Compilers and Formal Languages

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Slides & Progs: KEATS (also homework is there)

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2 Regular Expressions, Derivatives	7 Compilation, JVM
3 Automata, Regular Languages	8 Compiling Functional Languages
4 Lexing, Tokenising	9 Optimisations
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(Basic) Regular Expressions

$$
\begin{array}{rcll}
r & ::= & \mathbf{0} & \text{nothing} \\
  & \mid & \mathbf{1} & \text{empty string / "" / []} \\
  & \mid & c & \text{character} \\
  & \mid & r_1 \cdot r_2 & \text{sequence} \\
  & \mid & r_1 + r_2 & \text{alternative / choice} \\
  & \mid & r^* & \text{star (zero or more)}
\end{array}
$$

How about ranges $[a\text{-}z]$, r^+ and $\sim r$? Do they increase the set of languages we can recognise?

Negation

Assume you have an alphabet consisting of the letters a, b and c only. Find a (basic!) regular expression that matches all strings *except* ab and ac!

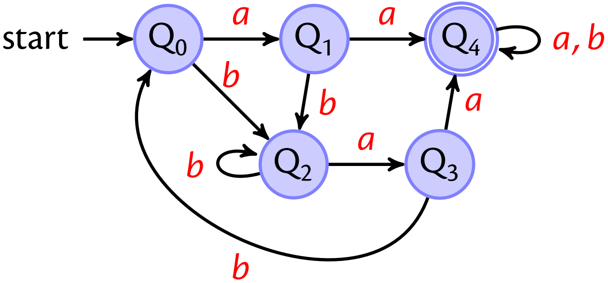
Automata

A **deterministic finite automaton**, DFA, consists of:

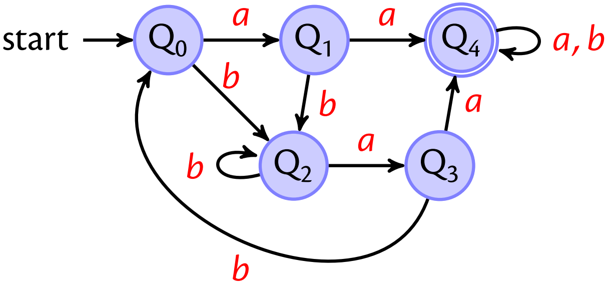
- an alphabet Σ
- a set of states Qs
- one of these states is the start state Q_0
- some states are accepting states F, and
- there is transition function δ

  which takes a state as argument and a character and produces a new state; this function might not be everywhere defined $\Rightarrow$ partial function

$$A(\Sigma, Qs, Q_0, F, \delta)$$



- the start state can be an accepting state
- it is possible that there is no accepting state
- all states might be accepting (but this does not necessarily mean all strings are accepted)



for this automaton δ is the function

$$
\begin{array}{lll}
(Q_0, a) \rightarrow Q_1 & (Q_1, a) \rightarrow Q_4 & (Q_4, a) \rightarrow Q_4 \\
(Q_0, b) \rightarrow Q_2 & (Q_1, b) \rightarrow Q_2 & (Q_4, b) \rightarrow Q_4
\end{array} \quad \ldots
$$

Accepting a String

Given

$$A(\Sigma, Qs, Q_0, F, \delta)$$

you can define

$$\widehat{\delta}(q, []) \stackrel{\text{def}}{=} q$$
$$\widehat{\delta}(q, c :: s) \stackrel{\text{def}}{=} \widehat{\delta}(\delta(q, c), s)$$

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$$\widehat{\delta}(q, c :: s) \stackrel{\text{def}}{=} \widehat{\delta}(\delta(q, c), s)$$

Whether a string s is accepted by A?

$$\widehat{\delta}(\mathrm{Q}_0, s) \in F$$

Regular Languages

A **language** is a set of strings.

A **regular expression** specifies a language.

A language is **regular** iff there exists a regular expression that recognises all its strings.

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not all languages are regular, e.g. $a^n b^n$ is not

Regular Languages (2)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

Non-Deterministic Finite Automata

A non-deterministic finite automaton (NFA) consists again of:

- a finite set of states
- some these states are the start states
- some states are accepting states, and
- there is transition relation

$$(Q_1, a) \rightarrow Q_2 \qquad \ldots$$
$$(Q_1, a) \rightarrow Q_3$$

Non-Deterministic Finite Automata

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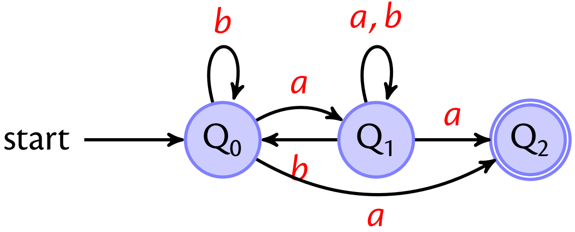
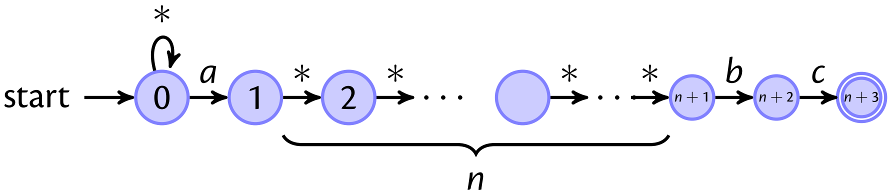
$$(Q_1, a) \rightarrow Q_2 \quad (Q_1, a) \rightarrow Q_3 \quad \ldots \qquad (Q_1, a) \rightarrow \{Q_2, Q_3\}$$

