Automata and Formal Languages (10)

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Slides: KEATS (also home work is there)

There are more problems, than there are programs.

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There must be a problem for which there is no program.

Revision: Proofs

Subsets

$$A \subseteq B$$

$$\forall e.\ e \in A \Rightarrow e \in B$$

Subsets

$A \subseteq B$ and $B \subseteq A$

then $A = B$

Injective Function

f is an injective function iff

$$\forall x y.\ f(x) = f(y) \Rightarrow x = y$$

Cardinality

$|A| \stackrel{\text{def}}{=}$ "how many elements"

$A \subseteq B \Rightarrow |A| \leq |B|$

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$|A| \stackrel{\text{def}}{=}$ "how many elements"

$A \subseteq B \Rightarrow |A| \leq |B|$

if there is an injective function $f : A \to B$ then $|A| \leq |B|$

Natural Numbers

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\ldots\}$$

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$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\ldots\}$

A is countable iff $|A| \leq |\mathbb{N}|$

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

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$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\{1, 3, 5......\}$

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$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\{1, 3, 5......\}$
$\mathbb{E} \stackrel{\text{def}}{=}$ even numbers $\{0, 2, 4......\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{N} \stackrel{\text{def}}{=}$ positive numbers $\{0, 1, 2, 3, \ldots\ldots\}$

$-\mathbb{N} \stackrel{\text{def}}{=}$ negative numbers $\{0, -1, -2, -3, \ldots\ldots\}$

A is countable if there exists an injective $f : A \to \mathbb{N}$

A is uncountable if there does not exist an injective $f : A \to \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$
uncountable: $|A| > |\mathbb{N}|$

A is countable if there exists an injective $f : A \rightarrow \mathbb{N}$

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Does there exist such an A ?

Halting Problem

Assume a program $\boldsymbol{H}$ that decides for all programs $\boldsymbol{A}$ and all input data $\boldsymbol{D}$ whether

- $\boldsymbol{H}(\boldsymbol{A}, \boldsymbol{D}) \stackrel{\text{def}}{=} 1$ iff $\boldsymbol{A}(\boldsymbol{D})$ terminates
- $\boldsymbol{H}(\boldsymbol{A}, \boldsymbol{D}) \stackrel{\text{def}}{=} 0$ otherwise

Halting Problem (2)

Given such a program $\boldsymbol{H}$ define the following program $\boldsymbol{C}$: for all programs $\boldsymbol{A}$

- $\boldsymbol{C}(\boldsymbol{A}) \stackrel{\text{def}}{=} 0$ iff $\boldsymbol{H}(\boldsymbol{A}, \boldsymbol{A}) = 0$
- $\boldsymbol{C}(\boldsymbol{A}) \stackrel{\text{def}}{=}$ loops otherwise

Contradiction

$H(C, C)$ is either 0 or 1.

- $H(C, C) = 1 \overset{\text{def}\, H}{\Rightarrow} C(C) \downarrow \overset{\text{def}\, C}{\Rightarrow} H(C, C) = 0$
- $H(C, C) = 0 \overset{\text{def}\, H}{\Rightarrow} C(C)$ loops $\overset{\text{def}\, C}{\Rightarrow}$ $H(C, C) = 1$

Contradiction in both cases. So H cannot exist.