# Compilers and Formal Languages (4)


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Slides: KEATS (also homework is there)

# Survey: Thanks!

*"...Thanks a million! Thanks without end!"*


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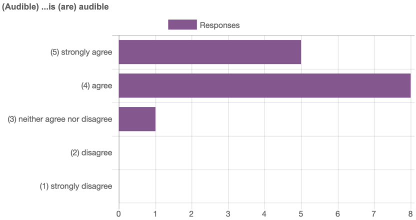


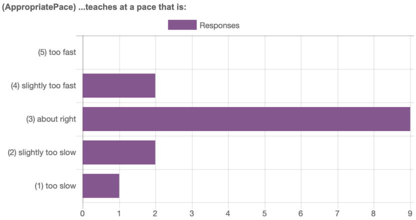

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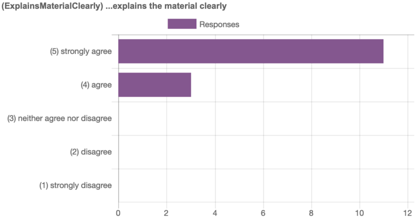


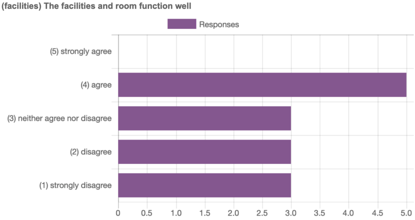

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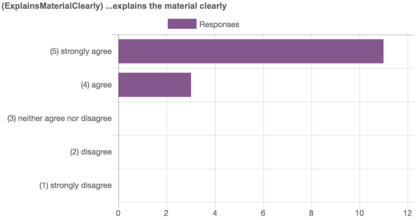
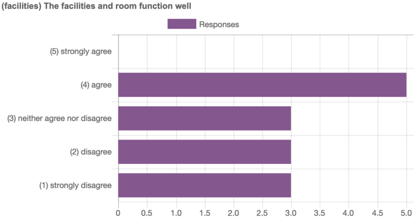
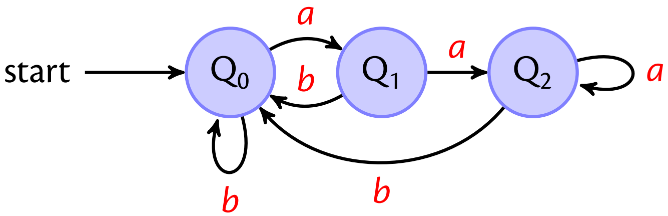



room too hot, 3h lecture




$$
\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\
Q_1 &= Q_0\, a \\
Q_2 &= Q_1\, a + Q_2\, a
\end{aligned}
$$

Arden's Lemma:

If $q = q\, r + s$ then $q = s\, r^*$

substitute $Q_1$ into $Q_0$ & $Q_2$:

$$Q_0 = \mathbf{1} + Q_0\, b + Q_0\, a\, b + Q_2\, b$$
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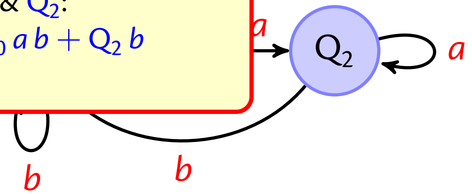


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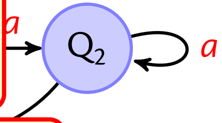


simplifying $Q_0$:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\,b) + Q_2\,b$$
$$Q_2 = Q_0\,a\,a + Q_2\,a$$

$$\begin{aligned} Q_0 &= \mathbf{1} + Q_0\,b + Q_1\,b + Q_2\,b \\ Q_1 &= Q_0\,a \\ Q_2 &= Q_1\,a + Q_2\,a \end{aligned}$$

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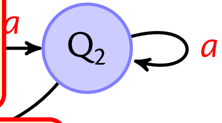


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Arden for $Q_2$:

$$Q_0 = \mathbf{1} + Q_0\, (b + a\, b) + Q_2\, b$$
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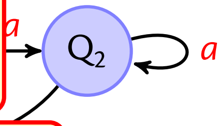


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Substitute $Q_2$ and simplify:

