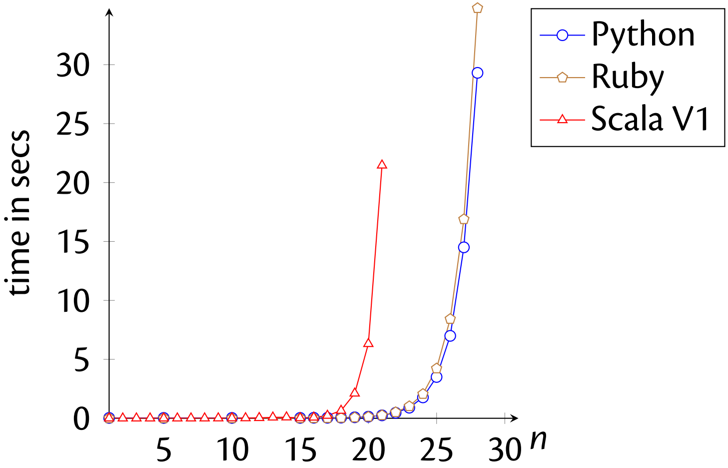
1. $der\, a\, r$
2. $der\, b\, (der\, a\, r)$
3. $der\, c\, (der\, b\, (der\, a\, r))$

4. finally check whether the last regular expression can match the empty string

$$a^{?\{n\}} \cdot a^{\{n\}}$$



Oops... $a^{?\{n\}} \cdot a^{\{n\}}$



A Problem

We represented the "n-times" $a^{\{n\}}$ as a sequence regular expression:

$$
\begin{array}{rl}
0: & \mathbf{1} \\
1: & a \\
2: & a \cdot a \\
3: & a \cdot a \cdot a \\
 & \ldots \\
13: & a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \\
 & \ldots \\
20: &
\end{array}
$$

This problem is aggravated with $a^?$ being represented as $a + \mathbf{1}$.

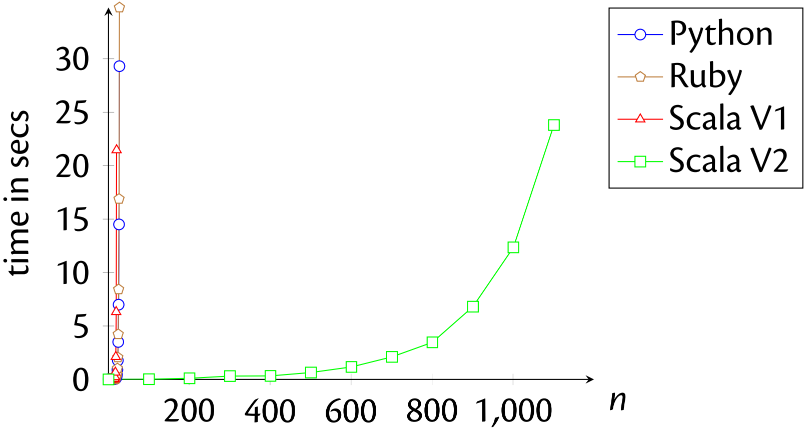
Solving the Problem

What happens if we extend our regular expressions with explicit constructors

$$
\begin{array}{rcl}
r & ::= & \ldots \\
  & | & r^{\{n\}} \\
  & | & r^{?}
\end{array}
$$

What is their meaning?
What are the cases for *nullable* and *der*?

Brzozowski: $a^{?\{n\}} \cdot a^{\{n\}}$



Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$
\begin{aligned}
der\, a\, r &= ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r \\
der\, b\, r &= ((\mathbf{0} \cdot b) + \mathbf{1}) \cdot r \\
der\, c\, r &= ((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r
\end{aligned}
$$

What are these regular expressions equivalent to?

