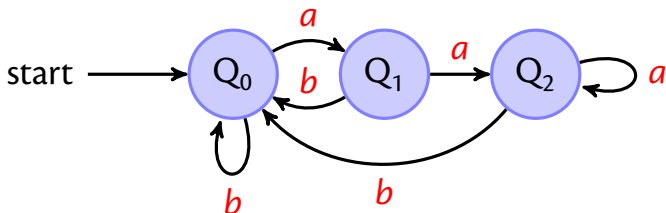


Compilers and Formal Languages (4)

Email: christian.urban at kcl.ac.uk
Office Hours: Thursdays 12 – 14
Location: N7.07 (North Wing, Bush House)
Slides & Progs: KEATS (also homework is there)



$$Q_0 = \mathbf{1} + Q_0 b + Q_1 b + Q_2 b$$

$$Q_1 = Q_0 a$$

$$Q_2 = Q_1 a + Q_2 a$$

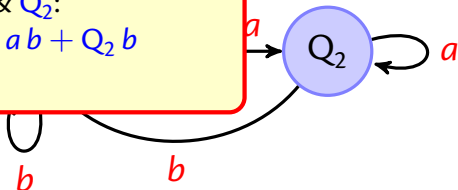
Arden's Lemma:

$$\text{If } q = qr + s \text{ then } q = sr^*$$

substitute Q_1 into Q_0 & Q_2 :

$$Q_0 = \mathbf{1} + Q_0 b + Q_0 a b + Q_2 b$$

$$Q_2 = Q_0 a a + Q_2 a$$



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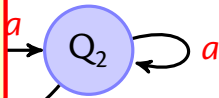
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simplifying Q_0 :

$$Q_0 = 1 + Q_0 (b + a b) + Q_2 b$$

$$Q_2 = Q_0 a a + Q_2 a$$

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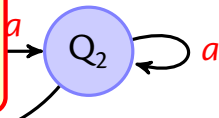
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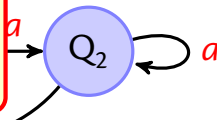
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Arden's Lemma

Substitute Q_2 and simplify:

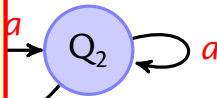
$$Q_0 = 1 + Q_0 (b + a b + a a (a^*) b)$$

$$\text{If } q = q r + s \text{ then } q = s r^*$$

substitute Q_1 into Q_0 & Q_2 :

$$Q_0 = 1 + Q_0 b + Q_0 a b + Q_2 b$$

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Arden's Lem

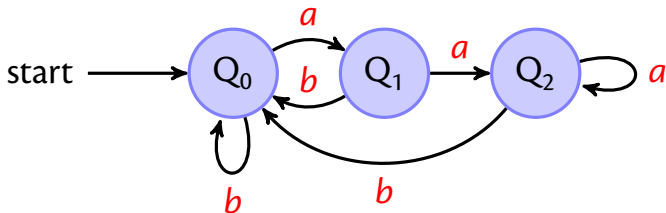
Substitute Q_2 and simplify:

$$Q_0 = 1 + Q_0 (b + a b + a a (a^*) b)$$

If

Arden again for Q_0 :

$$Q_0 = (b + a b + a a (a^*) b)^*$$



$$Q_0 = 1 + Q_0 b + Q_1 b + Q_2 b$$

$$Q_1 = Q_0 a$$

$$Q_2 = Q_1 a + Q_2 a$$

Arden's Lemma:

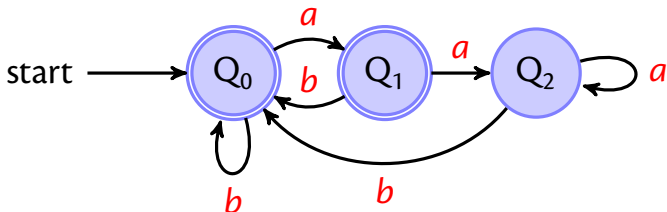
If $q =$

Finally:

$$Q_0 = (b + ab + aa(a^*)b)^*$$

$$Q_1 = (b + ab + aa(a^*)b)^* a$$

$$Q_2 = (b + ab + aa(a^*)b)^* aa(a^*)$$



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Coursework

- $\text{derc}(r^+) \stackrel{\text{def}}{=} \text{derc}(r \cdot r^*)$ given that $r^+ \stackrel{\text{def}}{=} r \cdot r^*$