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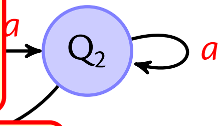


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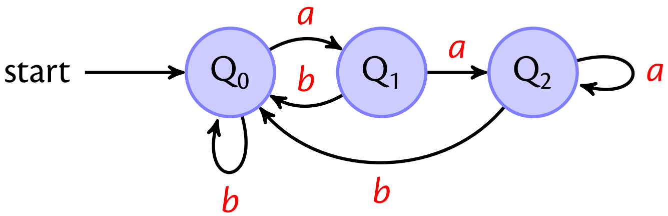
Arden again for $Q_0$:

$$Q_0 = (b + a\, b + a\, a\, (a^*)\, b)^*$$

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If $q =$

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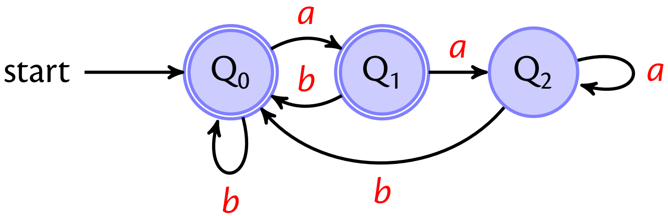


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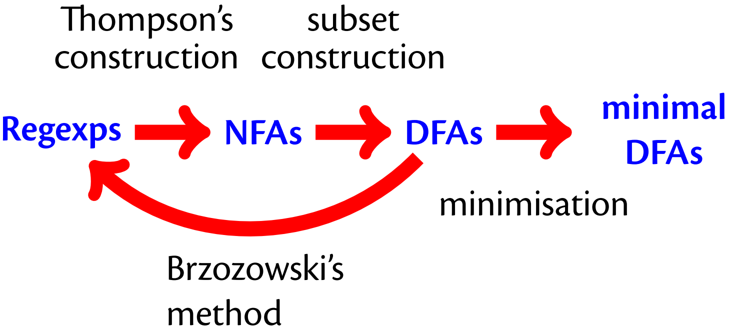
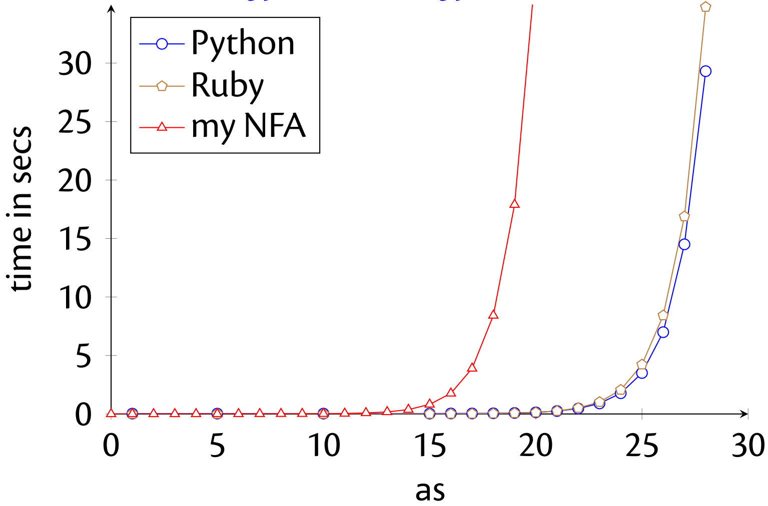
# Regexps and Automata



# $a^{?\{n\}} \cdot a^{\{n\}}$




The punchline is that many existing libraries do depth-first search in NFAs (backtracking).

# Regular Languages

Two equivalent definitions:

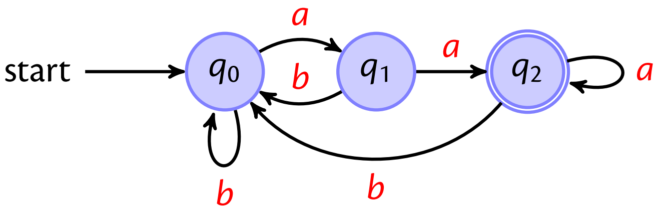
A language is regular iff there exists a regular expression that recognises all its strings.

A language is regular iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

# Negation

Regular languages are closed under negation:




But requires that the automaton is completed!

