Automata and Formal Languages (4)

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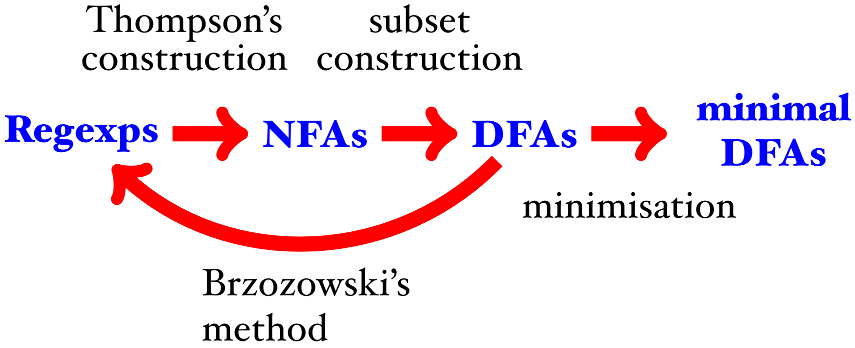
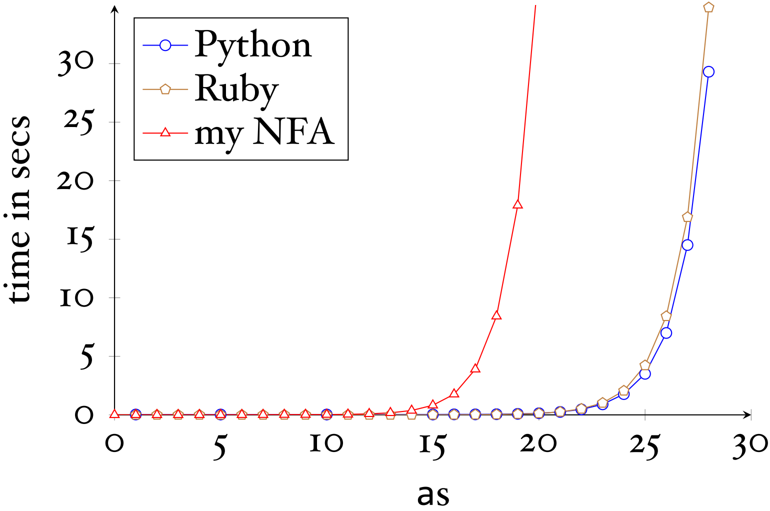
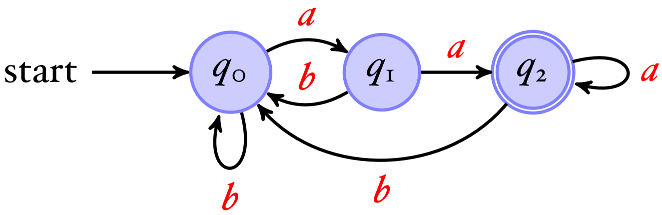
Regexps and Automata



$a^{?\{n\}} \cdot a^{\{n\}}$



DFA to Rexp



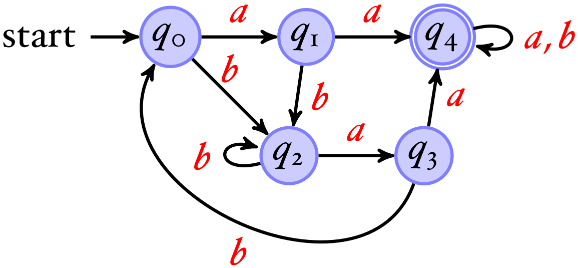
$$
\begin{aligned}
q_0 &= \epsilon + q_0\,b + q_1\,b + q_2\,b \qquad \text{(start state)}\\
q_1 &= q_0\,a\\
q_2 &= q_1\,a + q_2\,a
\end{aligned}
$$

Arden's Lemma:

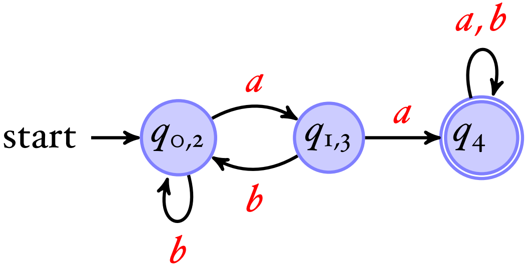
If $q = q\,r + s$ then $q = s\,r^*$

DFA Minimisation

1. Take all pairs (q, p) with $q \neq p$
2. Mark all pairs that accepting and non-accepting states
3. For all unmarked pairs (q, p) and all characters c test whether
   $$(\delta(q, c), \delta(p, c))$$
   are marked. If yes, then also mark (q, p).
4. Repeat last step until no change.
5. All unmarked pairs can be merged.



	q_0	q_1	q_2	q_3
q_1	★			
q_2		★		
q_3	★		★	
q_4	★	★	★	★



minimal automaton

Regular Languages

Two equivalent definitions:

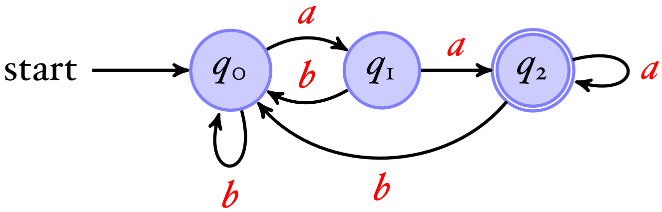
A language is regular iff there exists a regular expression that recognises all its strings.

A language is regular iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

Negation

Regular languages are closed under negation:



But requires that the automaton is completed!

Negation

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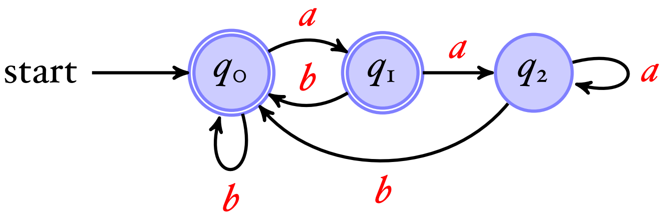

But requires that the automaton is completed!

The Goal of this Course

Write A Compiler

Today a lexer.

Survey: Thanks!

- **My Voice** “lecturer speaks in a low voice and is hard to hear him” “please use mic” “please use mic & lecture recording”
- **Pace** “faster pace” “a bit quick for me personally”
- **Recording** “please use recording class”
- **Module Name** “misleading”
- **Examples** “more examples”
- **Assessment** “really appreciate extension of first coursework”

Lexing

```
write "Fib";
read n;
minus1 := 0;
minus2 := 1;
while n > 0 do {
       temp := minus2;
       minus2 := minus1 + minus2;
       minus1 := temp;
       n := n - 1
};
write "Result";
write minus2
```

??

```
write "Input a number ";
read n;
while n > 1 do {
  if n % 2 == 0
  then n := n/2
  else n := 3*n+1;
};
write "Yes";
```

"if true then then 42 else +"

KEYWORD:

if, then, else,

WHITESPACE:

" ", \n,

IDENT:

$\mathtt{LETTER} \cdot (\mathtt{LETTER} + \mathtt{DIGIT} + \_)^*$

NUM:

$(\mathtt{NONZERODIGIT} \cdot \mathtt{DIGIT}^*) + \mathtt{0}$

OP:

+

COMMENT:

$/* \cdot \sim(\mathtt{ALL}^* \cdot (*/) \cdot \mathtt{ALL}^*) \cdot */$

”if true then then 42 else +”

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

”if true then then 42 else +”

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

There is one small problem with the tokenizer. How should we tokenize:

"x-3"

```
ID: ...
OP:
  "+", "-"
NUM:
  (NONZERODIGIT · DIGIT*) + "0"
NUMBER:
  NUM + ("-" · NUM)
```

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string abc.

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string abc.

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string abc.