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$$derc(r \cdot r^*) \stackrel{\text{def}}{=} (derc r) \cdot r^*$$

Coursework (2)

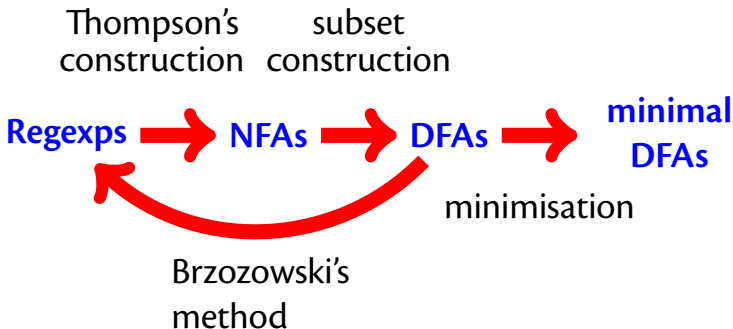
- CFUN(f: Char => Boolean)

CHAR(c: Char) $\stackrel{\text{def}}{=}$
CFUN(_ == c)

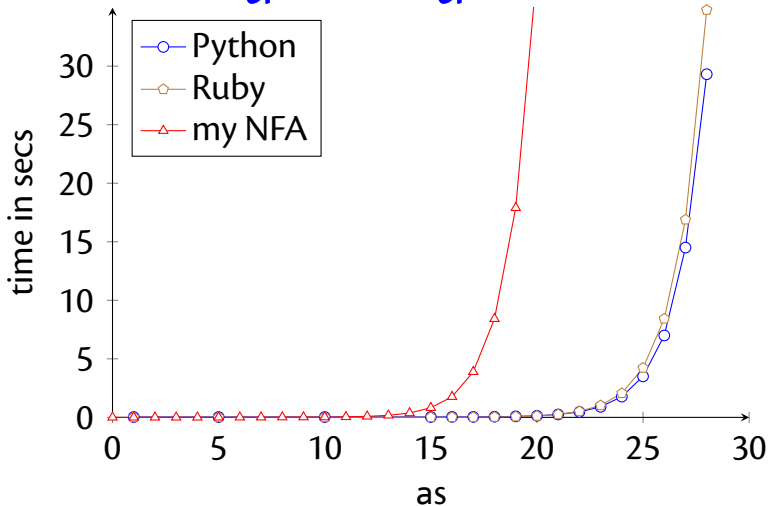
RANGE(cs: Set[Char]) $\stackrel{\text{def}}{=}$
CFUN(cs.contains(_))

ALL $\stackrel{\text{def}}{=}$
CFUN((c: Char) => true)

Regexps and Automata



$$a^{\{n\}} \cdot a^{\{n\}}$$



The punchline is that many existing libraries do depth-first search in NFAs (backtracking).

Regular Languages

Two equivalent definitions:

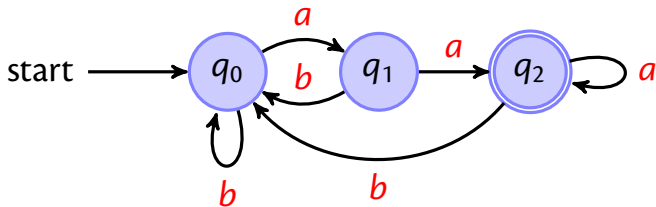
A language is **regular** iff there exists a regular expression that recognises all its strings.

A language is **regular** iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

Negation

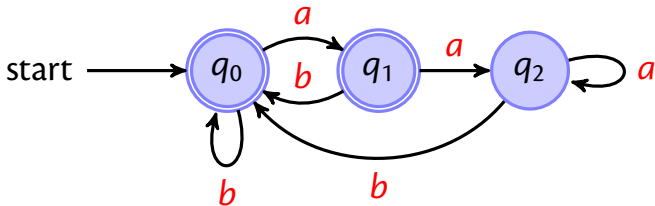
Regular languages are closed under negation:



But requires that the automaton is **completed!**

Negation

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But requires that the automaton is **completed!**

The Goal of this Course

Write a compiler



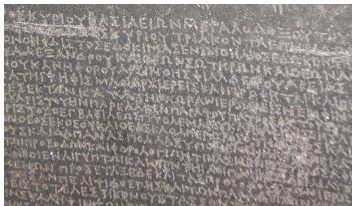
Today a lexer.

