

Compilers and Formal Languages (3)

Email: christian.urban at kcl.ac.uk

Office: S1.27 (1st floor Strand Building)

Slides: KEATS (also home work and course-
work is there)

Scala Book, Exams

- www.inf.kcl.ac.uk/urbanc/ProgInScala2ed.pdf
- homeworks (exam 80%)
- coursework (20%)

Regular Expressions

In programming languages they are often used to recognise:

- symbols, digits
- identifiers
- numbers (non-leading zeros)
- keywords
- comments

<http://www.regexper.com>

Last Week

Last week I showed you a regular expression matcher that works provably correct in all cases (we only started with the proving part though)

matches $s r$ if and only if $s \in L(r)$

by Janusz Brzozowski (1964)

The Derivative of a Rexp

$$\mathit{der} c (\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0}$$

$$\mathit{der} c (\mathbf{1}) \stackrel{\text{def}}{=} \mathbf{0}$$

$$\mathit{der} c (d) \stackrel{\text{def}}{=} \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}$$

$$\mathit{der} c (r_1 + r_2) \stackrel{\text{def}}{=} \mathit{der} c r_1 + \mathit{der} c r_2$$

$$\mathit{der} c (r_1 \cdot r_2) \stackrel{\text{def}}{=} \text{if } \mathit{nullable}(r_1) \\ \text{then } (\mathit{der} c r_1) \cdot r_2 + \mathit{der} c r_2 \\ \text{else } (\mathit{der} c r_1) \cdot r_2$$

$$\mathit{der} c (r^*) \stackrel{\text{def}}{=} (\mathit{der} c r) \cdot (r^*)$$

$$\mathit{ders} [] r \stackrel{\text{def}}{=} r$$

$$\mathit{ders} (c :: s) r \stackrel{\text{def}}{=} \mathit{ders} s (\mathit{der} c r)$$

Input: string abc and regular expression r

- 1 $der\ a\ r$
- 2 $der\ b\ (der\ a\ r)$
- 3 $der\ c\ (der\ b\ (der\ a\ r))$

Input: string *abc* and regular expression *r*

- 1 *der a r*
- 2 *der b (der a r)*
- 3 *der c (der b (der a r))*
- 4 finally check whether the last regular expression can match the empty string

We proved already

nullable(r) if and only if $\epsilon \in L(r)$

by induction on the regular expression r .

We proved already

nullable(r) if and only if $\epsilon \in L(r)$

by induction on the regular expression r .

Any Questions?

We need to prove

$$L(\text{der } c r) = \text{Der } c (L(r))$$

also by induction on the regular expression r .

Proofs about Rexp

- P holds for $\mathbf{0}$, $\mathbf{1}$ and c
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2 .
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2 .
- P holds for r^* under the assumption that P already holds for r .

Proofs about Natural Numbers and Strings

- P holds for 0 and
- P holds for $n + 1$ under the assumption that P already holds for n

- P holds for $[]$ and
- P holds for $c::s$ under the assumption that P already holds for s

Regular Expressions

$r ::= \mathbf{0}$	null
$\mathbf{1}$	empty string / "" / []
c	character
$r_1 \cdot r_2$	sequence
$r_1 + r_2$	alternative / choice
r^*	star (zero or more)

How about ranges $[a-z]$, r^+ and $\sim r$? Do they increase the set of languages we can recognise?

Negation of Regular Expr's

- $\sim r$ (everything that r cannot recognise)
- $L(\sim r) \stackrel{\text{def}}{=} UNIV - L(r)$
- $nullable(\sim r) \stackrel{\text{def}}{=} \text{not } (nullable(r))$
- $derc(\sim r) \stackrel{\text{def}}{=} \sim (derc r)$

Negation of Regular Expr's

- $\sim r$ (everything that r cannot recognise)
- $L(\sim r) \stackrel{\text{def}}{=} UNIV - L(r)$
- $nullable(\sim r) \stackrel{\text{def}}{=} \text{not } (nullable(r))$
- $derc(\sim r) \stackrel{\text{def}}{=} \sim (derc r)$

Used often for recognising comments:

$$/ \cdot * \cdot (\sim ([a-z]^* \cdot * \cdot / \cdot [a-z]^*)) \cdot * \cdot /$$

Negation

Assume you have an alphabet consisting of the letters *a*, *b* and *c* only. Find a (basic!) regular expression that matches all strings *except* *ab* and *ac*!

