Compilers and Formal Languages

Email: christian.urban at kcl.ac.uk
Office: N7.07 (North Wing, Bush House)
Slides: KEATS (also home work is there)

There are more problems, than there are programs.

There are more problems, than there are programs.

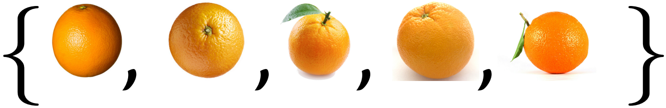
There must be a problem for which there is no program.

Subsets

If $A \subseteq B$ then A has fewer elements than B

$A \subseteq B$ and $B \subseteq A$

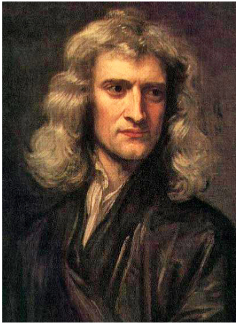
then $A = B$



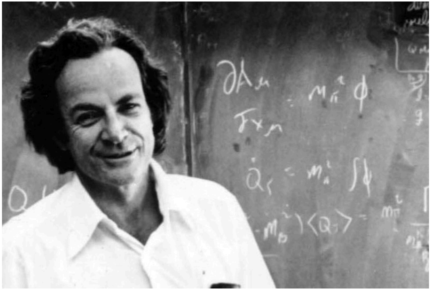
5 elements

3 elements

Newton vs Feynman



classical physics



quantum physics

The Goal of the Talk

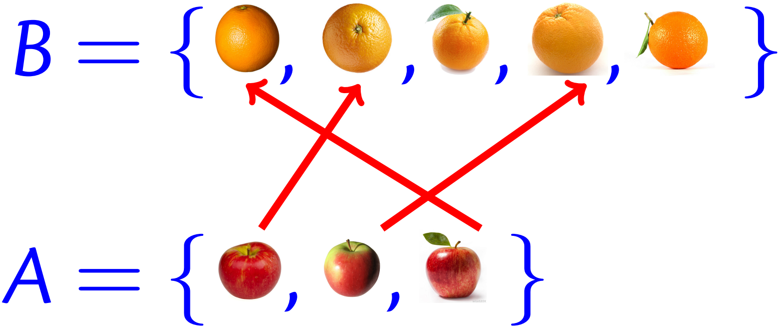
- show you that something very unintuitive happens with very large sets

- convince you that there are more **problems** than **programs**

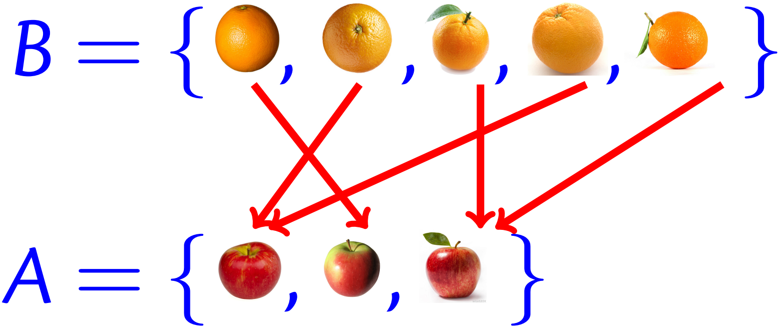
$B = \{$  $\}$

$A = \{$  $\}$

$$|A| = 5, |B| = 3$$



then $|A| \leq |B|$



for $=$ has to be a **one-to-one** mapping

Cardinality

$|A| \overset{\text{def}}{=}$ "how many elements"

$A \subseteq B \Rightarrow |A| \leq |B|$

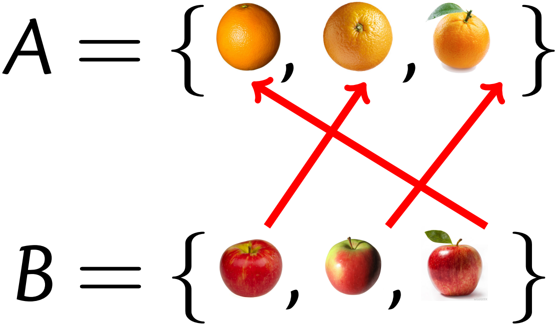
Cardinality

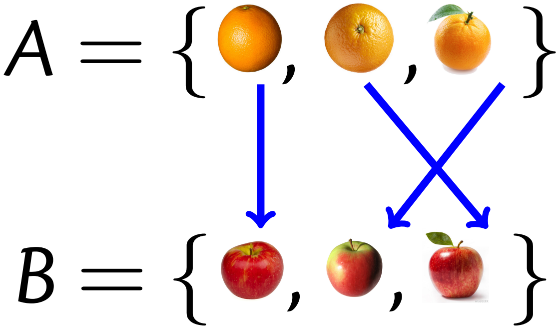
$|A| \overset{\text{def}}{=}$ "how many elements"

