Automata and Formal Languages (2)

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Languages

A language is a set of strings.

A regular expression specifies a set of strings, or language.

Strings

Different ways of writing strings:

$$\textit{"hello"} \qquad [h, e, l, l, o] \qquad h :: e :: l :: l :: o :: \textit{Nil}$$

$$\textit{""} \qquad [] \qquad \textit{Nil}$$

Strings

Different ways of writing strings:

$$"hello" \qquad [h, e, l, l, o] \qquad h :: e :: l :: l :: o :: Nil$$

$$"" \qquad [] \qquad Nil$$

The concatenation operation on strings and sets of strings:

$$"foo" \,@\, "bar" = "foobar"$$

$$A \,@\, B \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

Regular Expressions

Their inductive definition:

r	$::=$	$\varnothing$	null
	$\mid$	ϵ	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

Re

Their indu

```scala
abstract class Rexp

case object NULL extends Rexp
case object EMPTY extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

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	$\mid$	ϵ	empty string / "" / []
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The Meaning of a Regular Expression

$$
\begin{array}{rcl}
L(\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
L(\epsilon) & \stackrel{\text{def}}{=} & \{""\} \\
L(c) & \stackrel{\text{def}}{=} & \{"c"\} \\
L(r_1 + r_2) & \stackrel{\text{def}}{=} & L(r_1) \cup L(r_2) \\
L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & L(r_1) \mathbin{@} L(r_2) \\
L(r^*) & \stackrel{\text{def}}{=} & \bigcup_{n \geq 0} L(r)^n
\end{array}
$$

L is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

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L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & L(r_1) \,@\, L(r_2) \\
L(r^*) & \stackrel{\text{def}}{=} & \bigcup_{n \geq 0} L(r)^n
\end{array}
$$

$$
\begin{array}{rcl}
L(r)^0 & \stackrel{\text{def}}{=} & \{''''\} \\
L(r)^{n+1} & \stackrel{\text{def}}{=} & L(r) \,@\, L(r)^n
\end{array}
$$

L is a function from regular expressions to sets of strings

$L : \mathrm{Rexp} \Rightarrow \mathrm{Set}[\mathrm{String}]$

What is $L(a^*)$?

Reg Exp Equivalences

$$
\begin{array}{lrcl}
 & (a + b) + c & \equiv^? & a + (b + c) \\
 & a + a & \equiv^? & a \\
 & (a \cdot b) \cdot c & \equiv^? & a \cdot (b \cdot c) \\
 & a \cdot a & \equiv^? & a \\
 & \epsilon^* & \equiv^? & \epsilon \\
 & \varnothing^* & \equiv^? & \varnothing \\
\forall\, r. & r \cdot \epsilon & \equiv^? & r \\
\forall\, r. & r + \epsilon & \equiv^? & r \\
\forall\, r. & r + \varnothing & \equiv^? & r \\
\forall\, r. & r \cdot \varnothing & \equiv^? & r \\
 & c \cdot (a + b) & \equiv^? & (c \cdot a) + (c \cdot b) \\
 & a^* & \equiv^? & \epsilon + (a \cdot a^*)
\end{array}
$$

Reg Exp Equivalences

$$
\begin{array}{lrcll}
 & (a + b) + c & \equiv^? & a + (b + c) & \text{yes} \\
 & a + a & \equiv^? & a & \\
 & (a \cdot b) \cdot c & \equiv^? & a \cdot (b \cdot c) & \\
 & a \cdot a & \equiv^? & a & \\
 & \epsilon^* & \equiv^? & \epsilon & \\
 & \varnothing^* & \equiv^? & \varnothing & \\
\forall\ \text{r.} & r \cdot \epsilon & \equiv^? & r & \\
\forall\ \text{r.} & r + \epsilon & \equiv^? & r & \\
\forall\ \text{r.} & r + \varnothing & \equiv^? & r & \\
