Compilers and Formal Languages (4)

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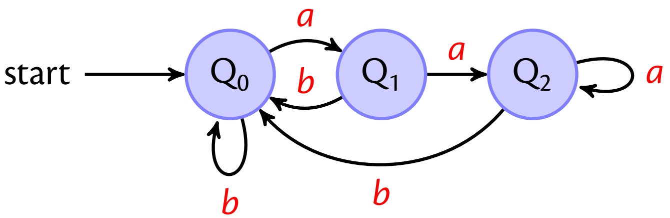
Slides & Progs:	KEATS (also homework is there)



$$
\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\
Q_1 &= Q_0\, a \\
Q_2 &= Q_1\, a + Q_2\, a
\end{aligned}
$$

Arden's Lemma:

If $q = q\, r + s$ then $q = s\, r^*$

substitute Q_1 into Q_0 & Q_2:

$$Q_0 = \mathbf{1} + Q_0\, b + Q_0\, a\, b + Q_2\, b$$
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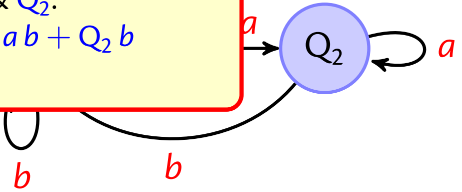

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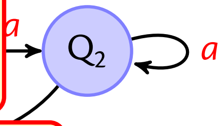

simplifying Q_0:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\, b) + Q_2\, b$$
$$Q_2 = Q_0\, a\, a + Q_2\, a$$

$$\begin{aligned} Q_0 &= \mathbf{1} + Q_0\, b + Q_1\, b + Q_2\, b \\ Q_1 &= Q_0\, a \\ Q_2 &= Q_1\, a + Q_2\, a \end{aligned}$$

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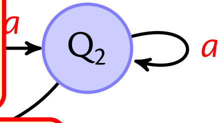

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Arden for Q_2:

$$Q_0 = \mathbf{1} + Q_0\, (b + a\, b) + Q_2\, b$$
$$Q_2 = Q_0\, a\, a\, (a^*)$$

Arden's Lemma:

$$\text{If } q = q\, r + s \;\text{ then }\; q = s\, r^*$$

substitute Q_1 into Q_0 & Q_2:

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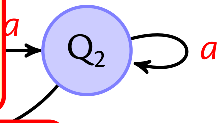

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Arden for Q_2:

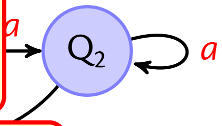
$$Q_0 = \mathbf{1} + Q_0\,(b + a\,b) + Q_2\,b$$
$$Q_2 = Q_0\,a\,a\,(a^*)$$

Substitute Q_2 and simplify:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\,b + a\,a\,(a^*)\,b)$$

Arden's Lem

If $q = q\,r + s$ then $q = s\,r^*$



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Arden's Lem

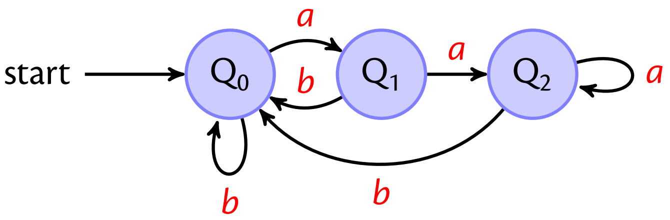
Substitute Q_2 and simplify:

$$Q_0 = \mathbf{1} + Q_0\,(b + a\,b + a\,a\,(a^*)\,b)$$

If

Arden again for Q_0:

$$Q_0 = (b + a\,b + a\,a\,(a^*)\,b)^*$$



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\begin{aligned}
Q_0 &= \mathbf{1} + Q_0\,b + Q_1\,b + Q_2\,b\\
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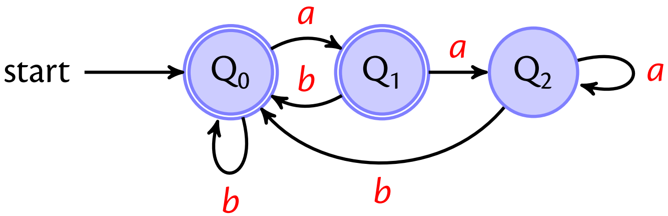
\end{aligned}
$$