Or, keywords are `if` and identifiers are letters followed by "letters + numbers + _"*

$$iffoo$$

POSIX: Two Rules

- Longest match rule (“maximal munch rule”): The longest initial substring matched by any regular expression is taken as the next token.
- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

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http://www.haskell.org/haskellwiki/Regex_Posix

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`http://www.haskell.org/haskellwiki/Regex_Posix`

traditional lexers are fast, but hairy

Sulzmann Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\ a} r_2$$

Sulzmann Matcher

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$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3$$

Sulzmann Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4$$

Sulzmann Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4\ nullable$$

Sulzmann Matcher

We want to match the string abc using r_1:

$$\begin{tikzcd} r_1 \arrow[r, "der\,a"] & r_2 \arrow[r, "der\,b"] & r_3 \arrow[r, "der\,c"] & r_4\ nullable \arrow[d] \\ & & & v_4 \end{tikzcd}$$

Sulzmann Matcher

We want to match the string *abc* using r_1:

$$
\begin{tikzcd}
r_1 \arrow[r, "der\,a"] & r_2 \arrow[r, "der\,b"] & r_3 \arrow[r, "der\,c"] & r_4 \; nullable \arrow[d] \\
 & & v_3 & v_4 \arrow[l, "inj\,c"']
\end{tikzcd}
$$

Sulzmann Matcher

We want to match the string *abc* using r_1:

$$
\begin{tikzcd}
r_1 \arrow[r, "der\ a"] & r_2 \arrow[r, "der\ b"] & r_3 \arrow[r, "der\ c"] & r_4\ nullable \arrow[d] \\
 & v_2 & v_3 \arrow[l, "inj\ b"'] & v_4 \arrow[l, "inj\ c"']
\end{tikzcd}
$$

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v_1 & v_2 \arrow[l, "inj\ a"'] & v_3 \arrow[l, "inj\ b"'] & v_4 \arrow[l, "inj\ c"']
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r_1 \arrow[r, "der\,a"] \arrow[d] & r_2 \arrow[r, "der\,b"] \arrow[d] & r_3 \arrow[r, "der\,c"] \arrow[d] & r_4\ nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\,a"] & v_3 \arrow[l, "inj\,b"] & v_4 \arrow[l, "inj\,c"]
\end{tikzcd}
$$

Regexes and Values

Regular expressions and their corresponding values:

$$
\begin{array}{rcl}
r & ::= & \varnothing \\
  & \mid & \epsilon \\
  & \mid & c \\
  & \mid & r_1 \cdot r_2 \\
  & \mid & r_1 + r_2 \\
  & & \\
  & \mid & r^*
\end{array}
\qquad
\begin{array}{rcl}
v & ::= & \\
  & & \mathit{Empty} \\
  & \mid & \mathit{Char}(c) \\
  & \mid & \mathit{Seq}(v_1, v_2) \\
  & \mid & \mathit{Left}(v) \\
  & \mid & \mathit{Right}(v) \\
  & \mid & [] \\
  & \mid & [v_1, \ldots v_n]
\end{array}
$$

```scala
abstract class Rexp
case object NULL extends Rexp
case object EMPTY extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp

abstract class Val
case object Empty extends Val
case class Char(c: Char) extends Val
case class Seq(v1: Val, v2: Val) extends Val
case class Left(v: Val) extends Val
case class Right(v: Val) extends Val
case class Stars(vs: List[Val]) extends Val
```

Mkeps

Finding a (posix) value for recognising the empty string:

$$
\begin{array}{lcl}
mkeps\,\epsilon & \stackrel{\text{def}}{=} & Empty \\
mkeps\,r_1 + r_2 & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } Left(mkeps(r_1)) \\
 & & \text{else } Right(mkeps(r_2)) \\
mkeps\,r_1 \cdot r_2 & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), mkeps(r_2)) \\
mkeps\,r^* & \stackrel{\text{def}}{=} & []
\end{array}
$$

Inject

Injecting ("Adding") a character to a value

$$
\begin{array}{ll}
inj\,(c)\;c\;Empty & \stackrel{\text{def}}{=} Char\;c \\
inj\,(r_1 + r_2)\;c\;Left(v) & \stackrel{\text{def}}{=} Left(inj\;r_1\;c\;v) \\
inj\,(r_1 + r_2)\;c\;Right(v) & \stackrel{\text{def}}{=} Right(inj\;r_2\;c\;v) \\
inj\,(r_1 \cdot r_2)\;c\;Seq(v_1, v_2) & \stackrel{\text{def}}{=} Seq(inj\;r_1\;c\;v_1, v_2) \\
inj\,(r_1 \cdot r_2)\;c\;Left(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} Seq(inj\;r_1\;c\;v_1, v_2) \\
inj\,(r_1 \cdot r_2)\;c\;Right(v) & \stackrel{\text{def}}{=} Seq(mkeps(r_1), inj\;r_2\;c\;v) \\
inj\,(r^*)\;c\;Seq(v, vs) & \stackrel{\text{def}}{=} inj\;r\;c\;v\;::\;vs
\end{array}
$$