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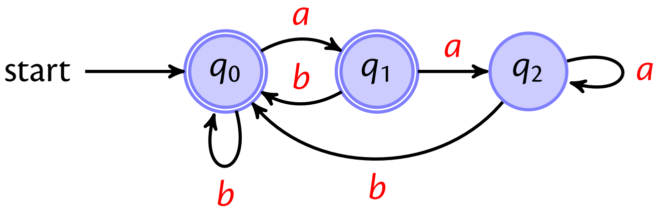


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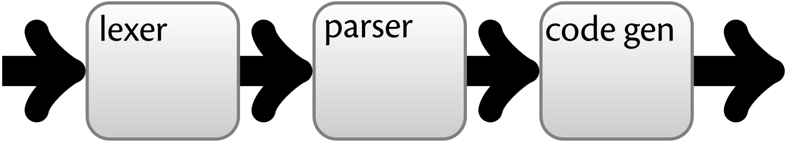
# The Goal of this Course

## Write a compiler




Today a lexer.

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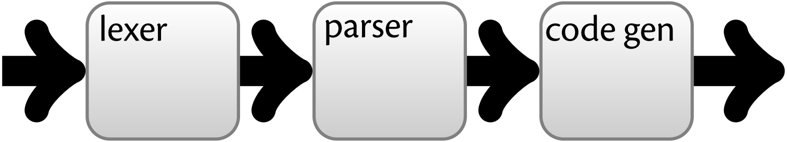


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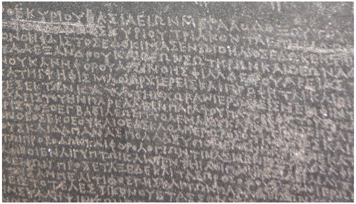

lexing ⇒ recognising words (Stone of Rosetta)

# Regular Expressions

In programming languages they are often used to recognise:

- symbols, digits
- identifiers
- numbers (non-leading zeros)
- keywords
- comments

`http://www.regexper.com`

# Lexing: Test Case

```
write "Fib";
read n;
minus1 := 0;
minus2 := 1;
while n > 0 do {
       temp := minus2;
       minus2 := minus1 + minus2;
       minus1 := temp;
       n := n - 1
};
write "Result";
write minus2
```

`"if true then then 42 else +"`

KEYWORD:

`if`, `then`, `else`,

WHITESPACE:

`" "`, `\n`,

IDENTIFIER:

$\texttt{LETTER} \cdot (\texttt{LETTER} + \texttt{DIGIT} + \_)^*$

NUM:

$(\texttt{NONZERODIGIT} \cdot \texttt{DIGIT}^*) + \texttt{0}$

OP:

`+`, `-`, `*`, `%`, `<`, `<=`

COMMENT:

$/* \cdot \sim(\texttt{ALL}^* \cdot (*/) \cdot \texttt{ALL}^*) \cdot */$

```
"if true then then 42 else +"
```

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

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"if true then then 42 else +"
```

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

There is one small problem with the tokenizer. How should we tokenize...?

"x-3"