Arden's Lemma:

If $q =$

Finally:

$$
\begin{aligned}
Q_0 &= (b + a\,b + a\,a\,(a^*)\,b)^*\\
Q_1 &= (b + a\,b + a\,a\,(a^*)\,b)^*\,a\\
Q_2 &= (b + a\,b + a\,a\,(a^*)\,b)^*\,a\,a\,(a^*)
\end{aligned}
$$



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Q_1 &= (b + a\, b + a\, a\, (a^*)\, b)^*\, a \\
Q_2 &= (b + a\, b + a\, a\, (a^*)\, b)^*\, a\, a\, (a^*)
\end{aligned}
$$

Coursework

- $der\,c\,(r^+) \stackrel{\text{def}}{=} der\,c(r \cdot r^*)$ given that $r^+ \stackrel{\text{def}}{=} r \cdot r^*$

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$$der\, c\, (r \cdot r^*) \stackrel{\text{def}}{=} \begin{array}{l} \text{if } nullable\, r \\ \text{then } (der\, c\, r) \cdot r^* + der\, c\, (r^*) \\ \text{else } (der\, c\, r) \cdot r^* \end{array}$$

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$$der\, c\, (r \cdot r^*) \quad \stackrel{\text{def}}{=} \quad (der\, c\, r) \cdot r^*$$

Coursework (2)

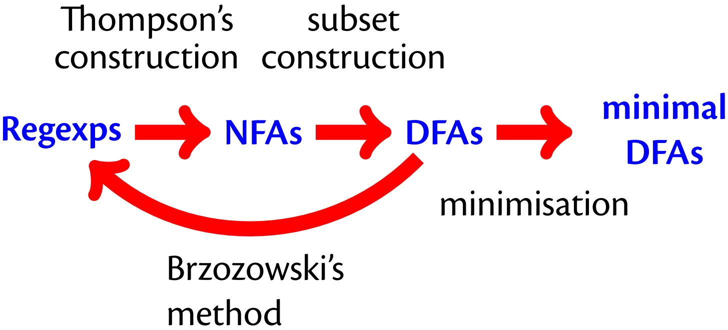
- `CFUN(f: Char => Boolean)`

$$\texttt{CHAR(c: Char)} \stackrel{\text{def}}{=} \texttt{CFUN(\_ == c)}$$

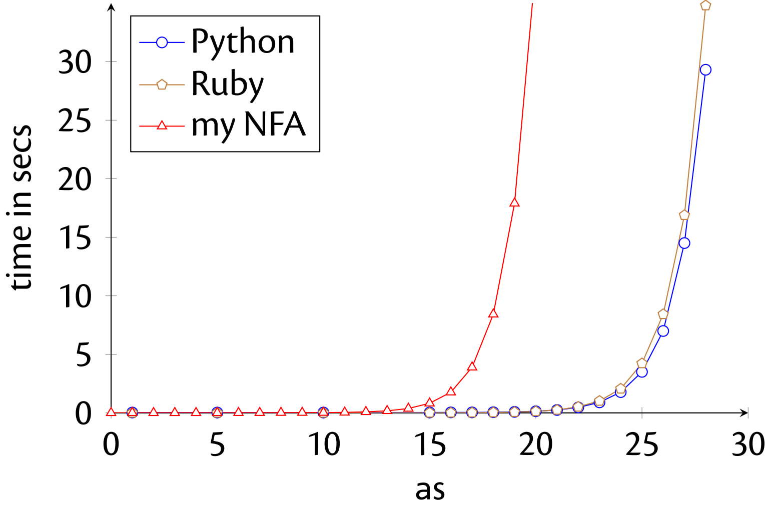
$$\texttt{RANGE(cs: Set[Char])} \stackrel{\text{def}}{=} \texttt{CFUN(cs.contains(\_))}$$

$$\texttt{ALL} \stackrel{\text{def}}{=} \texttt{CFUN((c: Char) => true)}$$

Regexps and Automata



$$a^{?\{n\}} \cdot a^{\{n\}}$$



The punchline is that many existing libraries do depth-first search in NFAs (backtracking).

