Compilers and Formal Languages

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Slides & Progs: KEATS (also homework is there)

1 Introduction, Languages	6 While-Language
2 Regular Expressions, Derivatives	7 Compilation, JVM
3 Automata, Regular Languages	8 Compiling Functional Languages
4 Lexing, Tokenising	9 Optimisations
5 Grammars, Parsing	10 LLVM

Static Single-Assignment

$$(1 + a) + (3 + (b * 5))$$

```
let tmp0 = add 1 a in
let tmp1 = mul b 5 in
let tmp2 = add 3 tmp1 in
let tmp3 = add tmp0 tmp2
  in tmp3
```

```
define i32 @fact (i32 %n) {
   %tmp_20 = icmp eq i32  %n, 0
   br i1 %tmp_20, label %if_branch_24, label %else_branch_25
 if_branch_24:
   ret i32 1
 else_branch_25:
   %tmp_22 = sub i32  %n, 1
   %tmp_23 = call i32 @fact (i32 %tmp_22)
   %tmp_21 = mul i32  %n, %tmp_23
   ret i32 %tmp_21
}
```

```
def fact(n) = if n == 0 then 1 else n * fact(n - 1)
```

```
br i1 %var, label %if_br, label %else_br

icmp eq i32  %x, %y      ; for equal
icmp sle i32 %x, %y      ;   signed less or equal
icmp slt i32 %x, %y      ;   signed less than
icmp ult i32 %x, %y      ;   unsigned less than

%var = call i32 @foo(...args...)
```

```scala
def fact(n: Int) : Int = {
  if (n == 0) 1 else n * fact(n - 1)
}


def factC(n: Int, ret: Int => Int) : Int = {
  if (n == 0) ret(1)
  else factC(n - 1, x => ret(n * x))
}

fact(10)
factC(10, identity)
```

```scala
def fibC(n: Int, ret: Int => Int) : Int = {
  if (n == 0 || n == 1) ret(1) else
  fibC(n - 1,
       r1 => fibC(n - 2,
        r2 => ret(r1 + r2)))
}

fibC(10, identity)
```

Are there more strings in

$L(a^*)$ **or** $L((a+b)^*)$**?**

Can you remember this HW?

(1) How many basic regular expressions are there to match the string *abcd*?

(2) How many if they cannot include $\mathbf{1}$ and $\mathbf{0}$?

(3) How many if they are also not allowed to contain stars?

(4) How many if they are also not allowed to contain $\_ + \_$?

There are more problems, than there are programs.

There are more problems, than there are programs.

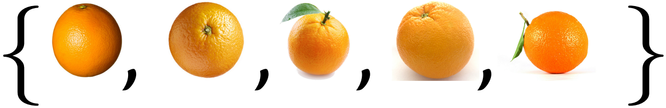
There must be a problem for which there is no program.

Subsets

If $A \subseteq B$ then A has fewer or equal elements than B

$A \subseteq B$ and $B \subseteq A$

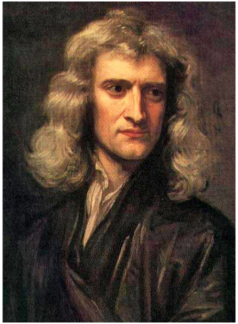
then $A = B$



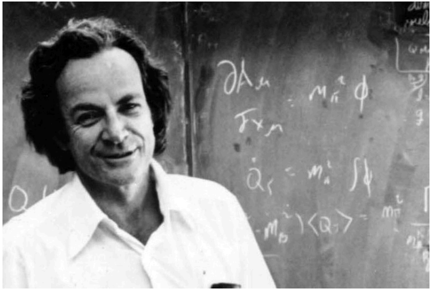
5 elements

3 elements

Newton vs Feynman



classical physics



quantum physics

The Goal of the Talk

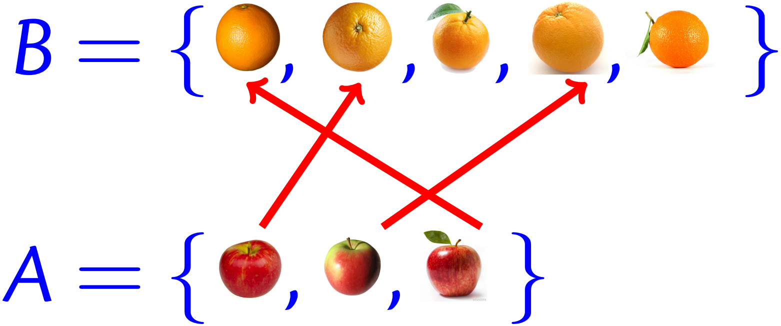
show you that something very unintuitive happens with very large sets