The Goal of this Course

Write a compiler



Today a lexer.



lexing \Rightarrow recognising words (Stone of Rosetta)

Regular Expressions

In programming languages they are often used to recognise:

- operands, digits
- identifiers
- numbers (non-leading zeros)
- keywords
- comments

<http://www.regexper.com>

Lexing: Test Case

```
write "Fib";  
read n;  
minus1 := 0;  
minus2 := 1;  
while n > 0 do {  
    temp := minus2;  
    minus2 := minus1 + minus2;  
    minus1 := temp;  
    n := n - 1  
};  
write "Result";  
write minus2
```


"if true then then 42 else +"

KEYWORD:

if, then, else,

WHITESPACE:

" ", \n,

IDENTIFIER:

LETTER · (LETTER + DIGIT + _)*

NUM:

(NONZERODIGIT · DIGIT*) + 0

OP:

+, -, *, %, <, <=

COMMENT:

/* · ~ (ALL* · (* /) · ALL*) · */

"if true then then 42 else +"

KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)

"if true then then 42 else +"

KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)

There is one small problem with the tokenizer. How should we tokenize...?

"x-3"

ID: ...

OP:

"+", "-"

NUM:

(NONZERODIGIT · DIGIT*) + ''0''

NUMBER:

NUM + ("-" · NUM)

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string *abc*.

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$$(ab + a) \cdot (c + bc)$$

and the string *abc*.

Or, keywords are **if** etc and identifiers are letters followed by “letters + numbers + _”*

if *iffoo*

POSIX: Two Rules

- Longest match rule (“maximal munch rule”): The longest initial substring matched by any regular expression is taken as the next token.
- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

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http://www.haskell.org/haskellwiki/Regex_Posix

traditional lexers are fast, but hairy

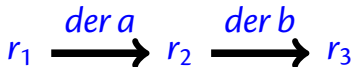
Sulzmann & Lu Matcher

We want to match the string *abc* using r_1 :



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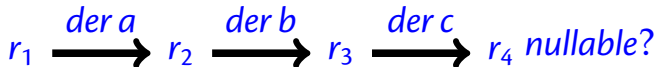
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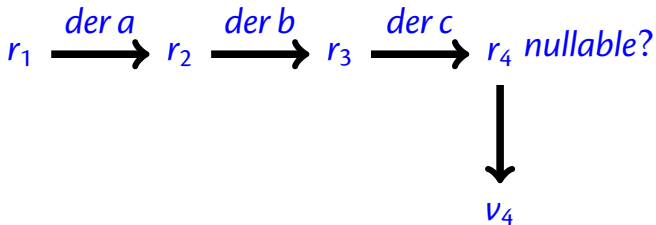
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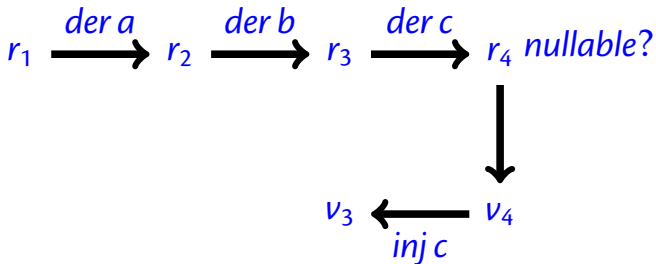
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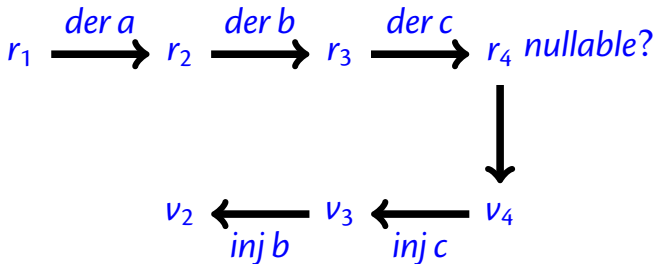
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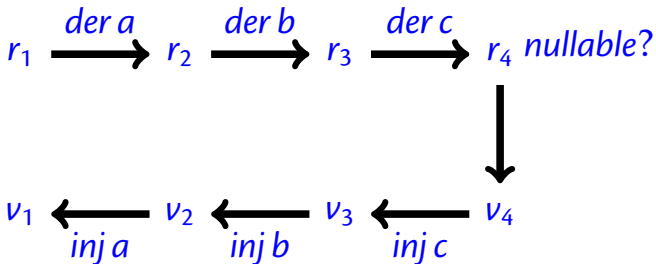
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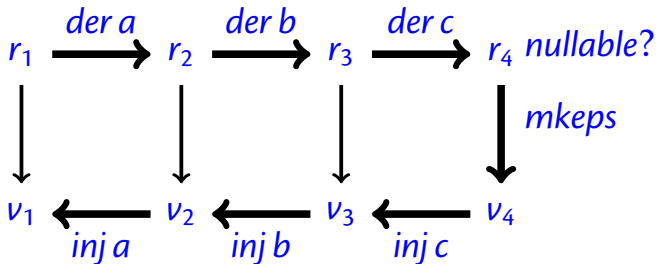
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Regexes and Values