Regular Languages

Two equivalent definitions:

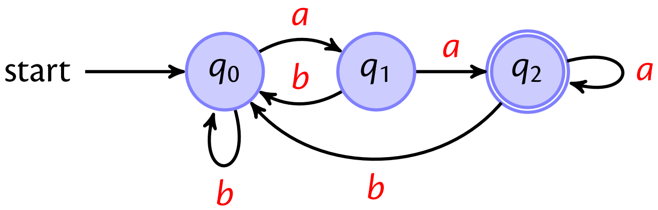
A language is regular iff there exists a regular expression that recognises all its strings.

A language is regular iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

Negation

Regular languages are closed under negation:



But requires that the automaton is completed!

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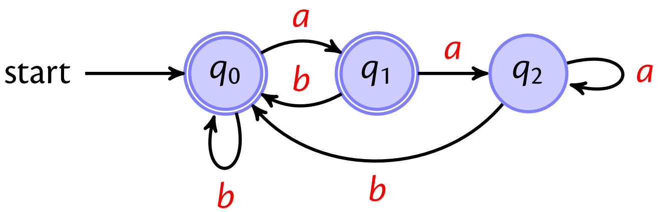

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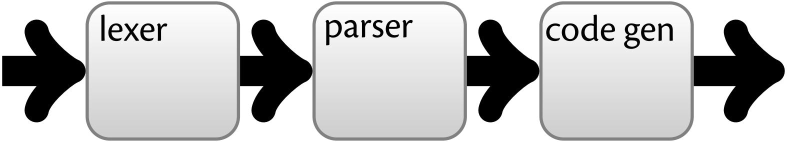
The Goal of this Course

Write a compiler



Today a lexer.

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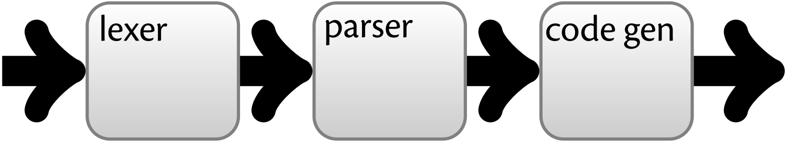

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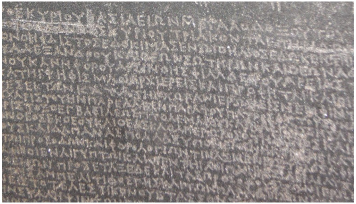

lexing ⇒ recognising words (Stone of Rosetta)

Regular Expressions

In programming languages they are often used to recognise:

- operands, digits
- identifiers
- numbers (non-leading zeros)
- keywords
- comments

`http://www.regexper.com`

Lexing: Test Case

```
write "Fib";
read n;
minus1 := 0;
minus2 := 1;
while n > 0 do {
       temp := minus2;
       minus2 := minus1 + minus2;
       minus1 := temp;
       n := n - 1
};
write "Result";
write minus2
```

```
"if true then then 42 else +"
```

KEYWORD:

if, then, else,

WHITESPACE:

" ", $\backslash$n,

IDENTIFIER:

$\mathrm{LETTER} \cdot (\mathrm{LETTER} + \mathrm{DIGIT} + \_)^*$

NUM:

$(\mathrm{NONZERODIGIT} \cdot \mathrm{DIGIT}^*) + 0$

OP:

+, -, *, %, <, <=

COMMENT:

$/* \cdot \sim(\mathrm{ALL}^* \cdot (*/) \cdot \mathrm{ALL}^*) \cdot */$

```
"if true then then 42 else +"
```

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

```
"if true then then 42 else +"
```

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

There is one small problem with the tokenizer. How should we tokenize...?

"x-3"

```
ID: ...
OP:
  "+", "-"
NUM:
  (NONZERODIGIT · DIGIT*) + ''0''
NUMBER:
  NUM + ("-" · NUM)
```

The same problem with

$$(ab + a) \cdot (c + bc)$$

and the string abc.

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and the string abc.

Or, keywords are **if** etc and identifiers are letters followed by "letters + numbers + _"*

$$if \qquad iffoo$$

POSIX: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as the next token.

- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

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`http://www.haskell.org/haskellwiki/Regex_Posix`

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`http://www.haskell.org/haskellwiki/Regex_Posix`

traditional lexers are fast, but hairy

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\ a} r_2$$

Sulzmann & Lu Matcher

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$$r_1 \xrightarrow{der\ a} r_2 \xrightarrow{der\ b} r_3$$

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\ a} r_2 \xrightarrow{der\ b} r_3 \xrightarrow{der\ c} r_4$$

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4 \;\; nullable?$$

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$
\begin{array}{ccccccc}
r_1 & \xrightarrow{der\, a} & r_2 & \xrightarrow{der\, b} & r_3 & \xrightarrow{der\, c} & r_4 \; nullable? \\
 & & & & & & \downarrow \\
 & & & & & & v_4
\end{array}
$$

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$\begin{tikzcd}
r_1 \arrow[r, "der\ a"] & r_2 \arrow[r, "der\ b"] & r_3 \arrow[r, "der\ c"] & r_4\ nullable? \arrow[d] \\
 & & v_3 & v_4 \arrow[l, "inj\ c"']
\end{tikzcd}$$

Sulzmann & Lu Matcher

We want to match the string abc using r_1:

$$
\begin{tikzcd}
r_1 \arrow[r, "der\ a"] & r_2 \arrow[r, "der\ b"] & r_3 \arrow[r, "der\ c"] & r_4 \ nullable? \arrow[d] \\
 & v_2 & v_3 \arrow[l, "inj\ b"'] & v_4 \arrow[l, "inj\ c"']
\end{tikzcd}
$$

Sulzmann & Lu Matcher

We want to match the string *abc* using r_1:

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r_1 \arrow[r, "der\ a"] & r_2 \arrow[r, "der\ b"] & r_3 \arrow[r, "der\ c"] & r_4\ nullable? \arrow[d] \\
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v_1 & v_2 \arrow[l, "inj\,a"] & v_3 \arrow[l, "inj\,b"] & v_4 \arrow[l, "inj\,c"]
\end{tikzcd}
$$

Regexes and Values

Regular expressions and their corresponding values:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
& | & \mathbf{1} \\
& | & c \\
& | & r_1 \cdot r_2 \\
& | & r_1 + r_2 \\
& & \\
& | & r^*
\end{array}
\qquad
\begin{array}{rcl}
v & ::= & \\
& & \mathit{Empty} \\
& | & \mathit{Char}(c) \\
& | & \mathit{Seq}(v_1, v_2) \\
& | & \mathit{Left}(v) \\
& | & \mathit{Right}(v) \\
& | & \mathit{Stars}\,[] \\
& | & \mathit{Stars}\,[v_1, \ldots\, v_n]
\end{array}
$$

```scala
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp

abstract class Val
case object Empty extends Val
case class Chr(c: Char) extends Val
case class Sequ(v1: Val, v2: Val) extends Val
case class Left(v: Val) extends Val
case class Right(v: Val) extends Val
case class Stars(vs: List[Val]) extends Val
```

r_1: $a \cdot (b \cdot c)$
r_2: $\mathbf{1} \cdot (b \cdot c)$
r_3: $(\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c)$
r_4: $(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$

$$\begin{tikzcd}
r_1 \arrow[r, "der\ a"] \arrow[d] & r_2 \arrow[r, "der\ b"] \arrow[d] & r_3 \arrow[r, "der\ c"] \arrow[d] & r_4\ nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\ a"] & v_3 \arrow[l, "inj\ b"] & v_4 \arrow[l, "inj\ c"]
\end{tikzcd}$$

r_1: $a \cdot (b \cdot c)$
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v_1 & v_2 \arrow[l, "inj\ a"] & v_3 \arrow[l, "inj\ b"] & v_4 \arrow[l, "inj\ c"]
\end{tikzcd}
$$

v_1: $Seq(Char(a), Seq(Char(b), Char(c)))$
v_2: $Seq(Empty, Seq(Char(b), Char(c)))$
v_3: $Right(Seq(Empty, Char(c)))$
v_4: $Right(Right(Empty))$