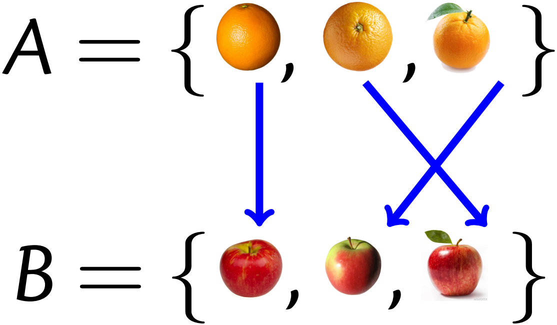
$A \subseteq B \Rightarrow |A| \leq |B|$

if there is an injective function $f : A \rightarrow B$ then $|A| \leq |B|$

$$\forall xy.\, f(x) = f(y) \Rightarrow x = y$$


A = { , , }
B = { , , }


A = { , , }
B = { , , }



then $|A| = |B|$

Natural Numbers

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\ldots\}$$

Natural Numbers

$\mathbb{N} \overset{\text{def}}{=} \{0, 1, 2, 3, ...\}$

A is countable iff $|A| \leq |\mathbb{N}|$

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

$x \mapsto x + 1$,
$|\mathbb{N} - \{0\}| = |\mathbb{N}|$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \{0,1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \overset{\text{def}}{=}$ odd numbers $\{1,3,5......\}$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \overset{\text{def}}{=}$ odd numbers $\{1, 3, 5 \ldots\ldots\}$
$\mathbb{E} \overset{\text{def}}{=}$ even numbers $\{0, 2, 4 \ldots\ldots\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$$\mathbb{N} \overset{\text{def}}{=} \text{positive numbers} \quad \{0, 1, 2, 3, \ldots\ldots\}$$

$$-\mathbb{N} \overset{\text{def}}{=} \text{negative numbers} \quad \{0, -1, -2, -3, \ldots\ldots\}$$

A is countable if there exists an injective $f : A \rightarrow \mathbb{N}$

A is uncountable if there does not exist an injective $f : A \rightarrow \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$
uncountable: $|A| > |\mathbb{N}|$

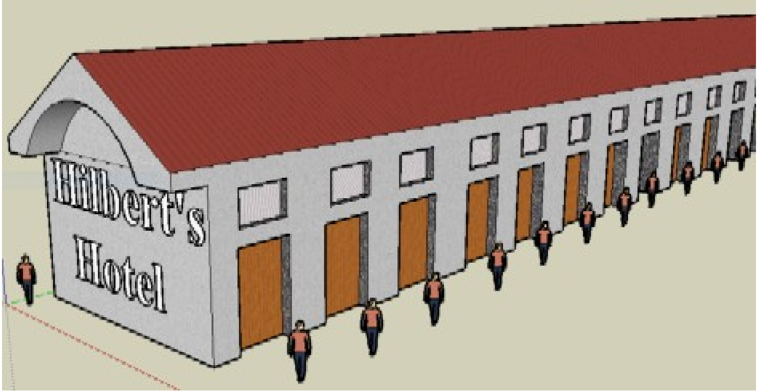
A is countable if there exists an injective $f : A \rightarrow \mathbb{N}$

A is uncountable if there does not exist an injective $f : A \rightarrow \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$
uncountable: $|A| > |\mathbb{N}|$

Does there exist such an A ?

Hilbert's Hotel



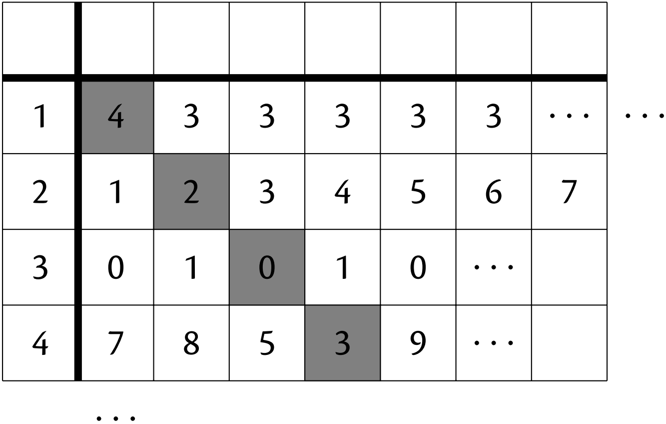
- ...has as many rooms as there are natural numbers