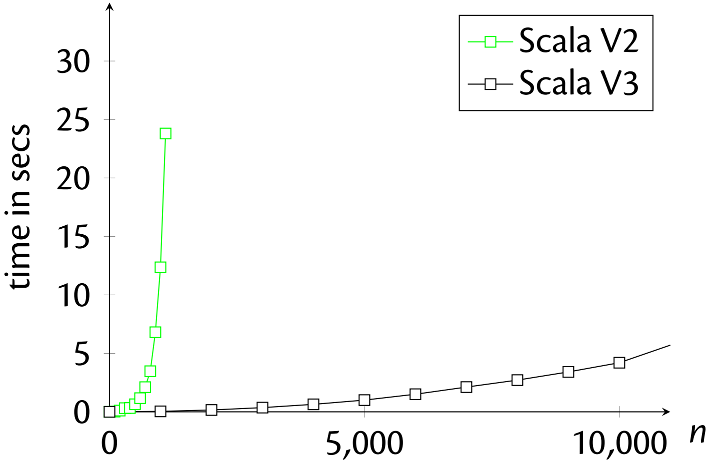
Simplification Rules

$$
\begin{array}{rcl}
r + \mathbf{0} & \Rightarrow & r \\
\mathbf{0} + r & \Rightarrow & r \\
r \cdot \mathbf{1} & \Rightarrow & r \\
\mathbf{1} \cdot r & \Rightarrow & r \\
r \cdot \mathbf{0} & \Rightarrow & \mathbf{0} \\
\mathbf{0} \cdot r & \Rightarrow & \mathbf{0} \\
r + r & \Rightarrow & r
\end{array}
$$

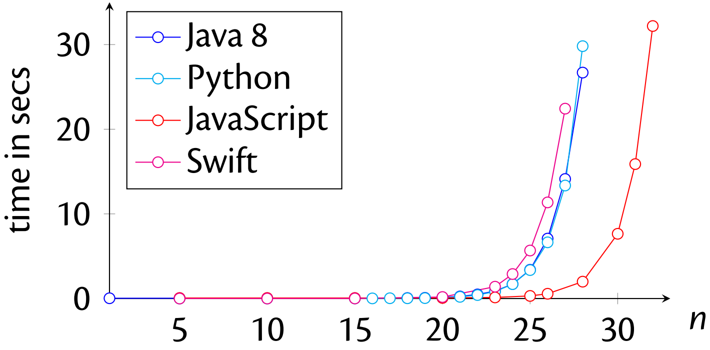
```
def ders(s: List[Char], r: Rexp) : Rexp = s match {
  case Nil => r
  case c::s => ders(s, simp(der(c, r)))
}
```

```scala
def simp(r: Rexp) : Rexp = r match {
  case ALT(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, r2s) => r2s
      case (r1s, ZERO) => r1s
      case (r1s, r2s) =>
        if (r1s == r2s) r1s else ALT(r1s, r2s)
    }
  }
  case SEQ(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, _) => ZERO
      case (_, ZERO) => ZERO
      case (ONE, r2s) => r2s
      case (r1s, ONE) => r1s
      case (r1s, r2s) => SEQ(r1s, r2s)
    }
  }
  case r => r
}
```