```
ID: ...
OP:
  "+", "-"
NUM:
  (NONZERODIGIT · DIGIT*) + ''0''
NUMBER:
  NUM + ("-" · NUM)
```

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string $abc$.

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The same problem with

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Or, keywords are `if` and identifiers are letters followed by "letters + numbers + _"*

$$if \qquad iffoo$$

# POSIX: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as the next token.

- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

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most posix matchers are buggy
`http://www.haskell.org/haskellwiki/Regex_Posix`

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traditional lexers are fast, but hairy

# Sulzmann & Lu Matcher

We want to match the string *abc* using $r_1$:

$$r_1 \xrightarrow{der\ a} r_2$$

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We want to match the string *abc* using $r_1$:

$$r_1 \xrightarrow{der\ a} r_2 \xrightarrow{der\ b} r_3 \xrightarrow{der\ c} r_4 \ \textit{nullable?}$$

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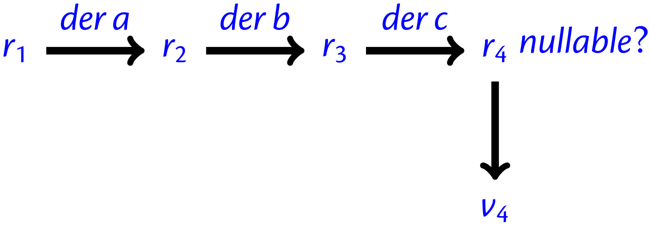

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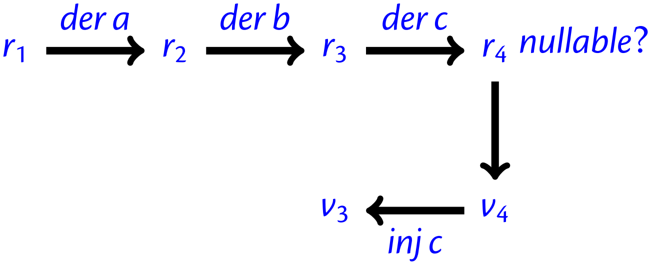

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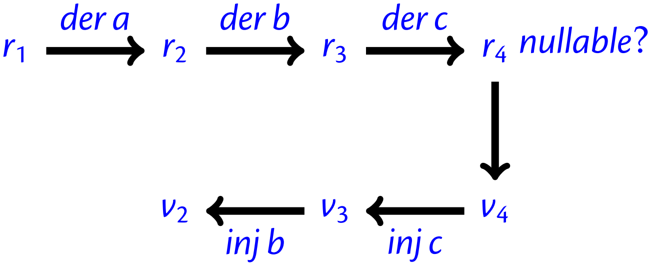

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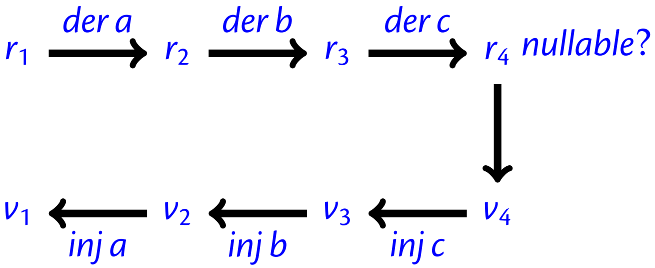

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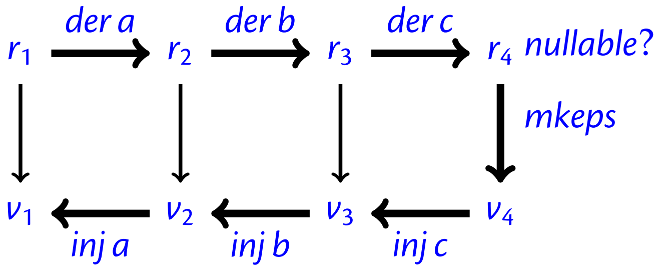


# Regexes and Values

Regular expressions and their corresponding values:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
  & | & \mathbf{1} \\
  & | & c \\
  & | & r_1 \cdot r_2 \\
  & | & r_1 + r_2 \\
  &   & \\
  & | & r^*
\end{array}
\qquad
\begin{array}{rcl}
v & ::= & \\
  &   & \mathit{Empty} \\
  & | & \mathit{Char}(c) \\
  & | & \mathit{Seq}(v_1, v_2) \\
  & | & \mathit{Left}(v) \\
  & | & \mathit{Right}(v) \\
  & | & \mathit{Stars}\,[] \\
  & | & \mathit{Stars}\,[v_1, \ldots\, v_n]
\end{array}
$$