An NFA Example



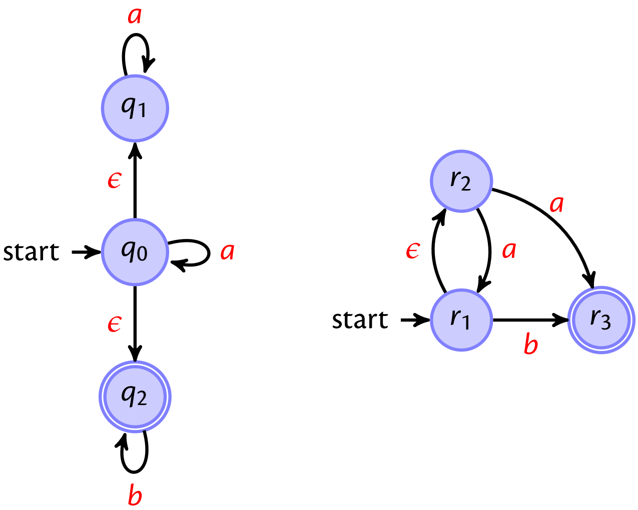
Another Example

For the regular expression $(.^*)a\,(.^{\{n\}})bc$

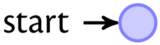


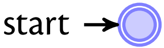
Note the star-transitions: accept any character.

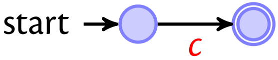
Two Epsilon NFA Examples



Rexp to NFA

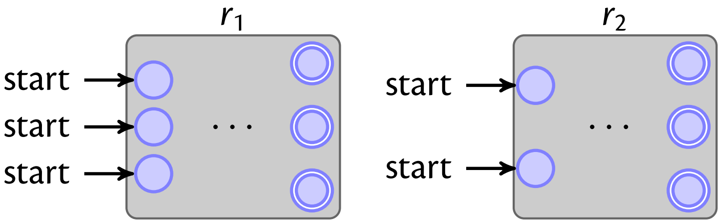
$\mathbf{0}$ 

$\mathbf{1}$ 

c 

Case $r_1 \cdot r_2$

By recursion we are given two automata:



We need to (1) change the accepting nodes of the first automaton into non-accepting nodes, and (2) connect them via ϵ-transitions to the starting state of the second automaton.

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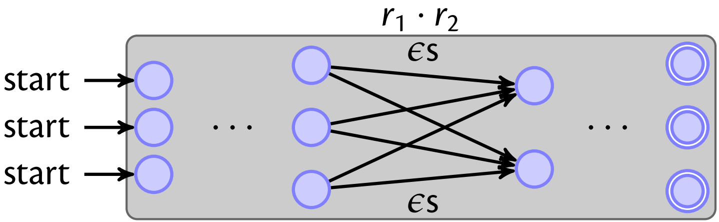

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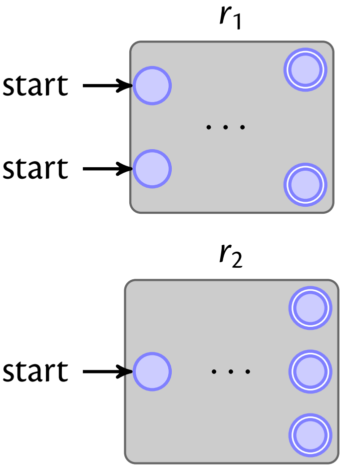

We can just put both automata together.

Case $r_1 + r_2$

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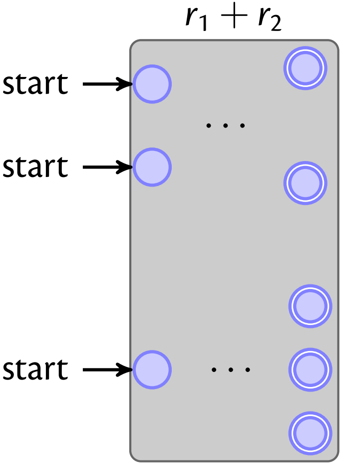
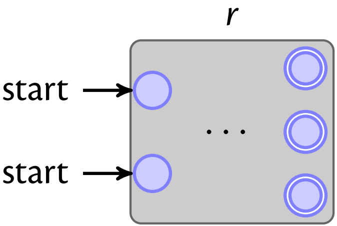


We can just put both automata together.