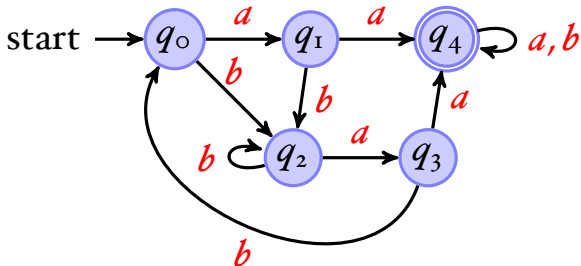
Automata

A **deterministic finite automaton**, DFA, consists of:

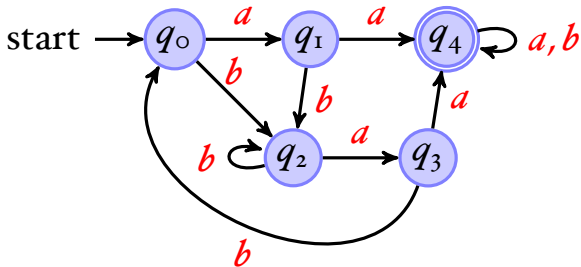
- a set of states \mathcal{Q}
- one of these states is the start state q_0
- some states are accepting states F , and
- there is transition function δ

which takes a state as argument and a character and produces a new state; this function might not be everywhere defined

$$A(\mathcal{Q}, q_0, F, \delta)$$



- the start state can be an accepting state
- it is possible that there is no accepting state
- all states might be accepting (but this does not necessarily mean all strings are accepted)



for this automaton δ is the function

$$\begin{array}{lll}
 (q_0, a) \rightarrow q_1 & (q_1, a) \rightarrow q_4 & (q_4, a) \rightarrow q_4 \\
 (q_0, b) \rightarrow q_2 & (q_1, b) \rightarrow q_2 & (q_4, b) \rightarrow q_4 \dots
 \end{array}$$

Accepting a String

Given

$$A(\mathcal{Q}, q_0, F, \delta)$$

you can define

$$\begin{aligned}\hat{\delta}(q, []) &\stackrel{\text{def}}{=} q \\ \hat{\delta}(q, c :: s) &\stackrel{\text{def}}{=} \hat{\delta}(\delta(q, c), s)\end{aligned}$$

Accepting a String

Given

$$A(\mathcal{Q}, q_0, F, \delta)$$

you can define

$$\begin{aligned}\hat{\delta}(q, []) &\stackrel{\text{def}}{=} q \\ \hat{\delta}(q, c :: s) &\stackrel{\text{def}}{=} \hat{\delta}(\delta(q, c), s)\end{aligned}$$

Whether a string s is accepted by A ?

$$\hat{\delta}(q_0, s) \in F$$

Regular Languages

A **language** is a set of strings.

A **regular expression** specifies a language.

A language is **regular** iff there exists a regular expression that recognises all its strings.

Regular Languages

A **language** is a set of strings.

A **regular expression** specifies a language.

A language is **regular** iff there exists a regular expression that recognises all its strings.

not all languages are regular, e.g. $a^n b^n$ is not

Regular Languages (2)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

Non-Deterministic Finite Automata

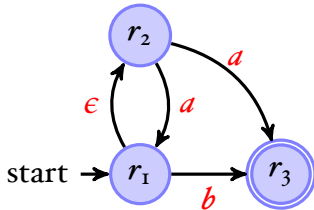
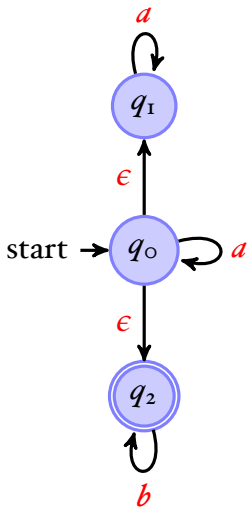
A non-deterministic finite automaton consists again of:

- a finite set of states
- one of these states is the start state
- some states are accepting states, and
- there is transition **relation**

$$\begin{aligned}(q_1, a) &\rightarrow q_2 \\ (q_1, a) &\rightarrow q_3\end{aligned}$$

$$(q_1, \epsilon) \rightarrow q_2$$

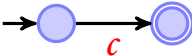
Two NFA Examples



Rexp to NFA

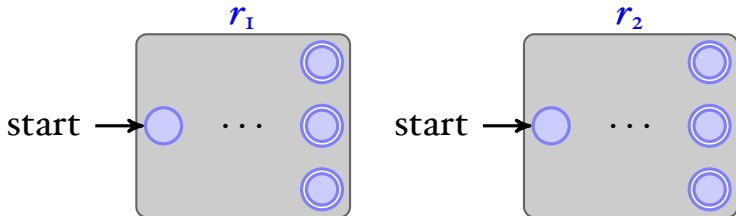
o start → 

1 start → 

c start → 

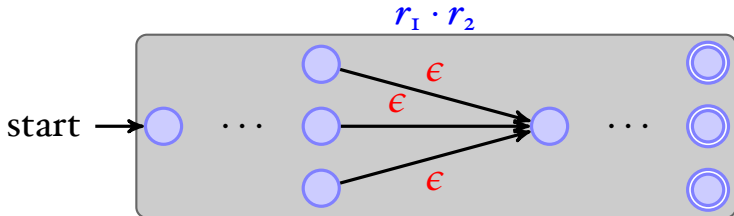
Case $r_1 \cdot r_2$

By recursion we are given two automata:



We need to (1) change the accepting nodes of the first automaton into non-accepting nodes, and (2) connect them via ϵ -transitions to the starting state of the second automaton.

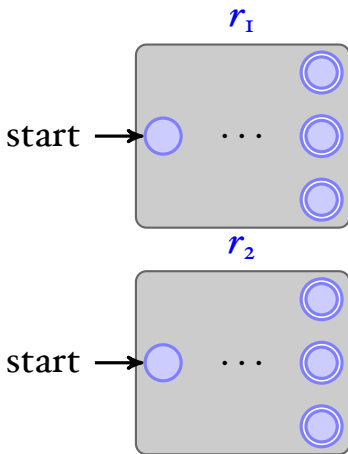
Case $r_1 \cdot r_2$



We need to (1) change the accepting nodes of the first automaton into non-accepting nodes, and (2) connect them via ϵ -transitions to the starting state of the second automaton.

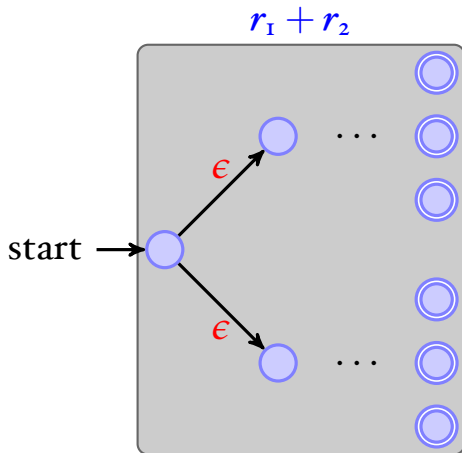
Case $r_1 + r_2$

By recursion we are given two automata:



We (1) need to introduce a new starting state and (2) connect it to the original two starting states.

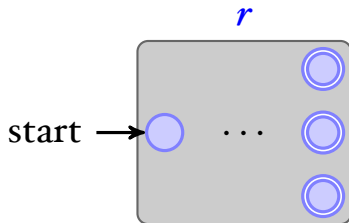
Case $r_1 + r_2$



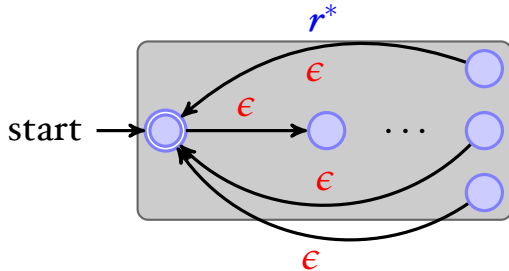
We (1) need to introduce a new starting state and (2) connect it to the original two starting states.