```scala
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp

abstract class Val
case object Empty extends Val
case class Chr(c: Char) extends Val
case class Sequ(v1: Val, v2: Val) extends Val
case class Left(v: Val) extends Val
case class Right(v: Val) extends Val
case class Stars(vs: List[Val]) extends Val
```

# Mkeps

Finding a (posix) value for recognising the empty string:

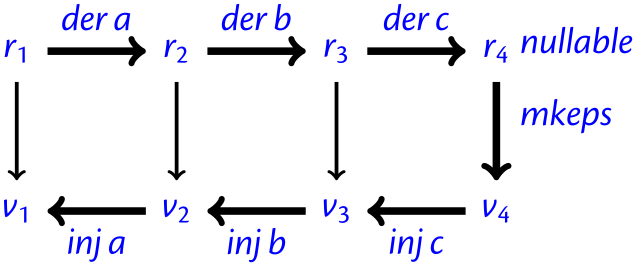
$$
\begin{array}{lcl}
mkeps\,(\mathbf{1}) & \stackrel{\text{def}}{=} & Empty \\
mkeps\,(r_1 + r_2) & \stackrel{\text{def}}{=} & \text{if}\; nullable(r_1) \\
 & & \text{then}\; Left(mkeps(r_1)) \\
 & & \text{else}\; Right(mkeps(r_2)) \\
mkeps\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), mkeps(r_2)) \\
mkeps\,(r^*) & \stackrel{\text{def}}{=} & Stars\;[]
\end{array}
$$

# Inject

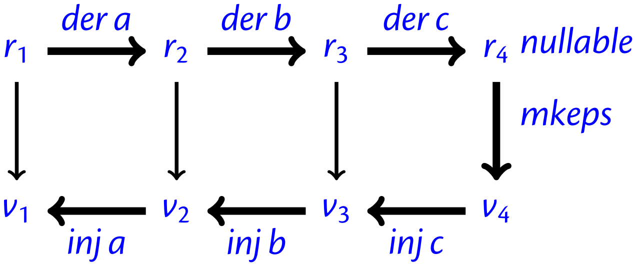
Injecting ("Adding") a character to a value

$$
\begin{array}{lcl}
inj\;(c)\;c\;(Empty) & \stackrel{\text{def}}{=} & Char\,c\\
inj\;(r_1 + r_2)\;c\;(Left(v)) & \stackrel{\text{def}}{=} & Left(inj\,r_1\,c\,v)\\
inj\;(r_1 + r_2)\;c\;(Right(v)) & \stackrel{\text{def}}{=} & Right(inj\,r_2\,c\,v)\\
inj\;(r_1 \cdot r_2)\;c\;(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & Seq(inj\,r_1\,c\,v_1, v_2)\\
inj\;(r_1 \cdot r_2)\;c\;(Left(Seq(v_1, v_2))) & \stackrel{\text{def}}{=} & Seq(inj\,r_1\,c\,v_1, v_2)\\
inj\;(r_1 \cdot r_2)\;c\;(Right(v)) & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), inj\,r_2\,c\,v)\\
inj\;(r^*)\;c\;(Seq(v, Stars\,vs)) & \stackrel{\text{def}}{=} & Stars\;(inj\,r\,c\,v\;::\;vs)
\end{array}
$$

*inj*: 1st arg $\mapsto$ a rexp; 2nd arg $\mapsto$ a character; 3rd arg $\mapsto$ a value


$r_1$
der a
$r_2$
der b
$r_3$
der c
$r_4$
nullable
mkeps
$v_1$
inj a
$v_2$
inj b
$v_3$
inj c
$v_4$

$r_1$: $a \cdot (b \cdot c)$
$r_2$: $\mathbf{1} \cdot (b \cdot c)$
$r_3$: $(\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c)$
$r_4$: $(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$



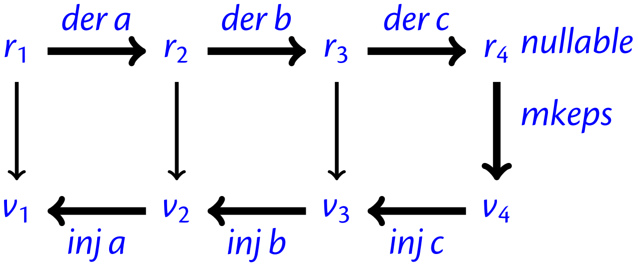

$v_1$: $Seq(Char(a), Seq(Char(b), Char(c)))$
$v_2$: $Seq(Empty, Seq(Char(b), Char(c)))$
$v_3$: $Right(Seq(Empty, Char(c)))$
$v_4$: $Right(Right(Empty))$

# Flatten

Obtaining the string underlying a value:

$$
\begin{array}{lcl}
|Empty| & \stackrel{\text{def}}{=} & [] \\
|Char(c)| & \stackrel{\text{def}}{=} & [c] \\
|Left(v)| & \stackrel{\text{def}}{=} & |v| \\
|Right(v)| & \stackrel{\text{def}}{=} & |v| \\
|Seq(v_1, v_2)| & \stackrel{\text{def}}{=} & |v_1| \mathbin{@} |v_2| \\
|[v_1, \ldots, v_n]| & \stackrel{\text{def}}{=} & |v_1| \mathbin{@} \ldots \mathbin{@} |v_n|
\end{array}
$$

$r_1$: $a \cdot (b \cdot c)$
$r_2$: $\mathbf{1} \cdot (b \cdot c)$
$r_3$: $(\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c)$
$r_4$: $(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$



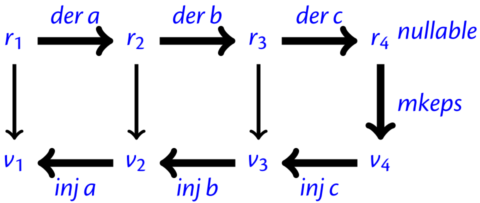

$v_1$: $Seq(Char(a), Seq(Char(b), Char(c)))$
$v_2$: $Seq(Empty, Seq(Char(b), Char(c)))$
$v_3$: $Right(Seq(Empty, Char(c)))$
$v_4$: $Right(Right(Empty))$

$|v_1|$: $abc$
$|v_2|$: $bc$
$|v_3|$: $c$
$|v_4|$: $[]$

# Lexing

$$\begin{aligned} lex\; r\; [] &\stackrel{\text{def}}{=} \text{if}\; nullable(r)\; \text{then}\; mkeps(r)\; \text{else}\; error \\ lex\; r\; c :: s &\stackrel{\text{def}}{=} inj\; r\; c\; lex(der(c, r), s) \end{aligned}$$

*lex*: returns a value



# Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

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- new regex: $(x : r)$ new value: $Rec(x, v)$
- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\, c\, (x : r) \stackrel{\text{def}}{=} (x : der\, c\, r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\, (x : r)\, c\, v \stackrel{\text{def}}{=} Rec(x, inj\, r\, c\, v)$