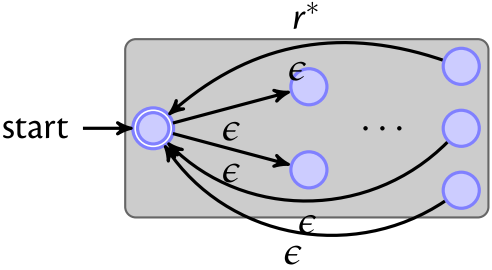
Case r^*

By recursion we are given an automaton for r:



Case r^*

By recursion we are given an automaton for r:



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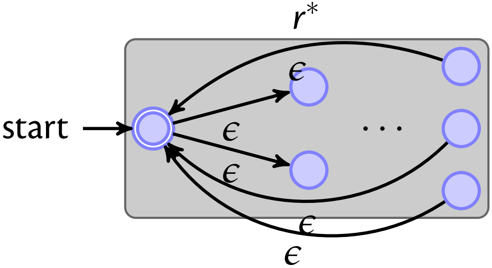
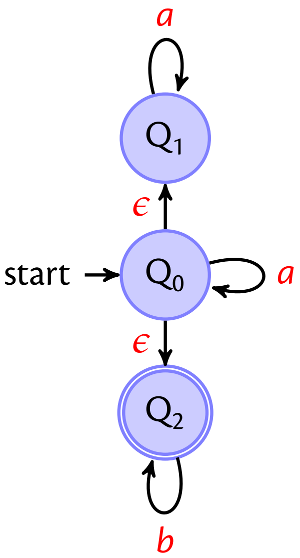


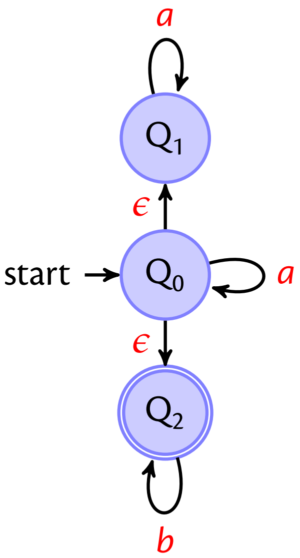
Why can't we just have an epsilon transition from the accepting states to the starting state?

Subset Construction



nodes	a	b
$\{\}$		
$\{0\}$		
$\{1\}$		
$\{2\}$		
$\{0, 1\}$		
$\{0, 2\}$		
$\{1, 2\}$		
$\{0, 1, 2\}$		

Subset Construction



nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$		
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$\{1,2\}$		
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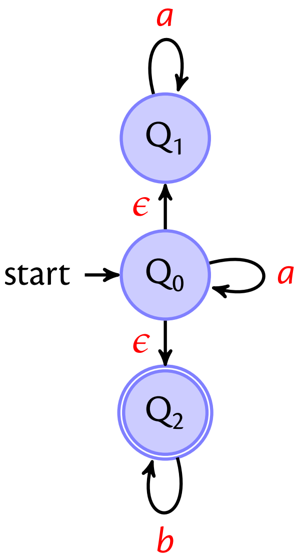
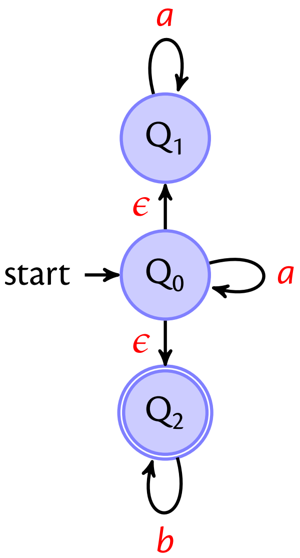
Subset Construction



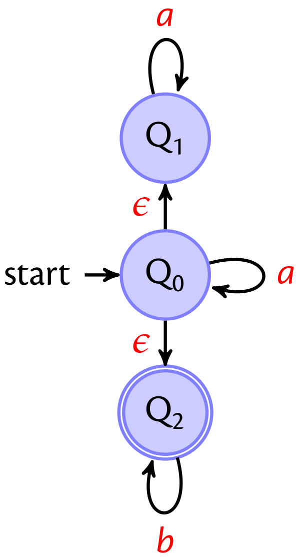
nodes	a	b
{}	{}	{}
{0}	{0, 1, 2}	{2}
{1}	{1}	{}
{2}	{}	{2}
{0, 1}		
{0, 2}		
{1, 2}		
{0, 1, 2}		