Flatten

Obtaining the string underlying a value:

$$
\begin{array}{lcl}
|Empty| & \stackrel{\text{def}}{=} & [] \\
|Char(c)| & \stackrel{\text{def}}{=} & [c] \\
|Left(v)| & \stackrel{\text{def}}{=} & |v| \\
|Right(v)| & \stackrel{\text{def}}{=} & |v| \\
|Seq(v_1, v_2)| & \stackrel{\text{def}}{=} & |v_1| \,@\, |v_2| \\
|[v_1, \ldots, v_n]| & \stackrel{\text{def}}{=} & |v_1| \,@ \ldots @\, |v_n|
\end{array}
$$

$$
\begin{array}{ll}
r_1: & a \cdot (b \cdot c) \\
r_2: & \mathbf{1} \cdot (b \cdot c) \\
r_3: & (\mathbf{0} \cdot (b \cdot c)) + (\mathbf{1} \cdot c) \\
r_4: & (\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})
\end{array}
$$

$$
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r_1 \arrow[r, "der\ a"] \arrow[d] & r_2 \arrow[r, "der\ b"] \arrow[d] & r_3 \arrow[r, "der\ c"] \arrow[d] & r_4\ \mathit{nullable} \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\ a"'] & v_3 \arrow[l, "inj\ b"'] & v_4 \arrow[l, "inj\ c"']
\end{tikzcd}
$$

$$
\begin{array}{ll}
v_1: & Seq(Char(a), Seq(Char(b), Char(c))) \\
v_2: & Seq(Empty, Seq(Char(b), Char(c))) \\
v_3: & Right(Seq(Empty, Char(c))) \\
v_4: & Right(Right(Empty))
\end{array}
$$

$$
\begin{array}{ll}
|v_1|: & abc \\
|v_2|: & bc \\
|v_3|: & c \\
|v_4|: & []
\end{array}
$$

Mkeps

Finding a (posix) value for recognising the empty string:

$$
\begin{array}{lcl}
mkeps\,(\mathbf{1}) & \stackrel{\text{def}}{=} & Empty \\
mkeps\,(r_1 + r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } Left(mkeps(r_1)) \\
 & & \text{else } Right(mkeps(r_2)) \\
mkeps\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), mkeps(r_2)) \\
mkeps\,(r^*) & \stackrel{\text{def}}{=} & Stars\,[]
\end{array}
$$

Inject

Injecting ("Adding") a character to a value

$$
\begin{array}{lcl}
inj\ (c)\ c\ (Empty) & \stackrel{\text{def}}{=} & Char\, c \\
inj\ (r_1 + r_2)\ c\ (Left(v)) & \stackrel{\text{def}}{=} & Left(inj\, r_1\, c\, v) \\
inj\ (r_1 + r_2)\ c\ (Right(v)) & \stackrel{\text{def}}{=} & Right(inj\, r_2\, c\, v) \\
inj\ (r_1 \cdot r_2)\ c\ (Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & Seq(inj\, r_1\, c\, v_1, v_2) \\
inj\ (r_1 \cdot r_2)\ c\ (Left(Seq(v_1, v_2))) & \stackrel{\text{def}}{=} & Seq(inj\, r_1\, c\, v_1, v_2) \\
inj\ (r_1 \cdot r_2)\ c\ (Right(v)) & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), inj\, r_2\, c\, v) \\
inj\ (r^*)\ c\ (Seq(v, Stars\, vs)) & \stackrel{\text{def}}{=} & Stars\ (inj\, r\, c\, v\ ::\ vs)
\end{array}
$$

inj: 1st arg $\mapsto$ a rexp; 2nd arg $\mapsto$ a character; 3rd arg $\mapsto$ a value
result $\mapsto$ a value