Case r^*

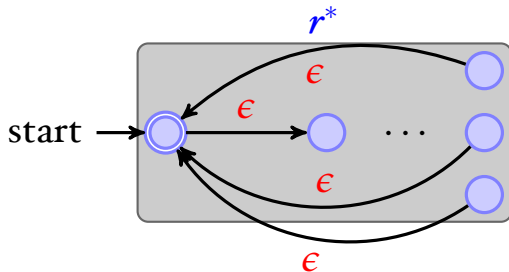
By recursion we are given an automaton for r :



Case r^*

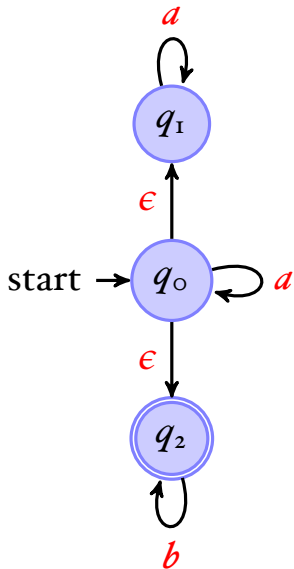


Case r^*



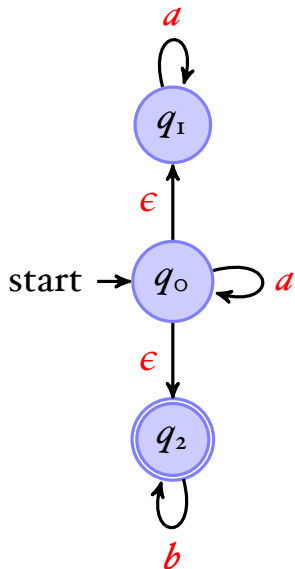
Why can't we just have an epsilon transition from the accepting states to the starting state?

Subset Construction



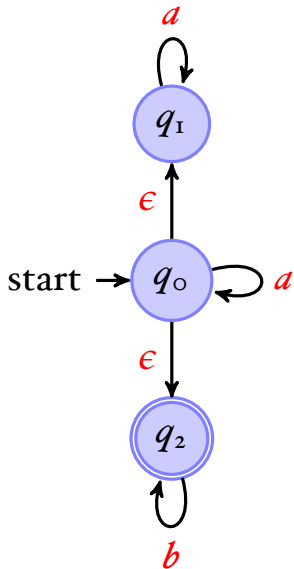
nodes	a	b
$\{\}$		
$\{0\}$		
$\{1\}$		
$\{2\}$		
$\{0, 1\}$		
$\{0, 2\}$		
$\{1, 2\}$		
$\{0, 1, 2\}$		

Subset Construction



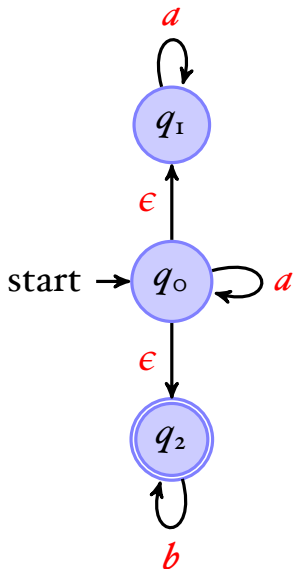
nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$		
$\{1\}$		
$\{2\}$		
$\{0, 1\}$		
$\{0, 2\}$		
$\{1, 2\}$		
$\{0, 1, 2\}$		

Subset Construction



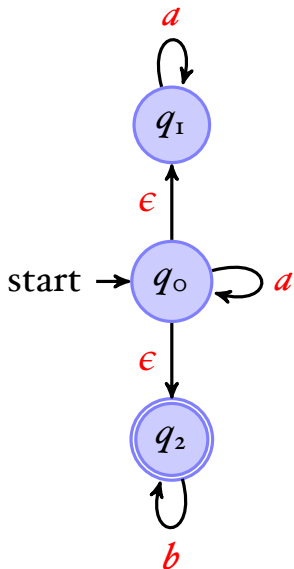
nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$	$\{0, 1, 2\}$	$\{2\}$
$\{1\}$	$\{1\}$	$\{\}$
$\{2\}$	$\{\}$	$\{2\}$
$\{0, 1\}$		
$\{0, 2\}$		
$\{1, 2\}$		
$\{0, 1, 2\}$		