Subset Construction



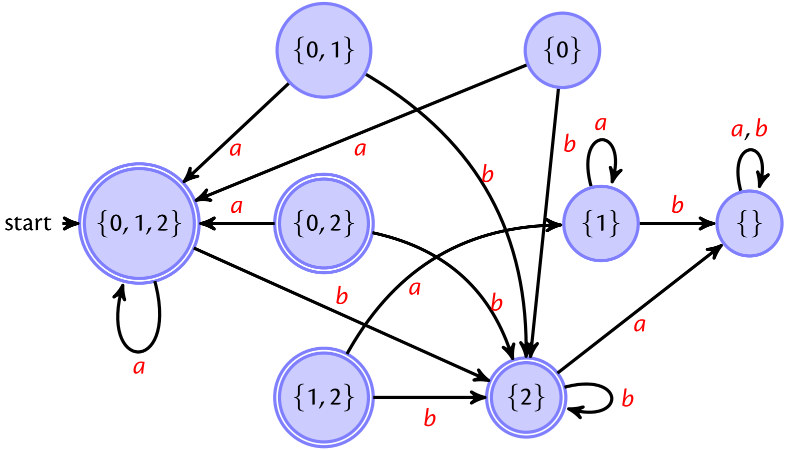
nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$	$\{0,1,2\}$	$\{2\}$
$\{1\}$	$\{1\}$	$\{\}$
$\{2\}$	$\{\}$	$\{2\}$
$\{0,1\}$	$\{0,1,2\}$	$\{2\}$
$\{0,2\}$	$\{0,1,2\}$	$\{2\}$
$\{1,2\}$	$\{1\}$	$\{2\}$
$\{0,1,2\}$	$\{0,1,2\}$	$\{2\}$

Subset Construction



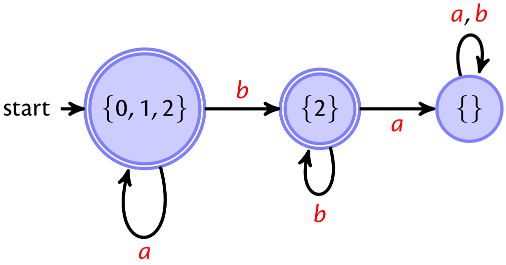
nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$	$\{0,1,2\}$	$\{2\}$
$\{1\}$	$\{1\}$	$\{\}$
$\{2\}$ *	$\{\}$	$\{2\}$
$\{0,1\}$	$\{0,1,2\}$	$\{2\}$
$\{0,2\}$ *	$\{0,1,2\}$	$\{2\}$
$\{1,2\}$ *	$\{1\}$	$\{2\}$
s: $\{0,1,2\}$ *	$\{0,1,2\}$	$\{2\}$

The Result

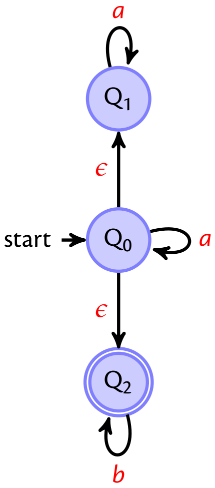


Removing Dead States

DFA:



(original) NFA:



Regexps and Automata

	Thompson's construction		subset construction	
Regexps	→	**NFAs**	→	**DFAs**

Regexps and Automata

Thompson's construction: **Regexps** → **NFAs**

subset construction: **NFAs** → **DFAs**

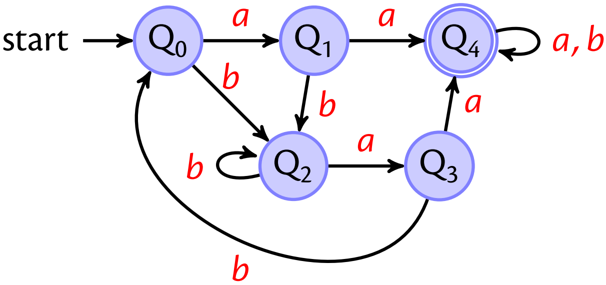
minimisation: **DFAs** → **minimal DFAs**

DFA Minimisation

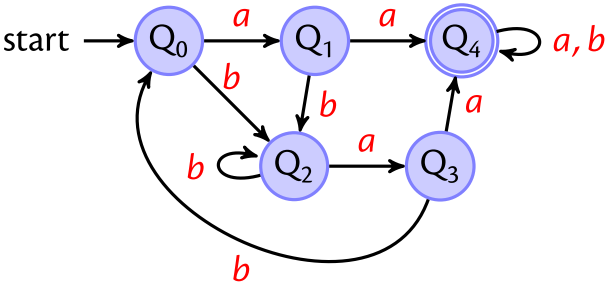
1. Take all pairs (q, p) with $q \neq p$
2. Mark all pairs that accepting and non-accepting states
3. For all unmarked pairs (q, p) and all characters c test whether

$$(\delta(q, c), \delta(p, c))$$

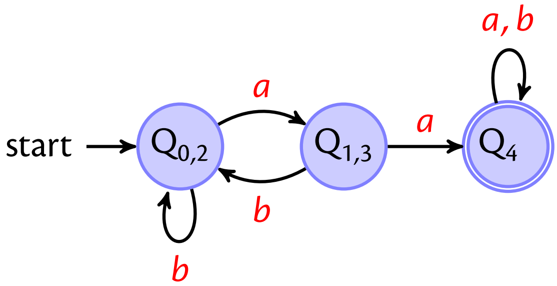
   are marked. If yes in at least one case, then also mark (q, p).
4. Repeat last step until no change.
5. All unmarked pairs can be merged.