inj: 1st arg $\mapsto$ a rexp; 2nd arg $\mapsto$ a character; 3rd arg $\mapsto$ a value

$$
\begin{tikzcd}
r_1 \arrow[r, "der\,a"] \arrow[d] & r_2 \arrow[r, "der\,b"] \arrow[d] & r_3 \arrow[r, "der\,c"] \arrow[d] & r_4 \; nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\,a"] & v_3 \arrow[l, "inj\,b"] & v_4 \arrow[l, "inj\,c"]
\end{tikzcd}
$$

$$
\begin{aligned}
r_1 &: \quad a \cdot (b \cdot c) \\
r_2 &: \quad \epsilon \cdot (b \cdot c) \\
r_3 &: \quad (\varnothing \cdot (b \cdot c)) + (\epsilon \cdot c) \\
r_4 &: \quad (\varnothing \cdot (b \cdot c)) + ((\varnothing \cdot c) + \epsilon)
\end{aligned}
$$

$$
\begin{tikzcd}
r_1 \arrow[r, "der\,a"] \arrow[d] & r_2 \arrow[r, "der\,b"] \arrow[d] & r_3 \arrow[r, "der\,c"] \arrow[d] & r_4 \; nullable \arrow[d, "mkeps"] \\
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\end{tikzcd}
$$

$$
\begin{aligned}
v_1 &: \quad Seq(Char(a), Seq(Char(b), Char(c))) \\
v_2 &: \quad Seq(Empty, Seq(Char(b), Char(c))) \\
v_3 &: \quad Right(Seq(Empty, Char(c))) \\
v_4 &: \quad Right(Right(Empty))
\end{aligned}
$$

Flatten

Obtaining the string underlying a value:

$$
\begin{array}{lcl}
|Empty| & \stackrel{\text{def}}{=} & [] \\
|Char(c)| & \stackrel{\text{def}}{=} & [c] \\
|Left(v)| & \stackrel{\text{def}}{=} & |v| \\
|Right(v)| & \stackrel{\text{def}}{=} & |v| \\
|Seq(v_1, v_2)| & \stackrel{\text{def}}{=} & |v_1| \mathbin{@} |v_2| \\
|[v_1, \ldots, v_n]| & \stackrel{\text{def}}{=} & |v_1| \mathbin{@} \ldots \mathbin{@} |v_n|
\end{array}
$$

r_1: $a \cdot (b \cdot c)$
r_2: $\epsilon \cdot (b \cdot c)$
r_3: $(\varnothing \cdot (b \cdot c)) + (\epsilon \cdot c)$
r_4: $(\varnothing \cdot (b \cdot c)) + ((\varnothing \cdot c) + \epsilon)$

$$\begin{tikzcd}
r_1 \arrow[r, "der\ a"] \arrow[d] & r_2 \arrow[r, "der\ b"] \arrow[d] & r_3 \arrow[r, "der\ c"] \arrow[d] & r_4\ \textit{nullable} \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\ a"] & v_3 \arrow[l, "inj\ b"] & v_4 \arrow[l, "inj\ c"]
\end{tikzcd}$$

v_1: $Seq(Char(a), Seq(Char(b), Char(c)))$
v_2: $Seq(Empty, Seq(Char(b), Char(c)))$
v_3: $Right(Seq(Empty, Char(c)))$
v_4: $Right(Right(Empty))$