Regular expressions and their corresponding values:

$r ::=$	0	$v ::=$	<i>Empty</i>
	1		<i>Char(c)</i>
	<i>c</i>		<i>Seq(v₁, v₂)</i>
	$r_1 \cdot r_2$		<i>Left(v)</i>
	$r_1 + r_2$		<i>Right(v)</i>
	r^*		<i>Stars []</i>
			<i>Stars [v₁, ... v_n]</i>

```
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

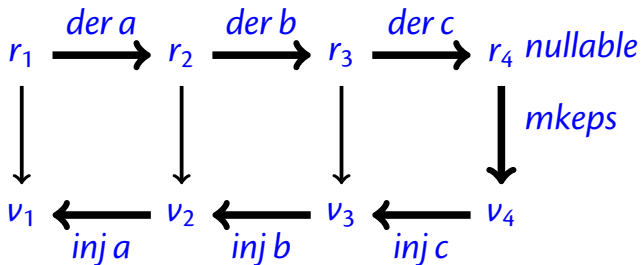
```
abstract class Val
case object Empty extends Val
case class Chr(c: Char) extends Val
case class Sequ(v1: Val, v2: Val) extends Val
case class Left(v: Val) extends Val
case class Right(v: Val) extends Val
case class Stars(vs: List[Val]) extends Val
```

$$r_1: a \cdot (b \cdot c)$$

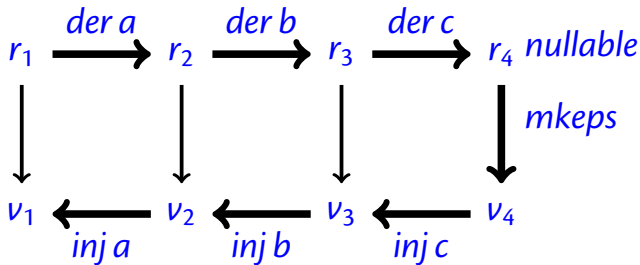
$$r_2: 1 \cdot (b \cdot c)$$

$$r_3: (0 \cdot (b \cdot c)) + (1 \cdot c)$$

$$r_4: (0 \cdot (b \cdot c)) + ((0 \cdot c) + 1)$$



$$\begin{aligned}
 r_1: & a \cdot (b \cdot c) \\
 r_2: & \mathbf{1} \cdot (b \cdot c) \\
 r_3: & (\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c) \\
 r_4: & (\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})
 \end{aligned}$$



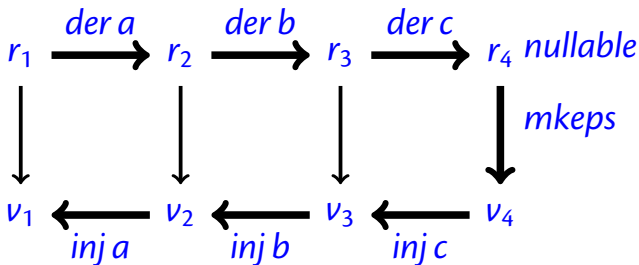
$$\begin{aligned}
 v_1: & \text{Seq}(\text{Char}(a), \text{Seq}(\text{Char}(b), \text{Char}(c))) \\
 v_2: & \text{Seq}(\text{Empty}, \text{Seq}(\text{Char}(b), \text{Char}(c))) \\
 v_3: & \text{Right}(\text{Seq}(\text{Empty}, \text{Char}(c))) \\
 v_4: & \text{Right}(\text{Right}(\text{Empty}))
 \end{aligned}$$

Flatten

Obtaining the string underlying a value:

$ Empty $	$\stackrel{\text{def}}{=} []$
$ Char(c) $	$\stackrel{\text{def}}{=} [c]$
$ Left(v) $	$\stackrel{\text{def}}{=} v $
$ Right(v) $	$\stackrel{\text{def}}{=} v $
$ Seq(v_1, v_2) $	$\stackrel{\text{def}}{=} v_1 @ v_2 $
$ [v_1, \dots, v_n] $	$\stackrel{\text{def}}{=} v_1 @ \dots @ v_n $

$r_1: a \cdot (b \cdot c)$
 $r_2: 1 \cdot (b \cdot c)$
 $r_3: (0 \cdot (b \cdot c)) + (1 \cdot c)$
 $r_4: (0 \cdot (b \cdot c)) + ((0 \cdot c) + 1)$



$v_1: \text{Seq}(\text{Char}(a), \text{Seq}(\text{Char}(b), \text{Char}(c)))$
 $v_2: \text{Seq}(\text{Empty}, \text{Seq}(\text{Char}(b), \text{Char}(c)))$
 $v_3: \text{Right}(\text{Seq}(\text{Empty}, \text{Char}(c)))$
 $v_4: \text{Right}(\text{Right}(\text{Empty}))$

$v_1: abc$
 $v_2: bc$
 $v_3: c$
 $v_4: []$

Mkeps

Finding a (posix) value for recognising the empty string:

$mkeps(1) \stackrel{\text{def}}{=} Empty$