Lexing

$$
\begin{aligned}
lex\; r\; [] &\stackrel{\text{def}}{=} \text{if } nullable(r) \text{ then } mkeps(r) \text{ else } error \\
lex\; r\; c :: s &\stackrel{\text{def}}{=} inj\; r\; c\; lex(der(c, r), s)
\end{aligned}
$$

lex: returns a value

$$
\begin{tikzcd}
r_1 \arrow[r, "der\,a"] \arrow[d] & r_2 \arrow[r, "der\,b"] \arrow[d] & r_3 \arrow[r, "der\,c"] \arrow[d] & r_4 \; nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\,a"] & v_3 \arrow[l, "inj\,b"] & v_4 \arrow[l, "inj\,c"]
\end{tikzcd}
$$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\, c\, (x : r) \stackrel{\text{def}}{=} (x : der\, c\, r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\, (x : r)\, c\, v \stackrel{\text{def}}{=} Rec(x, inj\, r\, c\, v)$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$
- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
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- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\, (x : r)\, c\, v \stackrel{\text{def}}{=} Rec(x, inj\, r\, c\, v)$

for extracting subpatterns $(z : ((x : ab) + (y : ba))$

- A regular expression for email addresses

$$(\text{name}: [a\text{-}z0\text{-}9\_\_\;.\,-]^{+})\cdot @ \cdot$$
$$(\text{domain}: [a\text{-}z0\text{-}9\;.\,-]^{+})\cdot . \cdot$$
$$(\text{top\_level}: [a\text{-}z\;.]^{\{2,6\}})$$

`christian.urban@kcl.ac.uk`

- the result environment:

$$[(name : \texttt{christian.urban}),$$
$$(domain : \texttt{kcl}),$$
$$(top\_level : \texttt{ac.uk})]$$

While Tokens

$$\texttt{WHILE\_REGS} \stackrel{\text{def}}{=} \big((\texttt{"k"} : \texttt{KEYWORD}) + \\ (\texttt{"i"} : \texttt{ID}) + \\ (\texttt{"o"} : \texttt{OP}) + \\ (\texttt{"n"} : \texttt{NUM}) + \\ (\texttt{"s"} : \texttt{SEMI}) + \\ (\texttt{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + \\ (\texttt{"b"} : (\texttt{BEGIN} + \texttt{END})) + \\ (\texttt{"w"} : \texttt{WHITESPACE})\big)^*$$

Simplification

- If we simplify after the derivative, then we are building the value for the simplified regular expression, but **not** for the original regular expression.

$$
\begin{tikzcd}
r_1 \arrow[r, "der\ a"] \arrow[d] & r_2 \arrow[r, "der\ b"] \arrow[d] & r_3 \arrow[r, "der\ c"] \arrow[d] & r_4 \ nullable \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\ a"] & v_3 \arrow[l, "inj\ b"] & v_4 \arrow[l, "inj\ c"]
\end{tikzcd}
$$

$$(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1}) \mapsto \mathbf{1}$$

Normally we would have

$$(\mathbf{0} \cdot (b \cdot c)) + ((\mathbf{0} \cdot c) + \mathbf{1})$$

and answer how this regular expression matches the empty string with the value

$$Right(Right(Empty))$$

But now we simplify this to $\mathbf{1}$ and would produce $Empty$ (see *mkeps*).

Rectification

rectification functions:

$$
\begin{array}{lcll}
r \cdot \mathbf{0} & \mapsto & \mathbf{0} & \\
\mathbf{0} \cdot r & \mapsto & \mathbf{0} & \\
r \cdot \mathbf{1} & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\mathbf{1} \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \mathbf{0} & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\mathbf{0} + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}
$$

Rectification

rectification functions:

$$
\begin{array}{lcll}
r \cdot \mathbf{0} & \mapsto & \mathbf{0} & \\
\mathbf{0} \cdot r & \mapsto & \mathbf{0} & \\
r \cdot \mathbf{1} & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty) \\
\mathbf{1} \cdot r & \mapsto & r & \lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v) \\
r + \mathbf{0} & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v) \\
\mathbf{0} + r & \mapsto & r & \lambda f_1 f_2 v.\, Right(f_2\, v) \\
r + r & \mapsto & r & \lambda f_1 f_2 v.\, Left(f_1\, v)
\end{array}
$$

old *simp* returns a rexp;
new *simp* returns a rexp and a rectification function.