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\end{array}
$$

Reg Exp Equivalences

	$(a + b) + c$	$\equiv^?$	$a + (b + c)$	yes
	$a + a$	$\equiv^?$	a	yes
	$(a \cdot b) \cdot c$	$\equiv^?$	$a \cdot (b \cdot c)$	
	$a \cdot a$	$\equiv^?$	a	
	ϵ^*	$\equiv^?$	ϵ	
	$\varnothing^*$	$\equiv^?$	$\varnothing$	
$\forall$ r.	$r \cdot \epsilon$	$\equiv^?$	r	
$\forall$ r.	$r + \epsilon$	$\equiv^?$	r	
$\forall$ r.	$r + \varnothing$	$\equiv^?$	r	
$\forall$ r.	$r \cdot \varnothing$	$\equiv^?$	r	
	$c \cdot (a + b)$	$\equiv^?$	$(c \cdot a) + (c \cdot b)$	
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\forall\ \text{r.} & r \cdot \epsilon & \equiv^? & r & \\
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\forall\ \text{r.} & r \cdot \varnothing & \equiv^? & r & \\
 & c \cdot (a + b) & \equiv^? & (c \cdot a) + (c \cdot b) & \\
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\end{array}
$$

Reg Exp Equivalences

	$(a + b) + c$	$\equiv^?$	$a + (b + c)$	yes
	$a + a$	$\equiv^?$	a	yes
	$(a \cdot b) \cdot c$	$\equiv^?$	$a \cdot (b \cdot c)$	yes
	$a \cdot a$	$\equiv^?$	a	no
	ϵ^*	$\equiv^?$	ϵ	
	$\varnothing^*$	$\equiv^?$	$\varnothing$	
$\forall$ r.	$r \cdot \epsilon$	$\equiv^?$	r	
$\forall$ r.	$r + \epsilon$	$\equiv^?$	r	
$\forall$ r.	$r + \varnothing$	$\equiv^?$	r	
$\forall$ r.	$r \cdot \varnothing$	$\equiv^?$	r	
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∀ r.	$r + \varnothing$	$\equiv^?$	r	
∀ r.	$r \cdot \varnothing$	$\equiv^?$	r	
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Reg Exp Equivalences

	$(a+b)+c$	$\equiv^?$	$a+(b+c)$	yes
	$a+a$	$\equiv^?$	a	yes
	$(a\cdot b)\cdot c$	$\equiv^?$	$a\cdot(b\cdot c)$	yes
	$a\cdot a$	$\equiv^?$	a	no
	ϵ^*	$\equiv^?$	ϵ	yes
	$\varnothing^*$	$\equiv^?$	$\varnothing$	no
$\forall$ r.	$r\cdot\epsilon$	$\equiv^?$	r	yes
$\forall$ r.	$r+\epsilon$	$\equiv^?$	r	
$\forall$ r.	$r+\varnothing$	$\equiv^?$	r	
$\forall$ r.	$r\cdot\varnothing$	$\equiv^?$	r	
	$c\cdot(a+b)$	$\equiv^?$	$(c\cdot a)+(c\cdot b)$	
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The Specification of Matching

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$$s \in L(r)$$

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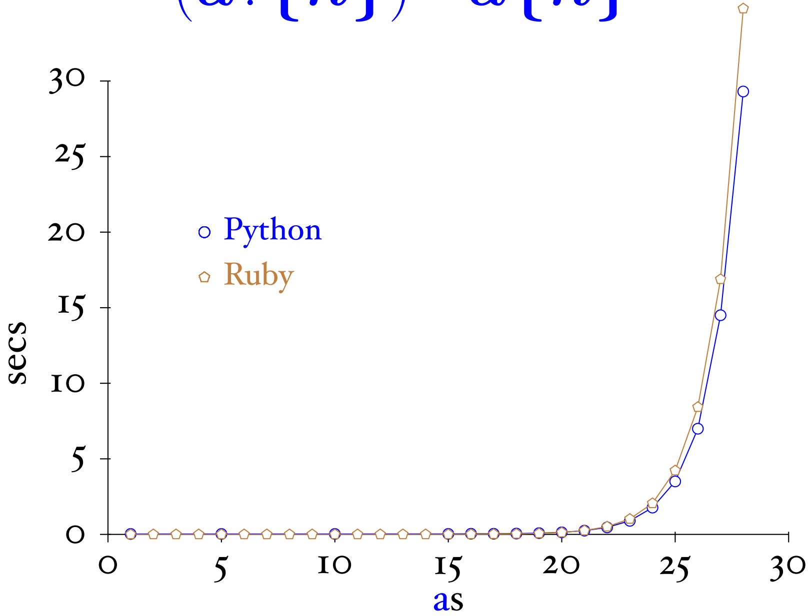
$(a?\{n\}) \cdot a\{n\}$



Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
  - $(a?\{n\}) \cdot a\{n\}$
  - $(a^+)^+$
  - $([a\text{-}z]^+)^*$
  - $(a + a \cdot a)^+$
  - $(a + a?)^+$

A Matching Algorithm

...whether a regular expression can match the empty string:

$$
\begin{array}{lcl}
nullable(\varnothing) & \stackrel{\text{def}}{=} & false \\
nullable(\epsilon) & \stackrel{\text{def}}{=} & true \\
nullable(c) & \stackrel{\text{def}}{=} & false \\
nullable(r_1 + r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \vee nullable(r_2) \\
nullable(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \wedge nullable(r_2) \\
nullable(r^*) & \stackrel{\text{def}}{=} & true
\end{array}
$$

A Matching Algorithm

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$$nullable(c) \stackrel{\text{def}}{=} false$$
$$nullable(r_1 + r_2) \stackrel{\text{def}}{=} nullable(r_1) \vee nullable(r_2)$$
$$nullable(r_1 \cdot r_2) \stackrel{\text{def}}{=} nullable(r_1) \wedge nullable(r_2)$$
$$nulla\ldots \stackrel{\text{def}}{=}$$