$|v_1|$: abc
$|v_2|$: bc
$|v_3|$: c
$|v_4|$: $[]$

Lexing

$$
\begin{array}{lcl}
lex\; r\; [] & \stackrel{\text{def}}{=} & \text{if } nullable(r) \text{ then } mkeps(r) \text{ else } error\\
lex\; r\; c :: s & \stackrel{\text{def}}{=} & inj\; r\; c\; lex(der(c,r), s)
\end{array}
$$

lex: returns a value

$$
\begin{tikzcd}
r_1 \arrow[r, "der\ a"] \arrow[d] & r_2 \arrow[r, "der\ b"] \arrow[d] & r_3 \arrow[r, "der\ c"] \arrow[d] & r_4\ nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\ a"] & v_3 \arrow[l, "inj\ b"] & v_4 \arrow[l, "inj\ c"]
\end{tikzcd}
$$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

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- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\,c\,(x : r) \stackrel{\text{def}}{=} (x : der\,c\,r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\,(x : r)\,c\,v \stackrel{\text{def}}{=} Rec(x, inj\,r\,c\,v)$

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for extracting subpatterns $(z : ((x : ab) + (y : ba))$

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for extracting subpatterns $(z : ((x : ab) + (y : ba))$

- Regular expression for email addresses

$$(\text{name: } [a\text{-}z0\text{-}9\_.-]^{+}) \cdot @ \cdot (\text{domain: } [a\text{-}z0\text{-}9.-]^{+}) \cdot . \cdot (\text{top\_level: } [a\text{-}z.]^{\{2,6\}})$$

`christian.urban@kcl.ac.uk`

- result environment:

$$[(\mathit{name} : \texttt{christian.urban}), (\mathit{domain} : \texttt{kcl}), (\mathit{top\_level} : \texttt{ac.uk})]$$

While Tokens

$$
\begin{array}{lcl}
\texttt{WHILE\_REGS} & \stackrel{\text{def}}{=} & ((\text{"k"} : \texttt{KEYWORD}) + \\
& & (\text{"i"} : \texttt{ID}) + \\
& & (\text{"o"} : \texttt{OP}) + \\
& & (\text{"n"} : \texttt{NUM}) + \\
& & (\text{"s"} : \texttt{SEMI}) + \\
& & (\text{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + \\
& & (\text{"b"} : (\texttt{BEGIN} + \texttt{END})) + \\
& & (\text{"w"} : \texttt{WHITESPACE}))^*
\end{array}
$$

Simplification

- If we simplify after the derivative, then we are builing the value for the simplified regular expression, but *not* for the original regular expression.

$$\begin{tikzcd}
r_1 \arrow[r, "der\ a"] \arrow[d] & r_2 \arrow[r, "der\ b"] \arrow[d] & r_3 \arrow[r, "der\ c"] \arrow[d] & r_4\ nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\ a"] & v_3 \arrow[l, "inj\ b"] & v_4 \arrow[l, "inj\ c"]
\end{tikzcd}$$

$$(\varnothing \cdot (b \cdot c)) + ((\varnothing \cdot c) + \epsilon) \mapsto \epsilon$$

Rectification

rectification
functions:

$$
\begin{array}{lcll}
r \cdot \varnothing & \mapsto & \varnothing & \\
\varnothing \cdot r & \mapsto & \varnothing & \\
r \cdot \epsilon & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\epsilon \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \varnothing & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\varnothing + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}
$$

Rectification

rectification functions:

$$
\begin{array}{lcll}
r \cdot \varnothing & \mapsto & \varnothing & \\
\varnothing \cdot r & \mapsto & \varnothing & \\
r \cdot \epsilon & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\epsilon \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \varnothing & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\varnothing + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}
$$

old *simp* returns a rexp;
new *simp* returns a rexp and a rectification fun.

Rectification

$simp(r)$:
  case $r = r_1 + r_2$
    let $(r_{1s}, f_{1s}) = simp(r_1)$
      $(r_{2s}, f_{2s}) = simp(r_2)$
    case $r_{1s} = \varnothing$: return $(r_{2s}, \lambda v.\, Right(f_{2s}(v)))$
    case $r_{2s} = \varnothing$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$
    case $r_{1s} = r_{2s}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$
    otherwise: return $(r_{1s} + r_{2s}, f_{alt}(f_{1s}, f_{2s}))$