Subset Construction



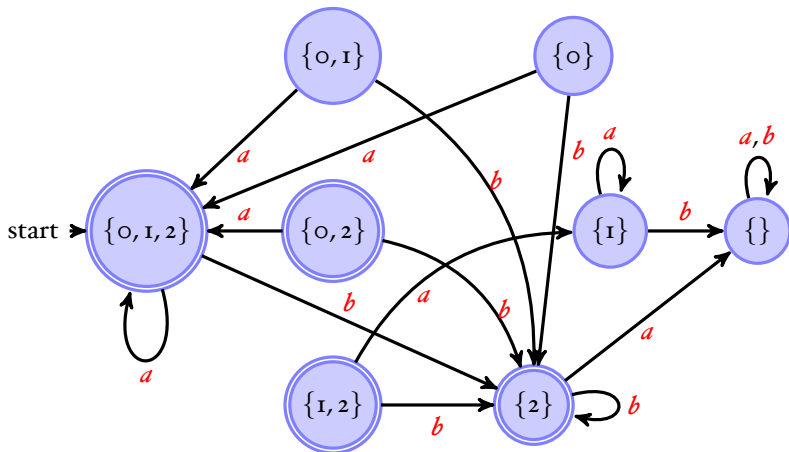
nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$	$\{0, 1, 2\}$	$\{2\}$
$\{1\}$	$\{1\}$	$\{\}$
$\{2\}$	$\{\}$	$\{2\}$
$\{0, 1\}$	$\{0, 1, 2\}$	$\{2\}$
$\{0, 2\}$	$\{0, 1, 2\}$	$\{2\}$
$\{1, 2\}$	$\{1\}$	$\{2\}$
$\{0, 1, 2\}$	$\{0, 1, 2\}$	$\{2\}$

Subset Construction



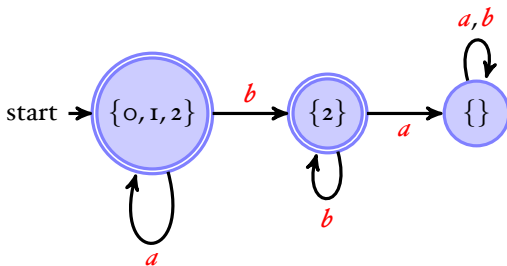
nodes	a	b
$\{\}$	$\{\}$	$\{\}$
$\{0\}$	$\{0, 1, 2\}$	$\{2\}$
$\{1\}$	$\{1\}$	$\{\}$
$\{2\}$ *	$\{\}$	$\{2\}$
$\{0, 1\}$	$\{0, 1, 2\}$	$\{2\}$
$\{0, 2\}$ *	$\{0, 1, 2\}$	$\{2\}$
$\{1, 2\}$ *	$\{1\}$	$\{2\}$
s: $\{0, 1, 2\}$ *	$\{0, 1, 2\}$	$\{2\}$

The Result

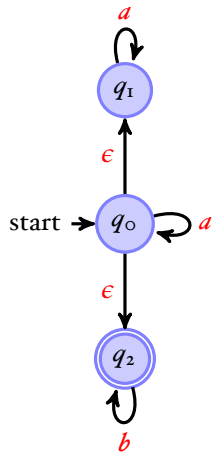


Removing Dead States

DFA:



(original) NFA:



Regexps and Automata

Thompson's construction subset construction

Regexps  **NFAs**  **DFAs**

Regexps and Automata

Thompson's construction subset construction

Regexps → **NFAs** → **DFAs** → **minimal DFAs**

minimisation

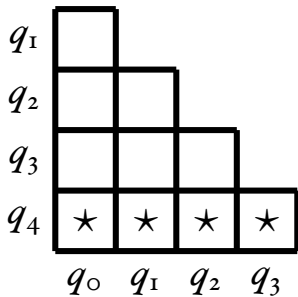
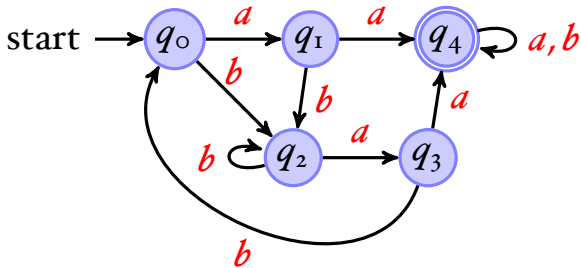
DFA Minimisation