convince you that there are more **problems** than **programs**

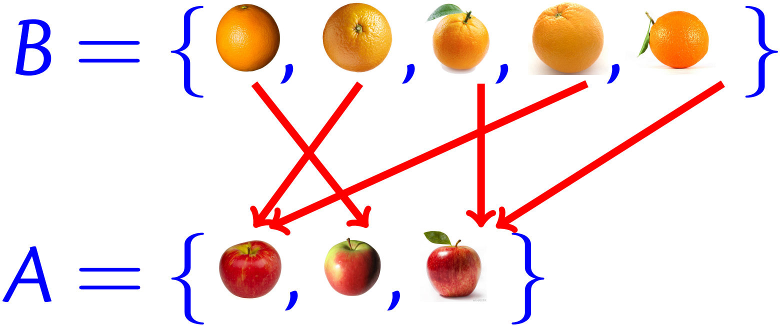
$B = \{$ , , , ,  $\}$

$A = \{$ , ,  $\}$

$$|A| = 5, |B| = 3$$



then $|A| \leq |B|$



for $=$ has to be a **one-to-one** mapping

Cardinality

$|A| \stackrel{\text{def}}{=}$ "how many elements"

$A \subseteq B \Rightarrow |A| \leq |B|$

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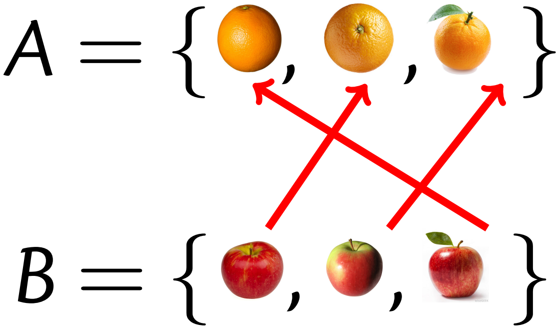
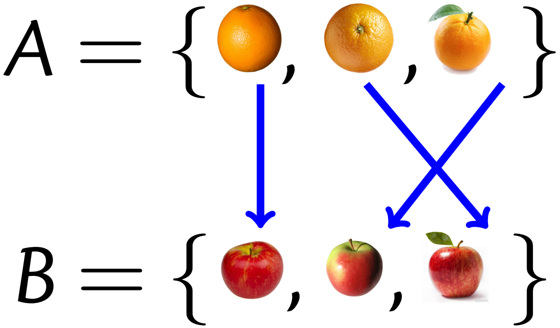
if there is an injective function $f : A \rightarrow B$ then $|A| \leq |B|$

$$\forall xy.\, f(x) = f(y) \Rightarrow x = y$$


A =
B =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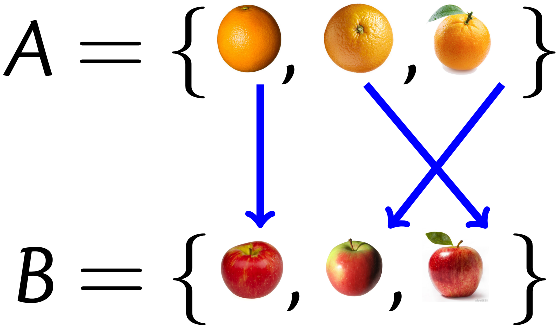
B =



then $|A| = |B|$

Natural Numbers

$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, 3, \ldots\ldots\}$$

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A is countable iff $|A| \leq |\mathbb{N}|$

First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

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$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$?