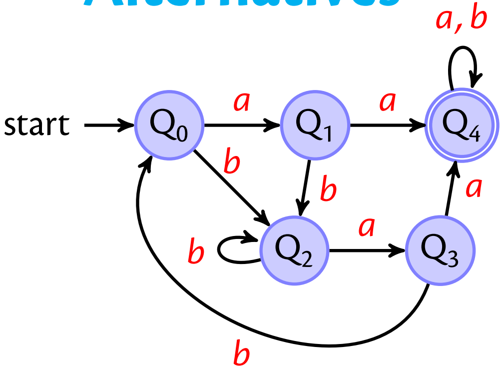
	Q_0	Q_1	Q_2	Q_3
Q_1				
Q_2				
Q_3				
Q_4	⋆	⋆	⋆	⋆



	Q_0	Q_1	Q_2	Q_3
Q_1	★			
Q_2		★		
Q_3	★		★	
Q_4	★	★	★	★

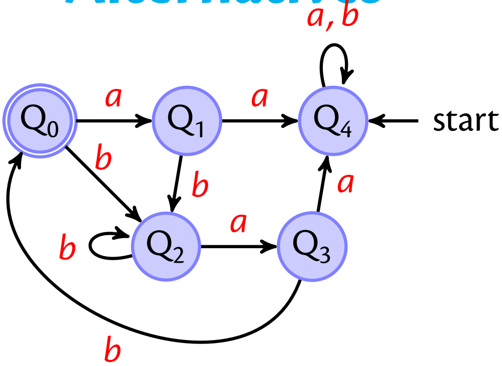


Alternatives



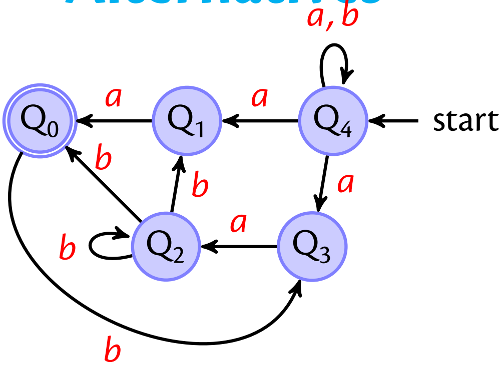
- exchange initial / accepting states

Alternatives



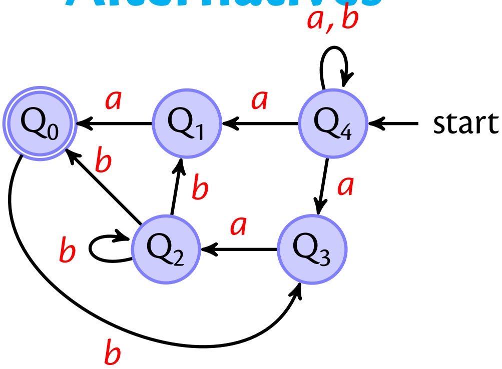
- exchange initial / accepting states
- reverse all edges

Alternatives



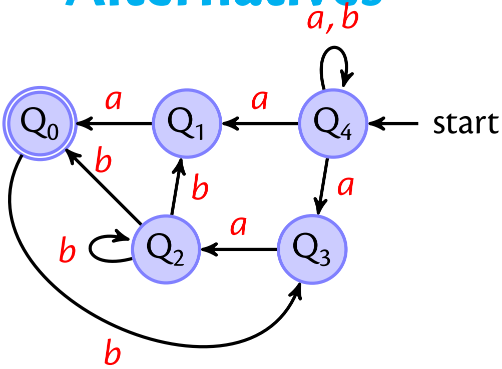
- exchange initial / accepting states
- reverse all edges
- subset construction $\Rightarrow$ DFA

Alternatives



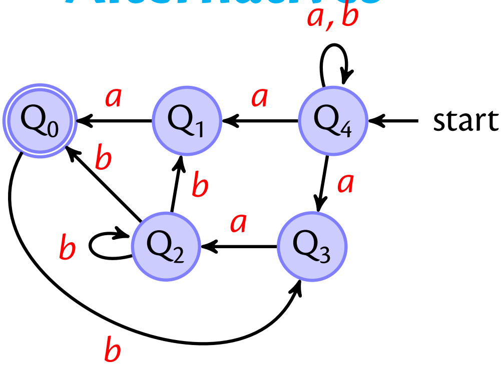
- exchange initial / accepting states
- reverse all edges
- subset construction ⇒ DFA
- remove dead states

Alternatives



- exchange initial / accepting states
- reverse all edges
- subset construction ⇒ DFA
- remove dead states
- repeat once more