$mkeps(r_1 + r_2) \stackrel{\text{def}}{=} \begin{cases} \text{if } nullable(r_1) \\ \text{then } Left(mkeps(r_1)) \\ \text{else } Right(mkeps(r_2)) \end{cases}$

$mkeps(r_1 \cdot r_2) \stackrel{\text{def}}{=} Seq(mkeps(r_1), mkeps(r_2))$

$mkeps(r^*) \stackrel{\text{def}}{=} Stars []$

Inject

Injecting (“Adding”) a character to a value

$inj(c) c (Empty)$	$\stackrel{\text{def}}{=} Char\ c$
$inj(r_1 + r_2) c (Left(v))$	$\stackrel{\text{def}}{=} Left(inj\ r_1\ c\ v)$
$inj(r_1 + r_2) c (Right(v))$	$\stackrel{\text{def}}{=} Right(inj\ r_2\ c\ v)$
$inj(r_1 \cdot r_2) c (Seq(v_1, v_2))$	$\stackrel{\text{def}}{=} Seq(inj\ r_1\ c\ v_1, v_2)$
$inj(r_1 \cdot r_2) c (Left(Seq(v_1, v_2)))$	$\stackrel{\text{def}}{=} Seq(inj\ r_1\ c\ v_1, v_2)$
$inj(r_1 \cdot r_2) c (Right(v))$	$\stackrel{\text{def}}{=} Seq(mkeps(r_1), inj\ r_2\ c\ v)$
$inj(r^*) c (Seq(v, Stars\ vs))$	$\stackrel{\text{def}}{=} Stars(inj\ r\ c\ v :: vs)$

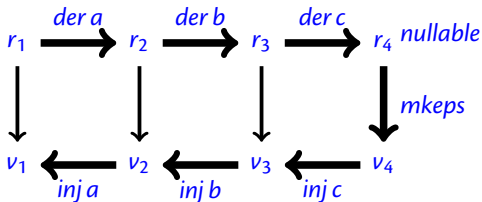
inj: 1st arg \mapsto a rexp; 2nd arg \mapsto a character; 3rd arg \mapsto a value
result \mapsto a value

Lexing

$\text{lex } r \ [] \stackrel{\text{def}}{=} \text{if nullable}(r) \text{ then mkeps}(r) \text{ else error}$

$\text{lex } r \ c :: s \stackrel{\text{def}}{=} \text{inj } r \ c \ \text{lex}(\text{der}(c, r), s)$

lex: returns a value



Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

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- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $derc(x : r) \stackrel{\text{def}}{=} derc r$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj(x : r) c v \stackrel{\text{def}}{=} Rec(x, inj r c v)$

Records

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- $inj(x : r) c v \stackrel{\text{def}}{=} Rec(x, inj r c v)$

for extracting subpatterns $(z : ((x : ab) + (y : ba)))$

- A regular expression for email addresses

(name: $[a-z0-9_.-]^+$).@.
(domain: $[a-z0-9.-]^+$) ..
(top_level: $[a-z.]{2,6}$)

christian.urban@kcl.ac.uk

- the result environment:

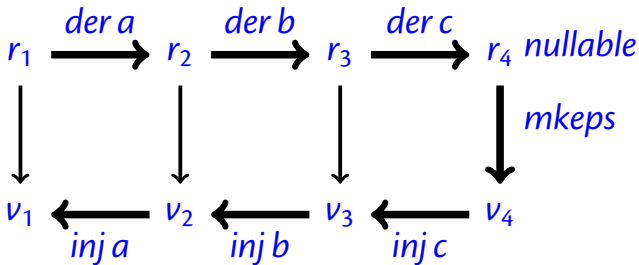
$[(name : christian.urban),$
 $(domain : kcl),$
 $(top_level : ac.uk)]$

While Tokens

WHILE_REGS $\stackrel{\text{def}}{=}$ (("k" : KEYWORD) +
("i" : ID) +
("o" : OP) +
("n" : NUM) +
("s" : SEMI) +
("p" : (LPAREN + RPAREN)) +
("b" : (BEGIN + END)) +
("w" : WHITESPACE))*