$x \mapsto x + 1$,

$|\mathbb{N} - \{0\}| = |\mathbb{N}|$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\quad \{1, 3, 5......\}$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \stackrel{\text{def}}{=}$ odd numbers $\quad \{1, 3, 5......\}$
$\mathbb{E} \stackrel{\text{def}}{=}$ even numbers $\quad \{0, 2, 4......\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{N} \stackrel{\text{def}}{=}$ positive numbers $\{0, 1, 2, 3, \ldots\ldots\}$

$-\mathbb{N} \stackrel{\text{def}}{=}$ negative numbers $\{0, -1, -2, -3, \ldots\ldots\}$

A is countable if there exists an injective $f : A \to \mathbb{N}$

A is uncountable if there does not exist an injective $f : A \to \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$
uncountable: $|A| > |\mathbb{N}|$

A is countable if there exists an injective $f : A \rightarrow \mathbb{N}$

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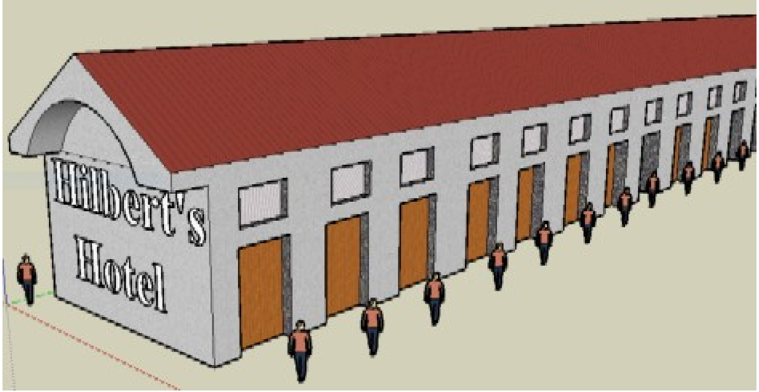
Does there exist such an A ?

Hilbert's Hotel



...has as many rooms as there are natural numbers

Real Numbers between 0 and 1

1	3	3	3	3	3	3	...
2	1	2	3	4	5	6	7
3	0	1	0	1	0	...	
4	7	8	5	3	9	...	

...

...

Real Numbers between 0 and 1

1	4	3	3	3	3	3	. . .
2	1	2	3	4	5	6	7
3	0	1	0	1	0	. . .	
4	7	8	5	3	9	. . .	

. . .

. . .

Real Numbers between 0 and 1

1	4	3	3	3	3	3	... ...
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...

Real Numbers between 0 and 1

1	4	3	3	3	3	3	. . .
2	1	3	3	4	5	6	7
3	0	1	1	1	0	. . .	
4	7	8	5	3	9	. . .	

. . .

. . .

Real Numbers between 0 and 1

1	4	3	3	3	3	3	…	…
2	1	3	3	4	5	6	7	
3	0	1	1	1	0	…		
4	7	8	5	4	9	…		
	…							

Real Numbers between 0 and 1

1	4	3	3	3	3	3	...
2	1	3	3	4	5	6	7
3	0	1	1	1	0	...	
4	7	8	5	4	9	...	

...

...

$$|\mathbb{N}| < |R|$$

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	...
1	0	1	0	1	0	1	...
2	0	0	0	1	1	0	0
3	0	0	0	0	0	...	
4	1	1	0	1	1	...	

...

...

The Set of Problems

$\aleph_0$

	0	1	2	3	4	5	...
1	0	1	0	1	0	1	...
2	0	0	0	1	1	0	0
3	0	0	0	0	0	...	
4	1	1	0	1	1	...	

...

...

$$|Progs| = |\mathbb{N}| < |Probs|$$

Halting Problem

Assume a program H that decides for all programs A and all input data D whether

$$H(A, D) \stackrel{\text{def}}{=} 1 \text{ iff } A(D) \text{ terminates}$$
$$H(A, D) \stackrel{\text{def}}{=} 0 \text{ otherwise}$$

Halting Problem (2)

Given such a program H define the following program C: for all programs A

$$C(A) \stackrel{\text{def}}{=} 0 \text{ iff } H(A, A) = 0$$

$$C(A) \stackrel{\text{def}}{=} \text{loops otherwise}$$

Contradiction

$H(C, C)$ is either 0 or 1.

$$H(C, C) = 1 \overset{\text{def}\, H}{\Rightarrow} C(C) \downarrow \overset{\text{def}\, C}{\Rightarrow} H(C, C) = 0$$

$$H(C, C) = 0 \overset{\text{def}\, H}{\Rightarrow} C(C) \text{ loops} \overset{\text{def}\, C}{\Rightarrow} H(C, C) = 1$$

Contradiction in both cases. So H cannot exist.

Take Home Points

there are sets that are more infinite than others

even with the most powerful computer we can imagine, there are problems that cannot be solved by any program

in CS we actually hit quite often such problems (halting problem)