# Records

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- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\,(x : r)\,c\,v \stackrel{\text{def}}{=} Rec(x, inj\,r\,c\,v)$

for extracting subpatterns $(z : ((x : ab) + (y : ba))$

- A regular expression for email addresses

$$(\text{name: } [a\text{-}z0\text{-}9\_\,.-]^+) \cdot @ \cdot$$
$$(\text{domain: } [a\text{-}z0\text{-}9\,.-]^+) \cdot . \cdot$$
$$(\text{top\_level: } [a\text{-}z\,.]^{\{2,6\}})$$

`christian.urban@kcl.ac.uk`

- the result environment:

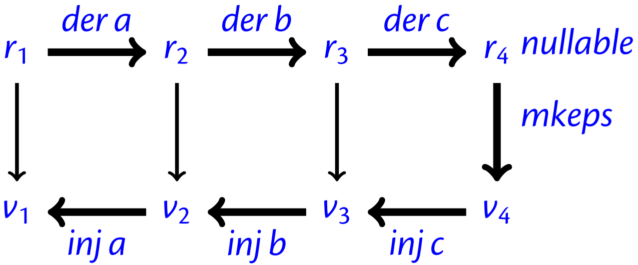
$$[(name : \texttt{christian.urban}),$$
$$(domain : \texttt{kcl}),$$
$$(top\_level : \texttt{ac.uk})]$$

# While Tokens

$$\texttt{WHILE\_REGS} \stackrel{\text{def}}{=} \Big((\texttt{"k"} : \texttt{KEYWORD}) + (\texttt{"i"} : \texttt{ID}) + (\texttt{"o"} : \texttt{OP}) + (\texttt{"n"} : \texttt{NUM}) + (\texttt{"s"} : \texttt{SEMI}) + (\texttt{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + (\texttt{"b"} : (\texttt{BEGIN} + \texttt{END})) + (\texttt{"w"} : \texttt{WHITESPACE})\Big)^*$$

# Simplification

- If we simplify after the derivative, then we are building the value for the simplified regular expression, but **not** for the original regular expression.




$$(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1}) \mapsto \mathbf{1}$$

Normally we would have

$$(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$$

and answer how this regular expression matches the empty string with the value

$$\textit{Right}(\textit{Right}(\textit{Empty}))$$

But now we simplify this to $\mathbf{1}$ and would produce $\textit{Empty}$ (see *mkeps*).

# Rectification

rectification functions:

$$
\begin{array}{lcll}
r \cdot \mathbf{0} & \mapsto & \mathbf{0} & \\
\mathbf{0} \cdot r & \mapsto & \mathbf{0} & \\
r \cdot \mathbf{1} & \mapsto & r & \lambda f_1 f_2\, v.\, Seq(f_1\, v, f_2\, Empty) \\
\mathbf{1} \cdot r & \mapsto & r & \lambda f_1 f_2\, v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \mathbf{0} & \mapsto & r & \lambda f_1 f_2\, v.\, Left(f_1\, v) \\
\mathbf{0} + r & \mapsto & r & \lambda f_1 f_2\, v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2\, v.\, Left(f_1\, v)
\end{array}
$$

# Rectification

rectification functions:

$$
\begin{array}{lcll}
r \cdot \mathbf{0} & \mapsto & \mathbf{0} & \\
\mathbf{0} \cdot r & \mapsto & \mathbf{0} & \\
r \cdot \mathbf{1} & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\mathbf{1} \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \mathbf{0} & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\mathbf{0} + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}
$$

old *simp* returns a rexp;
new *simp* returns a rexp and a rectification function.

# Rectification

$simp(r)$:

case $r = r_1 + r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(r_{2s}, \lambda v.\, Right(f_{2s}(v)))$

case $r_{2s} = \mathbf{0}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$

case $r_{1s} = r_{2s}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$

otherwise: return $(r_{1s} + r_{2s}, f_{alt}(f_{1s}, f_{2s}))$

$$f_{alt}(f_1, f_2) \stackrel{\text{def}}{=}$$
$$\lambda v.\ \text{case } v = Left(v'):\ \text{return } Left(f_1(v'))$$
$$\text{case } v = Right(v'):\ \text{return } Right(f_2(v'))$$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case ALT(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (r2s, F_RIGHT(f2s))
      case (_, ZERO) => (r1s, F_LEFT(f1s))
      case _ =>
         if (r1s == r2s) (r1s, F_LEFT(f1s))
         else (ALT (r1s, r2s), F_ALT(f1s, f2s))
    }
  }
  ...
}

def F_RIGHT(f: Val => Val) = (v:Val) => Right(f(v))
def F_LEFT(f: Val => Val) = (v:Val) => Left(f(v))
def F_ALT(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Right(v) => Right(f2(v))
    case Left(v) => Left(f1(v)) }
```