Brzozowski: $a^{?\{n\}} \cdot a^{\{n\}}$

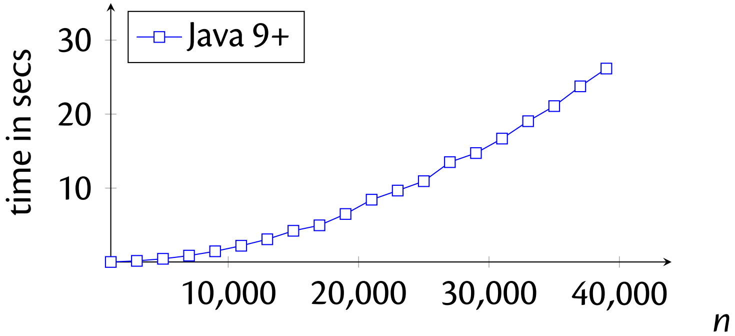


Another Example $(a^*)^* \cdot b$



Regex: $(a^*)^* \cdot b$
Strings of the form $\underbrace{a \ldots a}_{n}$

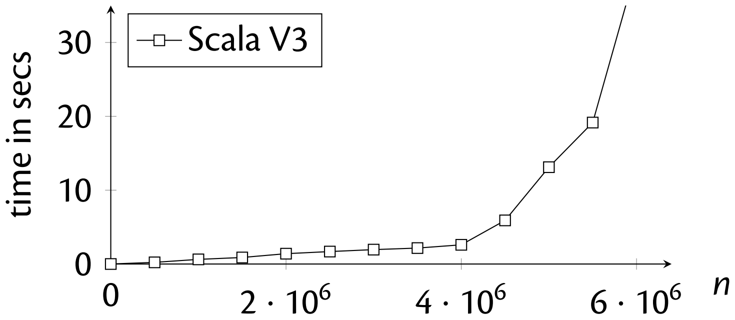
Same Example in Java 9+



Regex: $(a^*)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$

...and with Brzozowski



Regex: $(a^*)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$

What is good about this Alg.

- extends to most regular expressions, for example $\sim r$ (next slide)
- is easy to implement in a functional language (slide after)
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is relatively new work)
- we can prove its correctness...(several slides later)

Negation of Regular Expr's

- $\sim r$ (everything that r cannot recognise)
- $L(\sim r) \stackrel{\text{def}}{=} UNIV - L(r)$
- $nullable(\sim r) \stackrel{\text{def}}{=} not\ (nullable(r))$
- $der\, c\, (\sim r) \stackrel{\text{def}}{=} \sim (der\, c\, r)$

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- $der\, c\, (\sim r) \stackrel{\text{def}}{=} \sim (der\, c\, r)$

Used often for recognising comments:

$$/ \cdot * \cdot (\sim ([a\text{-}z]^* \cdot * \cdot / \cdot [a\text{-}z]^*)) \cdot * \cdot /$$

Coursework 1

- Submission on Friday 16 October @ 18:00
- source code needs to be submitted as well
- you can re-use my Scala code from KEATS
  and use any programming language you like
- https://nms.kcl.ac.uk/christian.urban/ProgInScala2ed.pdf

Proofs about Rexps

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
  & | & \mathbf{1} \\
  & | & c \\
  & | & r_1 \cdot r_2 \\
  & | & r_1 + r_2 \\
  & | & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\mathbf{0}$, $\mathbf{1}$ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$nullable(r) \text{ if and only if } [] \in L(r)$$

Proofs about Rexp (4)

$$
\begin{aligned}
rev(\mathbf{0}) &\stackrel{\text{def}}{=} \mathbf{0}\\
rev(\mathbf{1}) &\stackrel{\text{def}}{=} \mathbf{1}\\
rev(c) &\stackrel{\text{def}}{=} c\\
rev(r_1 + r_2) &\stackrel{\text{def}}{=} rev(r_1) + rev(r_2)\\
rev(r_1 \cdot r_2) &\stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1)\\
rev(r^*) &\stackrel{\text{def}}{=} rev(r)^*
\end{aligned}
$$

We can prove

$$L(rev(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on r.

Correctness Proof for our Matcher

- We started from

$$\begin{array}{rl} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\, s\, (L(r)) \end{array}$$

Correctness Proof for our Matcher

- We started from

$$\begin{array}{rl} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\, s\, (L(r)) \end{array}$$

- if we can show $Ders\, s\, (L(r)) = L(ders\, s\, r)$ we have

$$\begin{array}{rl} \Leftrightarrow & [] \in L(ders\, s\, r) \\ \Leftrightarrow & nullable(ders\, s\, r) \\ \stackrel{\text{def}}{=} & matcher\, s\, r \end{array}$$

Proofs about Rexp (5)

Let $Der\,c\,A$ be the set defined as

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c :: s \in A\}$$

We can prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

by induction on r.