Simplification

- If we simplify after the derivative, then we are building the value for the simplified regular expression, but **not** for the original regular expression.



$$(0 \cdot (b \cdot c)) + ((0 \cdot c) + 1) \mapsto 1$$

Normally we would have

$$(0 \cdot (b \cdot c)) + ((0 \cdot c) + 1)$$

and answer how this regular expression matches the empty string with the value

$$\textit{Right}(\textit{Right}(\textit{Empty}))$$

But now we simplify this to **1** and would produce *Empty* (see *mkeps*).

Rectification

rectification
functions:

$$r \cdot \mathbf{0} \mapsto \mathbf{0}$$

$$\mathbf{0} \cdot r \mapsto \mathbf{0}$$

$$r \cdot \mathbf{1} \mapsto r \quad \lambda f_1 f_2 v. \text{Seq}(f_1 v, f_2 \text{Empty})$$

$$\mathbf{1} \cdot r \mapsto r \quad \lambda f_1 f_2 v. \text{Seq}(f_1 \text{Empty}, f_2 v)$$

$$r + \mathbf{0} \mapsto r \quad \lambda f_1 f_2 v. \text{Left}(f_1 v)$$

$$\mathbf{0} + r \mapsto r \quad \lambda f_1 f_2 v. \text{Right}(f_2 v)$$

$$r + r \mapsto r \quad \lambda f_1 f_2 v. \text{Left}(f_1 v)$$

Rectification

rectification
functions:

$$r \cdot \mathbf{0} \mapsto \mathbf{0}$$

$$\mathbf{0} \cdot r \mapsto \mathbf{0}$$

$$r \cdot \mathbf{1} \mapsto r \quad \lambda f_1 f_2 v. \text{Seq}(f_1 v, f_2 \text{Empty})$$

$$\mathbf{1} \cdot r \mapsto r \quad \lambda f_1 f_2 v. \text{Seq}(f_1 \text{Empty}, f_2 v)$$

$$r + \mathbf{0} \mapsto r \quad \lambda f_1 f_2 v. \text{Left}(f_1 v)$$

$$\mathbf{0} + r \mapsto r \quad \lambda f_1 f_2 v. \text{Right}(f_2 v)$$

$$r + r \mapsto r \quad \lambda f_1 f_2 v. \text{Left}(f_1 v)$$

old *simp* returns a rexp;

new *simp* returns a rexp and a rectification function.

Rectification

$\text{simp}(r)$:

case $r = r_1 + r_2$

let $(r_{1s}, f_{1s}) = \text{simp}(r_1)$

$(r_{2s}, f_{2s}) = \text{simp}(r_2)$

case $r_{1s} = \mathbf{0}$: return $(r_{2s}, \lambda v. \text{Right}(f_{2s}(v)))$

case $r_{2s} = \mathbf{0}$: return $(r_{1s}, \lambda v. \text{Left}(f_{1s}(v)))$

case $r_{1s} = r_{2s}$: return $(r_{1s}, \lambda v. \text{Left}(f_{1s}(v)))$

otherwise: return $(r_{1s} + r_{2s}, f_{\text{alt}}(f_{1s}, f_{2s}))$

$f_{\text{alt}}(f_1, f_2) \stackrel{\text{def}}{=}$

$\lambda v. \text{case } v = \text{Left}(v') : \text{return } \text{Left}(f_1(v'))$

$\text{case } v = \text{Right}(v') : \text{return } \text{Right}(f_2(v'))$