Alternatives



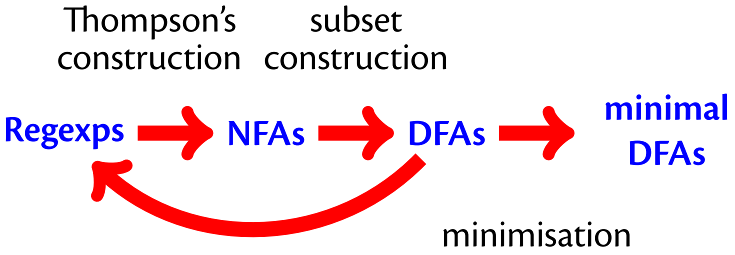
- exchange initial / accepting states
- reverse all edges
- subset construction $\Rightarrow$ DFA
- remove dead states
- repeat once more $\Rightarrow$ minimal DFA

Regexps and Automata

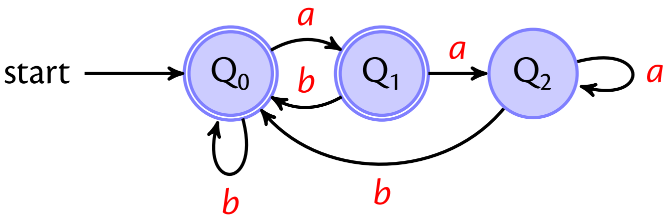
Thompson's construction | subset construction | minimisation

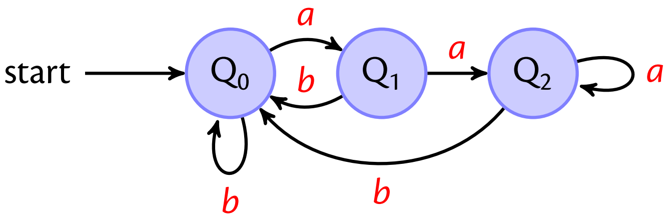
Regexps → **NFAs** → **DFAs** → **minimal DFAs**

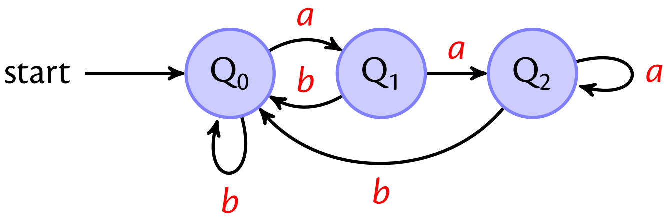
Regexps and Automata



DFA to Rexp

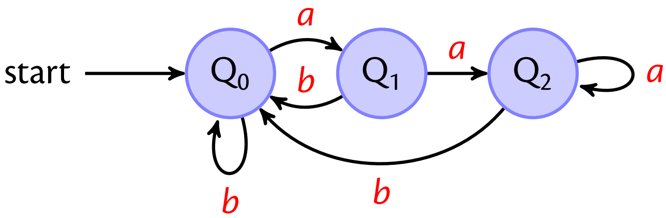


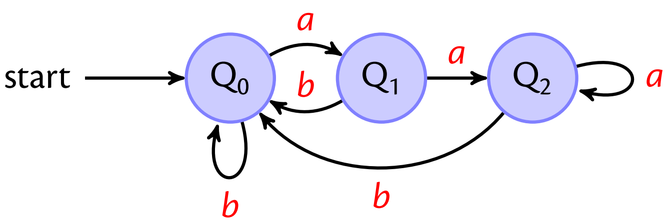

start
Q0
Q1
Q2
a
b
a
a
b
b



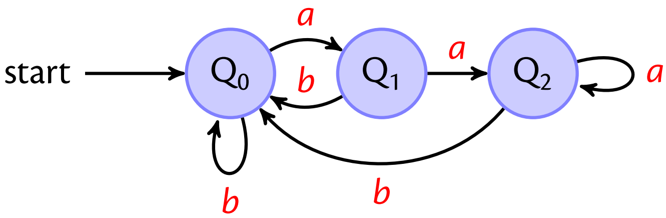
You know how to solve since school days, no?

$$
\begin{aligned}
Q_0 &= 2\,Q_0 + 3\,Q_1 + 4\,Q_2\\
Q_1 &= 2\,Q_0 + 3\,Q_1 + 1\,Q_2\\
Q_2 &= 1\,Q_0 + 5\,Q_1 + 2\,Q_2
\end{aligned}
$$


start
Q_0
Q_1
Q_2
a
a
a
b
b
b



$$
\begin{aligned}
Q_0 &= Q_0\,b + Q_1\,b + Q_2\,b + \mathbf{1}\\
Q_1 &= Q_0\,a\\
Q_2 &= Q_1\,a + Q_2\,a
\end{aligned}
$$

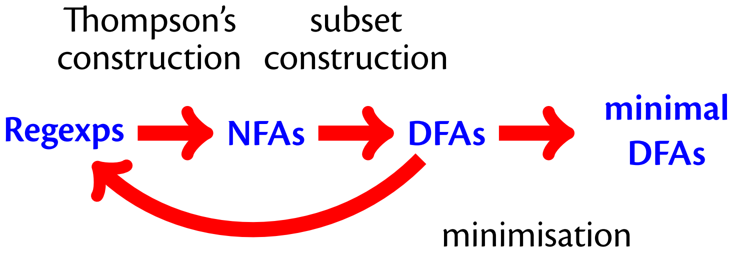


$$
\begin{aligned}
Q_0 &= Q_0\, b + Q_1\, b + Q_2\, b + \mathbf{1} \\
Q_1 &= Q_0\, a \\
Q_2 &= Q_1\, a + Q_2\, a
\end{aligned}
$$

Arden's Lemma:

If $q = q\, r + s$ then $q = s\, r^*$

Regexps and Automata



Regular Languages (3)

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or **equivalently**

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Why is every finite set of strings a regular language?