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c :: s$ under the assumption that P already holds for s

Proofs about Strings (2)

We can then prove

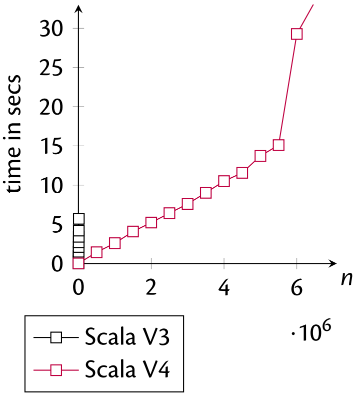
$$Ders\, s\, (L(r)) = L(ders\, s\, r)$$

We can finally prove

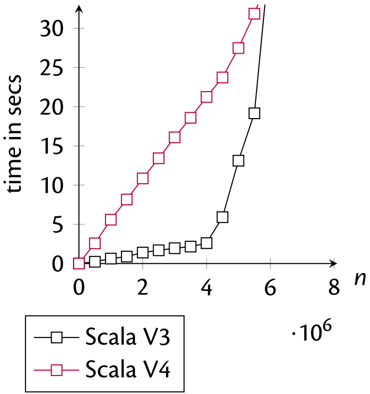
$$matcher\, s\, r \text{ if and only if } s \in L(r)$$

Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$



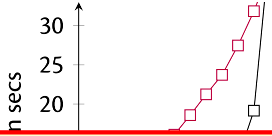
Graph: $(a^*)^* \cdot b$



Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$

Graph: $(a^*)^* \cdot b$




```scala
def ders2(s: List[Char], r: Rexp) : Rexp = (s, r) match {
  case (Nil, r) => r
  case (s, ZERO) => ZERO
  case (s, ONE) => if (s == Nil) ONE else ZERO
  case (s, CHAR(c)) => if (s == List(c)) ONE else
                       if (s == Nil) CHAR(c) else ZERO
  case (s, ALT(r1, r2)) => ALT(ders2(s, r2), ders2(s, r2))
  case (c::s, r) => ders2(s, simp(der(c, r)))
}
```

Another Homework Question

- How many basic regular expressions are there to match the string *abcd* ?

Another Homework Question

- How many basic regular expressions are there to match the string *abcd* ?
- How many if they cannot include $\mathbf{1}$ and $\mathbf{0}$?

Another Homework Question

- How many basic regular expressions are there to match the string *abcd* ?
- How many if they cannot include **1** and **0**?
- How many if they are also not allowed to contain stars?

Another Homework Question

- How many basic regular expressions are there to match the string *abcd* ?
- How many if they cannot include $\mathbf{1}$ and $\mathbf{0}$?
- How many if they are also not allowed to contain stars?
- How many if they are also not allowed to contain $\_ + \_$?

Questions?

Last Week

Last week I showed you a regular expression matcher that works provably correct in all cases (we only started with the proving part though)

$$\textit{matcher}\ s\ r \quad \text{if and only if} \quad s \in L(r)$$

by Janusz Brzozowski (1964)

Proofs about Rexp

- P holds for $\mathbf{0}$, $\mathbf{1}$ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

We proved

$$nullable(r) \;\text{ if and only if }\; [] \in L(r)$$

by induction on the regular expression r.

We proved

$$nullable(r) \;\; \text{if and only if} \;\; [] \in L(r)$$

by induction on the regular expression r.

Any Questions?

Proofs about Natural Numbers and Strings

- P holds for 0 and
- P holds for $n + 1$ under the assumption that P already holds for n

- P holds for $[]$ and
- P holds for $c :: s$ under the assumption that P already holds for s

Correctness Proof for our Matcher

- We started from

$$
\begin{array}{rl}
& s \in L(r) \\
\Leftrightarrow & [] \in Ders\, s\, (L(r))
\end{array}
$$

Correctness Proof for our Matcher

- We started from

$$\begin{array}{rl} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\, s\, (L(r)) \end{array}$$

- **if** we can show $Ders\, s\, (L(r)) = L(ders\, s\, r)$ we have

$$\begin{array}{rl} \Leftrightarrow & [] \in L(ders\, s\, r) \\ \Leftrightarrow & nullable(ders\, s\, r) \\ \stackrel{\text{def}}{=} & matcher\, s\, r \end{array}$$

We need to prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

also by induction on the regular expression r.