```

def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case ALT(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (r2s, F_RIGHT(f2s))
      case (_, ZERO) => (r1s, F_LEFT(f1s))
      case _ =>
        if (r1s == r2s) (r1s, F_LEFT(f1s))
        else (ALT (r1s, r2s), F_ALT(f1s, f2s))
    }
  }
  ...
}

```

```

def F_RIGHT(f: Val => Val) = (v:Val) => Right(f(v))
def F_LEFT(f: Val => Val) = (v:Val) => Left(f(v))
def F_ALT(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Right(v) => Right(f2(v))
    case Left(v) => Left(f1(v)) }

```

Rectification

$simp(r):...$

case $r = r_1 \cdot r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{2s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{1s} = \mathbf{1}$: return $(r_{2s}, \lambda v. Seq(f_{1s}(Empty), f_{2s}(v)))$

case $r_{2s} = \mathbf{1}$: return $(r_{1s}, \lambda v. Seq(f_{1s}(v), f_{2s}(Empty)))$

otherwise: return $(r_{1s} \cdot r_{2s}, f_{seq}(f_{1s}, f_{2s}))$

$f_{seq}(f_1, f_2) \stackrel{\text{def}}{=}$

$\lambda v. \text{case } v = Seq(v_1, v_2): \text{return } Seq(f_1(v_1), f_2(v_2))$

```

def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case SEQ(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (ZERO, F_ERROR)
      case (_, ZERO) => (ZERO, F_ERROR)
      case (ONE, _) => (r2s, F_SEQ_Void1(f1s, f2s))
      case (_, ONE) => (r1s, F_SEQ_Void2(f1s, f2s))
      case _ => (SEQ(r1s, r2s), F_SEQ(f1s, f2s))
    }
  }
  ...
}

```

```

def F_SEQ_Void1(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(Void), f2(v))

```

```

def F_SEQ_Void2(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(v), f2(Void))

```