# Rectification

$simp(r)$:...

case $r = r_1 \cdot r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{2s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{1s} = \mathbf{1}$: return $(r_{2s}, \lambda v.\, Seq(f_{1s}(Empty), f_{2s}(v)))$

case $r_{2s} = \mathbf{1}$: return $(r_{1s}, \lambda v.\, Seq(f_{1s}(v), f_{2s}(Empty)))$

otherwise: return $(r_{1s} \cdot r_{2s}, f_{seq}(f_{1s}, f_{2s}))$

$f_{seq}(f_1, f_2) \stackrel{\text{def}}{=}$

$\lambda v.$ case $v = Seq(v_1, v_2)$: return $Seq(f_1(v_1), f_2(v_2))$

```
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case SEQ(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (ZERO, F_ERROR)
      case (_, ZERO) => (ZERO, F_ERROR)
      case (ONE, _) => (r2s, F_SEQ_Void1(f1s, f2s))
      case (_, ONE) => (r1s, F_SEQ_Void2(f1s, f2s))
      case _ => (SEQ(r1s,r2s), F_SEQ(f1s, f2s))
    }
  }
  ...

def F_SEQ_Void1(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(Void), f2(v))
def F_SEQ_Void2(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(v), f2(Void))
def F_SEQ(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Sequ(v1, v2) => Sequ(f1(v1), f2(v2)) }
```

# Rectification Example

$$(b \cdot c) + (\mathbf{0} + \mathbf{1}) \mapsto (b \cdot c) + \mathbf{1}$$

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$$(\underline{b \cdot c}) + (\underline{\mathbf{0} + \mathbf{1}}) \mapsto (b \cdot c) + \mathbf{1}$$