Rectification

$simp(r)$:

case $r = r_1 + r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(r_{2s}, \lambda v.\, Right(f_{2s}(v)))$

case $r_{2s} = \mathbf{0}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$

case $r_{1s} = r_{2s}$: return $(r_{1s}, \lambda v.\, Left(f_{1s}(v)))$

otherwise: return $(r_{1s} + r_{2s}, f_{alt}(f_{1s}, f_{2s}))$

$$f_{alt}(f_1, f_2) \stackrel{\text{def}}{=}$$

$$\lambda v.\ \text{case } v = Left(v')\text{:} \quad \text{return } Left(f_1(v'))$$

$$\text{case } v = Right(v')\text{: return } Right(f_2(v'))$$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case ALT(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (r2s, F_RIGHT(f2s))
      case (_, ZERO) => (r1s, F_LEFT(f1s))
      case _ =>
        if (r1s == r2s) (r1s, F_LEFT(f1s))
        else (ALT (r1s, r2s), F_ALT(f1s, f2s))
    }
  }
  ...
}

def F_RIGHT(f: Val => Val) = (v:Val) => Right(f(v))
def F_LEFT(f: Val => Val) = (v:Val) => Left(f(v))
def F_ALT(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Right(v) => Right(f2(v))
    case Left(v) => Left(f1(v)) }
```

Rectification

$simp(r)$:...

case $r = r_1 \cdot r_2$

let $(r_{1s}, f_{1s}) = simp(r_1)$

$(r_{2s}, f_{2s}) = simp(r_2)$

case $r_{1s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{2s} = \mathbf{0}$: return $(\mathbf{0}, f_{error})$

case $r_{1s} = \mathbf{1}$: return $(r_{2s}, \lambda v.\, Seq(f_{1s}(Empty), f_{2s}(v)))$

case $r_{2s} = \mathbf{1}$: return $(r_{1s}, \lambda v.\, Seq(f_{1s}(v), f_{2s}(Empty)))$

otherwise: return $(r_{1s} \cdot r_{2s}, f_{seq}(f_{1s}, f_{2s}))$

$f_{seq}(f_1, f_2) \stackrel{\text{def}}{=}$

$\lambda v.$ case $v = Seq(v_1, v_2)$: return $Seq(f_1(v_1), f_2(v_2))$

```scala
def simp(r: Rexp): (Rexp, Val => Val) = r match {
  case SEQ(r1, r2) => {
    val (r1s, f1s) = simp(r1)
    val (r2s, f2s) = simp(r2)
    (r1s, r2s) match {
      case (ZERO, _) => (ZERO, F_ERROR)
      case (_, ZERO) => (ZERO, F_ERROR)
      case (ONE, _) => (r2s, F_SEQ_Void1(f1s, f2s))
      case (_, ONE) => (r1s, F_SEQ_Void2(f1s, f2s))
      case _ => (SEQ(r1s,r2s), F_SEQ(f1s, f2s))
    }
  }
  ...