Real Numbers between 0 and 1

1	3	3	3	3	3	3	…	…
2	1	2	3	4	5	6	7	
3	0	1	0	1	0	…		
4	7	8	5	3	9	…		

…

Real Numbers between 0 and 1



1	4	3	3	3	3	3	· · ·
2	1	2	3	4	5	6	7
3	0	1	0	1	0	· · ·	
4	7	8	5	3	9	· · ·	

· · ·

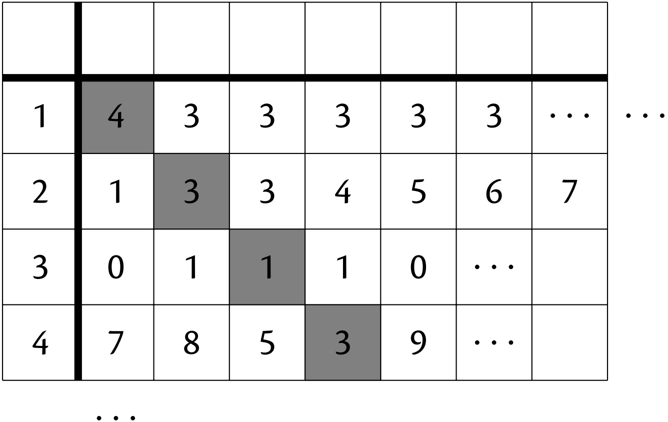
· · ·

Real Numbers between 0 and 1

1	4	3	3	3	3	3	. . .	. . .
2	1	3	3	4	5	6	7	
3	0	1	0	1	0	. . .		
4	7	8	5	3	9	. . .		

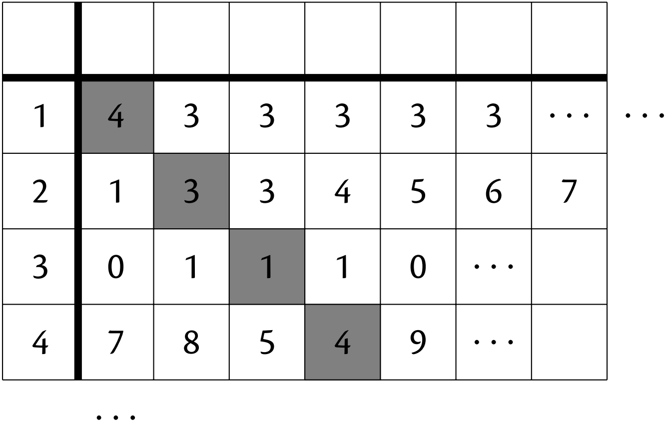
. . .

Real Numbers between 0 and 1



1	4	3	3	3	3	3	…	…
2	1	3	3	4	5	6	7	
3	0	1	1	1	0	…		
4	7	8	5	3	9	…		
	…							

Real Numbers between 0 and 1



1	4	3	3	3	3	3	…	…
2	1	3	3	4	5	6	7	
3	0	1	1	1	0	…		
4	7	8	5	4	9	…		

…

Real Numbers between 0 and 1

1	4	3	3	3	3	3	…
2	1	3	3	4	5	6	7
3	0	1	1	1	0	…	
4	7	8	5	4	9	…	

…

…

$$|\mathbb{N}| < |R|$$

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	$\cdots$
1	0	1	0	1	0	1	$\cdots$
2	0	0	0	1	1	0	0
3	0	0	0	0	0	$\cdots$	
4	1	1	0	1	1	$\cdots$	

$\cdots$

$\cdots$

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	$\cdots$
1	0	1	0	1	0	1	$\cdots$
2	0	0	0	1	1	0	0
3	0	0	0	0	0	$\cdots$	
4	1	1	0	1	1	$\cdots$	

$\cdots$

$\cdots$

$$|\mathrm{Progs}| = |\mathbb{N}| < |\mathrm{Probs}|$$

Halting Problem

Assume a program H that decides for all programs A and all input data D whether

- $H(A, D) \overset{\text{def}}{=} 1$ iff $A(D)$ terminates
- $H(A, D) \overset{\text{def}}{=} 0$ otherwise

Halting Problem (2)

Given such a program H define the following program C: for all programs A

- $C(A) \overset{\text{def}}{=} 0$ iff $H(A, A) = 0$
- $C(A) \overset{\text{def}}{=}$ loops otherwise

Contradiction

$H(C, C)$ is either 0 or 1.

- $H(C, C) = 1 \overset{\text{def } H}{\Rightarrow} C(C) \downarrow \overset{\text{def } C}{\Rightarrow} H(C, C) = 0$
- $H(C, C) = 0 \overset{\text{def } H}{\Rightarrow} C(C)$ loops $\overset{\text{def } C}{\Rightarrow}$ $H(C, C) = 1$

Contradiction in both cases. So H cannot exist.

Take Home Points

- there are sets that are more infinite than others
- even with the most powerful computer we can imagine, there are problems that cannot be solved by any program
- in CS we actually hit quite often such problems (halting problem)