$f_{alt}(f_1, f_2) \stackrel{\text{def}}{=}$
  $\lambda v.$ case $v = Left(v')$: return $Left(f_1(v'))$
    case $v = Right(v')$: return $Right(f_2(v'))$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case ALT(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (NULL, _) => (r2s, F_RIGHT(f2s))
      case (_, NULL) => (r1s, F_LEFT(f1s))
      case _ =>
         if (r1s == r2s) (r1s, F_LEFT(f1s))
         else (ALT (r1s, r2s), F_ALT(f1s, f2s))
    }
  }
  ...
}

def F_RIGHT(f: Val => Val) = (v:Val) => Right(f(v))
def F_LEFT(f: Val => Val) = (v:Val) => Left(f(v))
def F_ALT(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Right(v) => Right(f2(v))
    case Left(v) => Left(f1(v)) }
```

Rectification

$simp(r)$:...
  case $r = r_1 \cdot r_2$
    let $(r_{1s}, f_{1s}) = simp(r_1)$
      $(r_{2s}, f_{2s}) = simp(r_2)$
    case $r_{1s} = \varnothing$: return $(\varnothing, f_{error})$
    case $r_{2s} = \varnothing$: return $(\varnothing, f_{error})$
    case $r_{1s} = \epsilon$: return $(r_{2s}, \lambda v.\, Seq(f_{1s}(Empty), f_{2s}(v)))$
    case $r_{2s} = \epsilon$: return $(r_{1s}, \lambda v.\, Seq(f_{1s}(v), f_{2s}(Empty)))$
    otherwise: return $(r_{1s} \cdot r_{2s}, f_{seq}(f_{1s}, f_{2s}))$

$f_{seq}(f_1, f_2) \stackrel{\text{def}}{=}$
  $\lambda v.$ case $v = Seq(v_1, v_2)$: return $Seq(f_1(v_1), f_2(v_2))$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case SEQ(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (NULL, _) => (NULL, F_ERROR)
      case (_, NULL) => (NULL, F_ERROR)
      case (EMPTY, _) => (r2s, F_SEQ_Void1(f1s, f2s))
      case (_, EMPTY) => (r1s, F_SEQ_Void2(f1s, f2s))
      case _ => (SEQ(r1s,r2s), F_SEQ(f1s, f2s))
    }
  }
  ...

def F_SEQ_Void1(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(Void), f2(v))
def F_SEQ_Void2(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(v), f2(Void))
def F_SEQ(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Sequ(v1, v2) => Sequ(f1(v1), f2(v2)) }
```

Lexing with Simplification

$$
\begin{array}{ll}
\mathit{lex}\; r\; [] & \stackrel{\text{def}}{=} \text{if}\; \mathit{nullable}(r)\; \text{then}\; \mathit{mkeps}(r)\; \text{else}\; \mathit{error}\\
\mathit{lex}\; r\; c :: s & \stackrel{\text{def}}{=} \text{let}\; (r', \mathit{frect}) = \mathit{simp}(\mathit{der}(c, r))\\
& \quad \mathit{inj}\; r\; c\; (\mathit{frect}(\mathit{lex}(r', s)))
\end{array}
$$

$$
\begin{tikzcd}
r_1 \arrow[r, "der\,a"] \arrow[d] & r_2 \arrow[r, "der\,b"] \arrow[d] & r_3 \arrow[r, "der\,c"] \arrow[d] & r_4\ \mathit{nullable} \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\,a"'] & v_3 \arrow[l, "inj\,b"'] & v_4 \arrow[l, "inj\,c"']
\end{tikzcd}
$$

$$
\begin{array}{lcl}
\mathit{zeroable}(\varnothing) & \stackrel{\text{def}}{=} & \mathit{true} \\
\mathit{zeroable}(\epsilon) & \stackrel{\text{def}}{=} & \mathit{false} \\
\mathit{zeroable}(c) & \stackrel{\text{def}}{=} & \mathit{false} \\
\mathit{zeroable}(r_1 + r_2) & \stackrel{\text{def}}{=} & \mathit{zeroable}(r_1) \wedge \mathit{zeroable}(r_2) \\
\mathit{zeroable}(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \mathit{zeroable}(r_1) \vee \mathit{zeroable}(r_2) \\
\mathit{zeroable}(r^*) & \stackrel{\text{def}}{=} & \mathit{false}
\end{array}
$$

$$\mathit{zeroable}(r) \text{ if and only if } L(r) = \varnothing$$