def F_SEQ_Void1(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(Void), f2(v))
def F_SEQ_Void2(f1: Val => Val, f2: Val => Val) =
  (v:Val) => Sequ(f1(v), f2(Void))
def F_SEQ(f1: Val => Val, f2: Val => Val) =
  (v:Val) => v match {
    case Sequ(v1, v2) => Sequ(f1(v1), f2(v2)) }
```

Rectification Example

$$(b \cdot c) + (\mathbf{0} + \mathbf{1}) \mapsto (b \cdot c) + \mathbf{1}$$

Rectification Example

$$(\underline{b \cdot c}) + (\underline{\mathbf{0} + \mathbf{1}}) \mapsto (b \cdot c) + \mathbf{1}$$

Rectification Example

$$(\underline{b \cdot c}) + (\underline{\mathbf{0} + \mathbf{1}}) \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{rcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$\begin{array}{l} f_{alt}(f_{s1}, f_{s2}) \stackrel{\text{def}}{=} \\ \quad \lambda v.\ \text{case } v = Left(v'):\ \ \text{return } Left(f_{s1}(v')) \\ \qquad\ \ \text{case } v = Right(v'): \text{return } Right(f_{s2}(v')) \end{array}$$

Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$\begin{array}{ll} \lambda v. & \text{case } v = Left(v'): \quad \text{return } Left(v') \\ & \text{case } v = Right(v'): \text{return } Right(Right(v')) \end{array}$$

Rectification Example

$$\underline{(b \cdot c) + (\mathbf{0} + \mathbf{1})} \mapsto (b \cdot c) + \mathbf{1}$$

$$\begin{array}{lcl} f_{s1} & = & \lambda v.v \\ f_{s2} & = & \lambda v.Right(v) \end{array}$$

$$\begin{array}{ll} \lambda v.\ \text{case}\ v = Left(v'): & \text{return}\ Left(v') \\ \quad\ \text{case}\ v = Right(v'): & \text{return}\ Right(Right(v')) \end{array}$$

mkeps simplified case: $Right(Empty)$

rectified case: $Right(Right(Empty))$

Lexing with Simplification

$$
\begin{array}{lcl}
\mathit{lex}\;r\;[] & \stackrel{\text{def}}{=} & \text{if}\;\mathit{nullable}(r)\;\text{then}\;\mathit{mkeps}(r)\;\text{else}\;\mathit{error}\\
\mathit{lex}\;r\;c::s & \stackrel{\text{def}}{=} & \text{let}\;(r',\mathit{frect}) = \mathit{simp}(\mathit{der}(c,r))\\
 & & \mathit{inj}\;r\;c\;(\mathit{frect}(\mathit{lex}(r',s)))
\end{array}
$$

$$
\begin{tikzcd}
r_1 \arrow[r, "der\,a"] \arrow[d] & r_2 \arrow[r, "der\,b"] \arrow[d] & r_3 \arrow[r, "der\,c"] \arrow[d] & r_4\ \mathit{nullable} \arrow[d, "mkeps"] \\
v_1 & v_2 \arrow[l, "inj\,a"] & v_3 \arrow[l, "inj\,b"] & v_4 \arrow[l, "inj\,c"]
\end{tikzcd}
$$

Environments

Obtaining the “recorded” parts of a value:

$$
\begin{array}{lcl}
env(Empty) & \stackrel{\text{def}}{=} & [] \\
env(Char(c)) & \stackrel{\text{def}}{=} & [] \\
env(Left(v)) & \stackrel{\text{def}}{=} & env(v) \\
env(Right(v)) & \stackrel{\text{def}}{=} & env(v) \\
env(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} & env(v_1) \mathbin{@} env(v_2) \\
env(Stars\,[v_1, \ldots, v_n]) & \stackrel{\text{def}}{=} & env(v_1) \mathbin{@} \ldots \mathbin{@} env(v_n) \\
env(Rec(x : v)) & \stackrel{\text{def}}{=} & (x : |v|) :: env(v)
\end{array}
$$

While Tokens

$$
\begin{aligned}
\texttt{WHILE\_REGS} \stackrel{\text{def}}{=} \Big(&(\texttt{"k"} : \texttt{KEYWORD}) + \\
&(\texttt{"i"} : \texttt{ID}) + \\
&(\texttt{"o"} : \texttt{OP}) + \\
&(\texttt{"n"} : \texttt{NUM}) + \\
&(\texttt{"s"} : \texttt{SEMI}) + \\
&(\texttt{"p"} : (\texttt{LPAREN} + \texttt{RPAREN})) + \\
&(\texttt{"b"} : (\texttt{BEGIN} + \texttt{END})) + \\
&(\texttt{"w"} : \texttt{WHITESPACE})\Big)^*
\end{aligned}
$$

```
"if true then then 42 else +"
```

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

```
"if true then then 42 else +"
```

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

Lexer: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as next token.
- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

$$
\begin{array}{lcl}
\mathit{zeroable}(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathit{true} \\
\mathit{zeroable}(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathit{false} \\
\mathit{zeroable}(c) & \stackrel{\text{def}}{=} & \mathit{false} \\
\mathit{zeroable}(r_1 + r_2) & \stackrel{\text{def}}{=} & \mathit{zeroable}(r_1) \wedge \mathit{zeroable}(r_2) \\
\mathit{zeroable}(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \mathit{zeroable}(r_1) \vee \mathit{zeroable}(r_2) \\
\mathit{zeroable}(r^*) & \stackrel{\text{def}}{=} & \mathit{false}
\end{array}
$$

$\mathit{zeroable}(r)$ if and only if $L(r) = \{\}$