```

def F_SEQ(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Sequ(v1, v2) => Sequ(f1(v1), f2(v2)) }

```


Rectification Example

$$(b \cdot c) + (\mathbf{0} + \mathbf{1}) \mapsto (b \cdot c) + \mathbf{1}$$

Rectification Example

$$(\underline{b \cdot c}) + (\underline{0 + 1}) \mapsto (b \cdot c) + 1$$

Rectification Example

$$(\underline{b \cdot c}) + (\underline{0 + 1}) \mapsto (b \cdot c) + 1$$

$$\begin{aligned} f_{s1} &= \lambda v. v \\ f_{s2} &= \lambda v. \text{Right}(v) \end{aligned}$$

Rectification Example

$$\underline{(b \cdot c) + (0 + 1)} \mapsto (b \cdot c) + 1$$

$$f_{s1} = \lambda v.v$$

$$f_{s2} = \lambda v.Right(v)$$

$$f_{alt}(f_{s1}, f_{s2}) \stackrel{\text{def}}{=}$$

$\lambda v.$ case $v = Left(v')$: return $Left(f_{s1}(v'))$

case $v = Right(v')$: return $Right(f_{s2}(v'))$

Rectification Example

$$\underline{(b \cdot c) + (0 + 1)} \mapsto (b \cdot c) + 1$$

$$f_{s1} = \lambda v.v$$

$$f_{s2} = \lambda v.Right(v)$$

$\lambda v.$ case $v = Left(v')$: return $Left(v')$
case $v = Right(v')$: return $Right(Right(v'))$

Rectification Example

$$\underline{(b \cdot c) + (0 + 1)} \mapsto (b \cdot c) + 1$$

$$\begin{aligned} f_{s1} &= \lambda v.v \\ f_{s2} &= \lambda v.Right(v) \end{aligned}$$

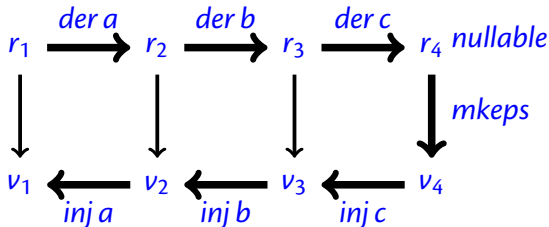
$\lambda v.$ case $v = Left(v')$: return $Left(v')$
case $v = Right(v')$: return $Right(Right(v'))$

mkeps simplified case: $Right(Empty)$
rectified case: $Right(Right(Empty))$

Lexing with Simplification

$\text{lex } r \ [] \stackrel{\text{def}}{=} \text{if nullable}(r) \text{ then } \text{mkeps}(r) \text{ else error}$

$\text{lex } r \ c \ :: \ s \stackrel{\text{def}}{=} \text{let } (r', \text{frect}) = \text{simp}(\text{der}(c, r))$
 $\text{inj } r \ c \ (\text{frect}(\text{lex}(r', s)))$



Environments

Obtaining the “recorded” parts of a value:

$env(Empty)$	$\stackrel{\text{def}}{=} []$
$env(Char(c))$	$\stackrel{\text{def}}{=} []$
$env(Left(v))$	$\stackrel{\text{def}}{=} env(v)$
$env(Right(v))$	$\stackrel{\text{def}}{=} env(v)$
$env(Seq(v_1, v_2))$	$\stackrel{\text{def}}{=} env(v_1) @ env(v_2)$
$env(Stars [v_1, \dots, v_n])$	$\stackrel{\text{def}}{=} env(v_1) @ \dots @ env(v_n)$
$env(Rec(x : v))$	$\stackrel{\text{def}}{=} (x : v) :: env(v)$

While Tokens

WHILE_REGS $\stackrel{\text{def}}{=} ((\text{"k"} : \text{KEYWORD}) +$
 $(\text{"i"} : \text{ID}) +$
 $(\text{"o"} : \text{OP}) +$
 $(\text{"n"} : \text{NUM}) +$
 $(\text{"s"} : \text{SEMI}) +$
 $(\text{"p"} : (\text{LPAREN} + \text{RPAREN})) +$
 $(\text{"b"} : (\text{BEGIN} + \text{END})) +$
 $(\text{"w"} : \text{WHITESPACE}))^*$

"if true then then 42 else +"

KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)

"if true then then 42 else +"

KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)

Lexer: Two Rules

- Longest match rule (“maximal munch rule”): The longest initial substring matched by any regular expression is taken as next token.
- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

$$\begin{aligned}
\text{zeroable}(\mathbf{0}) &\stackrel{\text{def}}{=} \text{true} \\
\text{zeroable}(\mathbf{1}) &\stackrel{\text{def}}{=} \text{false} \\
\text{zeroable}(c) &\stackrel{\text{def}}{=} \text{false} \\
\text{zeroable}(r_1 + r_2) &\stackrel{\text{def}}{=} \text{zeroable}(r_1) \wedge \text{zeroable}(r_2) \\
\text{zeroable}(r_1 \cdot r_2) &\stackrel{\text{def}}{=} \text{zeroable}(r_1) \vee \text{zeroable}(r_2) \\
\text{zeroable}(r^*) &\stackrel{\text{def}}{=} \text{false}
\end{aligned}$$

$\text{zeroable}(r)$ if and only if $L(r) = \{\}$