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$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

# Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{rcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$f_{alt}(f_{s1}, f_{s2}) \stackrel{\text{def}}{=}$$

$$\begin{array}{ll} \lambda v.\ \text{case}\ v = Left(v'): & \text{return}\ Left(f_{s1}(v')) \\ \quad\ \ \text{case}\ v = Right(v'): & \text{return}\ Right(f_{s2}(v')) \end{array}$$

# Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$\begin{array}{ll} \lambda v. & \text{case } v = Left(v'): \quad \text{return } Left(v') \\ & \text{case } v = Right(v'): \text{return } Right(Right(v')) \end{array}$$

# Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

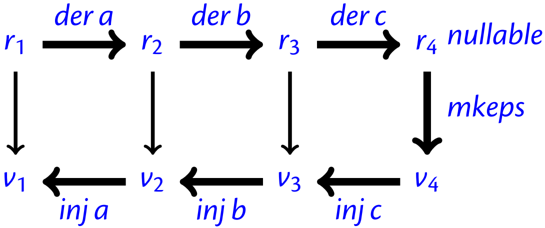
$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$\begin{array}{ll} \lambda v.\ \text{case } v = Left(v')\text{:} & \text{return } Left(v') \\ \quad\ \text{case } v = Right(v')\text{:} & \text{return } Right(Right(v')) \end{array}$$

| | |
|---|---|
| *mkeps* simplified case: | $Right(Empty)$ |
| rectified case: | $Right(Right(Empty))$ |

# Lexing with Simplification

$$\begin{array}{lcl} lex\; r\; [] & \stackrel{\text{def}}{=} & \text{if}\; nullable(r)\; \text{then}\; mkeps(r)\; \text{else}\; error \\ lex\; r\; c :: s & \stackrel{\text{def}}{=} & \text{let}\; (r', frect) = simp(der(c, r)) \\ & & inj\; r\; c\; (frect(lex(r', s))) \end{array}$$



# Environments

Obtaining the "recorded" parts of a value:

$$
\begin{array}{lcl}
env(Empty) & \stackrel{\text{def}}{=} & [] \\
env(Char(c)) & \stackrel{\text{def}}{=} & [] \\
env(Left(v)) & \stackrel{\text{def}}{=} & env(v) \\
env(Right(v)) & \stackrel{\text{def}}{=} & env(v) \\
env(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & env(v_1) \mathbin{@} env(v_2) \\
env(Stars\,[v_1, \ldots, v_n]) & \stackrel{\text{def}}{=} & env(v_1) \mathbin{@} \ldots \mathbin{@} env(v_n) \\
env(Rec(x : v)) & \stackrel{\text{def}}{=} & (x : |v|) :: env(v)
\end{array}
$$

# While Tokens

$$\texttt{WHILE\_REGS} \stackrel{\text{def}}{=} \big((\texttt{"k"} : \texttt{KEYWORD}) + \\ (\texttt{"i"} : \texttt{ID}) + \\ (\texttt{"o"} : \texttt{OP}) + \\ (\texttt{"n"} : \texttt{NUM}) + \\ (\texttt{"s"} : \texttt{SEMI}) + \\ (\texttt{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + \\ (\texttt{"b"} : (\texttt{BEGIN} + \texttt{END})) + \\ (\texttt{"w"} : \texttt{WHITESPACE})\big)^*$$

```
"if true then then 42 else +"
```

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

```
"if true then then 42 else +"
```

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

# Lexer: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as next token.

- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

$$
\begin{array}{lcl}
\mathit{zeroable}(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathit{true}\\
\mathit{zeroable}(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathit{false}\\
\mathit{zeroable}(c) & \stackrel{\text{def}}{=} & \mathit{false}\\
\mathit{zeroable}(r_1 + r_2) & \stackrel{\text{def}}{=} & \mathit{zeroable}(r_1) \wedge \mathit{zeroable}(r_2)\\
\mathit{zeroable}(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \mathit{zeroable}(r_1) \vee \mathit{zeroable}(r_2)\\
\mathit{zeroable}(r^*) & \stackrel{\text{def}}{=} & \mathit{false}
\end{array}
$$

$\mathit{zeroable}(r)$ if and only if $L(r) = \{\}$