```
def nullable (r: Rexp) : Boolean = r match {
  case NULL => false
  case EMPTY => true
  case CHAR(_) => false
  case ALT(r1, r2) => nullable(r1) || nullable(r2)
  case SEQ(r1, r2) => nullable(r1) && nullable(r2)
  case STAR(_) => true
}
```

The Derivative of a Rexp

If r matches the string $c{::}s$, what is a regular expression that matches s?

$der\ c\ r$ gives the answer

The Derivative of a Rexp (2)

$$
\begin{array}{lcl}
der\,c\,(\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
der\,c\,(\epsilon) & \stackrel{\text{def}}{=} & \varnothing \\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if}\; c = d \;\text{then}\; \epsilon \;\text{else}\; \varnothing \\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2 \\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if}\; nullable(r_1) \\
 & & \text{then}\; (der\,c\,r_1) \cdot r_2 + der\,c\,r_2 \\
 & & \text{else}\; (der\,c\,r_1) \cdot r_2 \\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}
$$

The Derivative of a Rexp (2)

$$
\begin{array}{lcl}
der\, c\, (\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
der\, c\, (\epsilon) & \stackrel{\text{def}}{=} & \varnothing \\
der\, c\, (d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \epsilon \text{ else } \varnothing \\
der\, c\, (r_1 + r_2) & \stackrel{\text{def}}{=} & der\, c\, r_1 + der\, c\, r_2 \\
der\, c\, (r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } (der\, c\, r_1) \cdot r_2 + der\, c\, r_2 \\
 & & \text{else } (der\, c\, r_1) \cdot r_2 \\
der\, c\, (r^*) & \stackrel{\text{def}}{=} & (der\, c\, r) \cdot (r^*) \\
ders\, []\, r & \stackrel{\text{def}}{=} & r \\
ders\, (c :: s)\, r & \stackrel{\text{def}}{=} & ders\, s\, (der\, c\, r)
\end{array}
$$

The Derivative of a Rexp (2)

$$der\, c\, (\varnothing) \stackrel{\text{def}}{=} \varnothing$$

$$der\, c\, (\epsilon) \stackrel{\text{def}}{=} \varnothing$$

```
def der (r: Rexp, c: Char) : Rexp = r match {
  case NULL => NULL
  case EMPTY => NULL
  case CHAR(d) => if (c == d) EMPTY else NULL
  case ALT(r1, r2) => ALT(der(r1, c), der(r2, c))
  case SEQ(r1, r2) =>
    if (nullable(r1)) ALT(SEQ(der(r1, c), r2), der(r2, c))
    else SEQ(der(r1, c), r2)
  case STAR(r) => SEQ(der(r, c), STAR(r))
}

def ders (s: List[Char], r: Rexp) : Rexp = s match {
  case Nil => r
  case c::s => ders(s, der(c, r))
}
```

The Rexp Matcher

```
abstract class Parser[I, T] {
  def parse(ts: I): Set[(T, I)]

  def parse_all(ts: I) : Set[T] =
    for ((head, tail) <- parse(ts); if (tail.isEmpty))
      yield head

  def || (right : => Parser[I, T]) : Parser[I, T] =
      new AltParser(this, right)
  def ==>[S] (f: => T => S) : Parser [I, S] =
      new FunParser(this, f)
   def ~[S] (right : => Parser[I, S]) : Parser[I, (T, S)] =
      new SeqParser(this, right)
}
```

Proofs about Rexp

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \varnothing \\
  & \mid & \epsilon \\
  & \mid & c \\
  & \mid & r_1 \cdot r_2 \\
  & \mid & r_1 + r_2 \\
  & \mid & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\varnothing$, ϵ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$\text{nullable}(\text{r}) \text{ if and only if } \text{""} \in L(\text{r})$$

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

Let $\text{Der } c\ A$ be the set defined as

$$\text{Der } c\ A \stackrel{\text{def}}{=} \{\, s \mid c\!::\!s \in A \}$$

Assume that $L(\text{der } c\ r) = \text{Der } c\ (L(r))$. Prove that

$$\text{matcher}(r, s) \text{ if and only if } s \in L(r)$$

Regular Languages

A language (set of strings) is regular iff there exists a regular expression that recognises all its strings.

Automata

A deterministic finite automaton consists of:

- a set of states
- one of these states is the start state
- some states are accepting states, and
- there is transition function

  which takes a state as argument and a character and produces a new state

  this function might not always be defined