Given the function

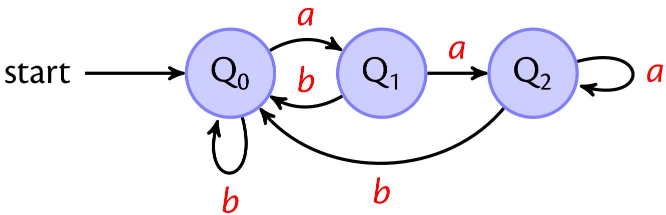
$$\begin{aligned}
rev(\mathbf{0}) &\stackrel{\text{def}}{=} \mathbf{0}\\
rev(\mathbf{1}) &\stackrel{\text{def}}{=} \mathbf{1}\\
rev(c) &\stackrel{\text{def}}{=} c\\
rev(r_1 + r_2) &\stackrel{\text{def}}{=} rev(r_1) + rev(r_2)\\
rev(r_1 \cdot r_2) &\stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1)\\
rev(r^*) &\stackrel{\text{def}}{=} rev(r)^*
\end{aligned}$$

and the set

$$Rev\, A \stackrel{\text{def}}{=} \{s^{-1} \mid s \in A\}$$

prove whether

$$L(rev(r)) = Rev(L(r))$$



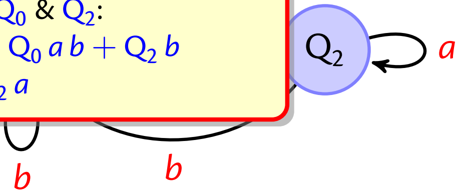
$$
\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\
Q_1 &= Q_0\, a \\
Q_2 &= Q_1\, a + Q_2\, a
\end{aligned}
$$

Arden's Lemma:

If $q = q\, r + s$ then $q = s\, r^*$

substitute Q_1 into Q_0 & Q_2:

$$Q_0 = \mathbf{1} + Q_0\, b + Q_0\, a\, b + Q_2\, b$$
$$Q_2 = Q_0\, a\, a + Q_2\, a$$



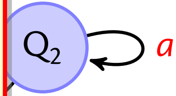
$$\begin{aligned} Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\ Q_1 &= Q_0\, a \\ Q_2 &= Q_1\, a + Q_2\, a \end{aligned}$$

Arden's Lemma:

If $q = q\, r + s$ then $q = s\, r^*$

substitute Q_1 into Q_0 & Q_2:

$$Q_0 = \mathbf{1} + Q_0\,b + Q_0\,a\,b + Q_2\,b$$
$$Q_2 = Q_0\,a\,a + Q_2\,a$$



simplifying Q_0:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\,b) + Q_2\,b$$
$$Q_2 = Q_0\,a\,a + Q_2\,a$$

$$
\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\,b + Q_1\,b + Q_2\,b\\
Q_1 &= Q_0\,a\\
Q_2 &= Q_1\,a + Q_2\,a
\end{aligned}
$$

Arden's Lemma:

If $q = q\,r + s$ then $q = s\,r^*$

substitute Q_1 into Q_0 & Q_2:

$$Q_0 = \mathbf{1} + Q_0\, b + Q_0\, a\, b + Q_2\, b$$
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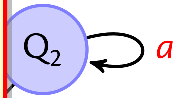

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Arden for Q_2:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\, b) + Q_2\, b$$
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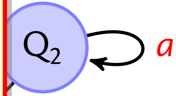

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Substitute Q_2 and simplify:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\, b + a\, a\,(a^*)\, b)$$

Arden's Lemma:

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substitute Q_1 into Q_0 & Q_2:

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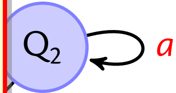

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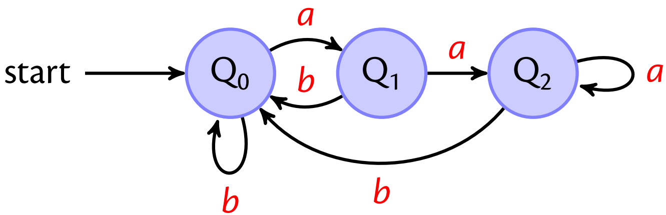
Arden's Lemma:

If $q = q\,r$

Arden again for Q_0:

$$Q_0 = (b + a\,b + a\,a\,(a^*)\,b)^*$$



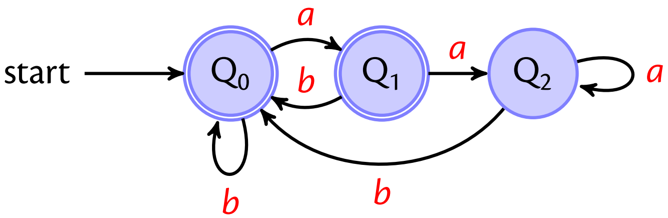
$$\begin{aligned} Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\ Q_1 &= Q_0\, a \\ Q_2 &= Q_1\, a + Q_2\, a \end{aligned}$$

Arden's Lemma:

If $q = q\, r + s$

Finally:

$$\begin{aligned} Q_0 &= (b + a\, b + a\, a\, (a^*)\, b)^* \\ Q_1 &= (b + a\, b + a\, a\, (a^*)\, b)^*\, a \\ Q_2 &= (b + a\, b + a\, a\, (a^*)\, b)^*\, a\, a\, (a^*) \end{aligned}$$



$$\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\
Q_1 &= Q_0\, a \\
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Arden's Lemma:

If $q = q\, r + s$

Finally:

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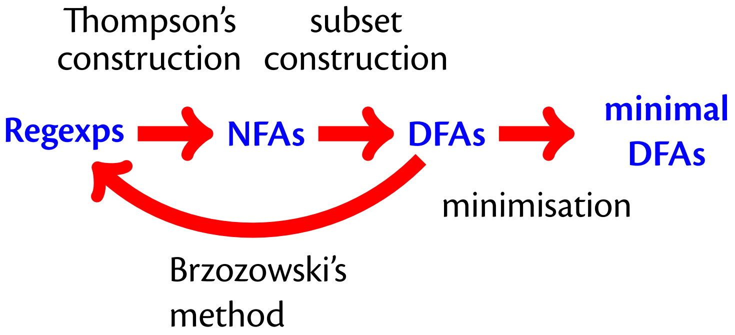
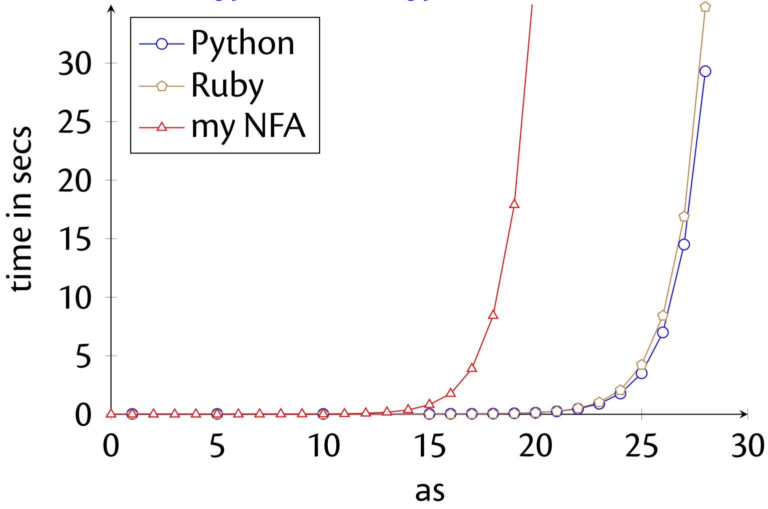
Q_0 &= (b + a\, b + a\, a\, (a^*)\, b)^* \\
Q_1 &= (b + a\, b + a\, a\, (a^*)\, b)^*\, a \\
Q_2 &= (b + a\, b + a\, a\, (a^*)\, b)^*\, a\, a\, (a^*)
\end{aligned}$$

Regexps and Automata



$$a^{?\{n\}} \cdot a^{\{n\}}$$



The punchline is that many existing libraries do depth-first search in NFAs (backtracking).

Regular Languages

Two equivalent definitions:

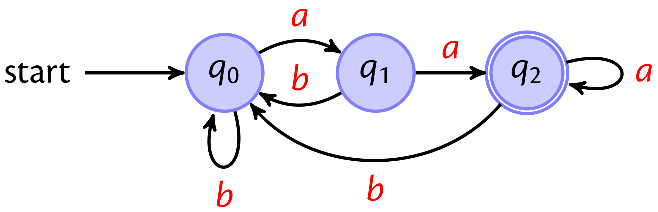
A language is regular iff there exists a regular expression that recognises all its strings.

A language is regular iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

Negation

Regular languages are closed under negation:



But requires that the automaton is completed!

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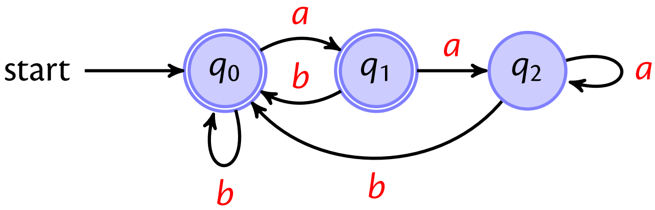

But requires that the automaton is completed!