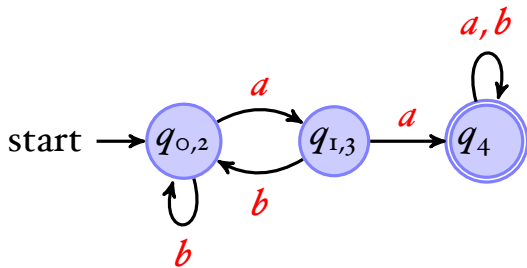
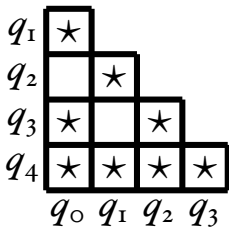
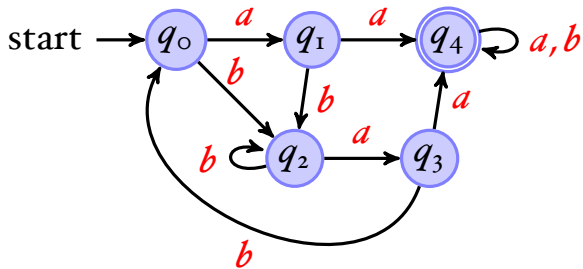
- 1 Take all pairs (q, p) with $q \neq p$
- 2 Mark all pairs that accepting and non-accepting states
- 3 For all unmarked pairs (q, p) and all characters c test whether

$$(\delta(q, c), \delta(p, c))$$

are marked. If yes in at least one case, then also mark (q, p) .

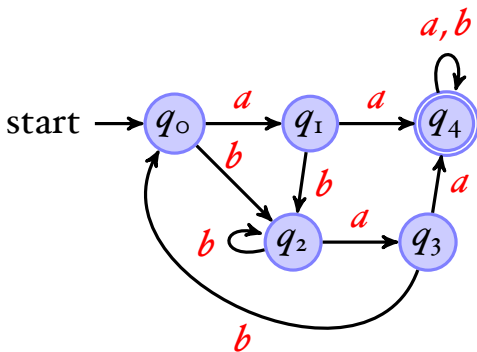
- 4 Repeat last step until no change.
- 5 All unmarked pairs can be merged.



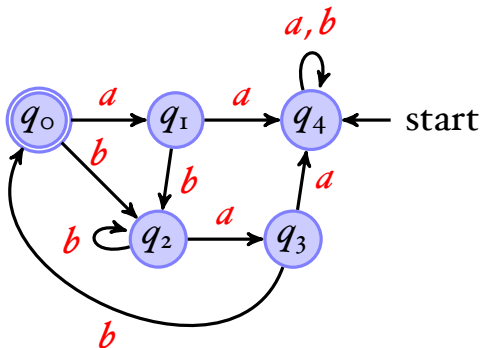


minimal automaton

Alternatives

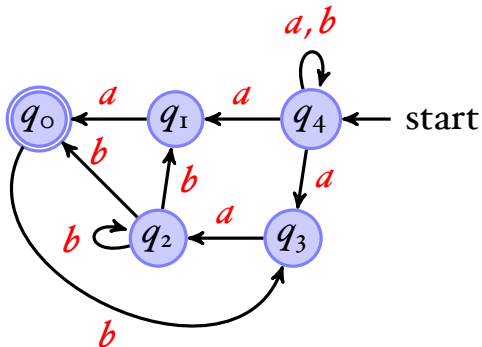


Alternatives



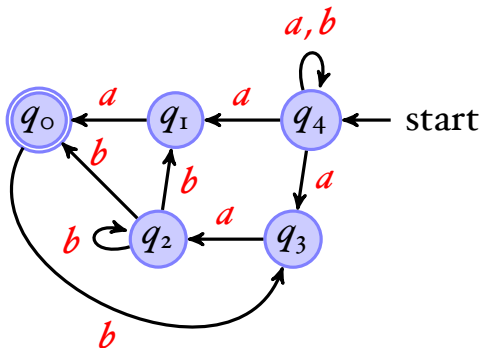
- exchange initial / accepting states

Alternatives



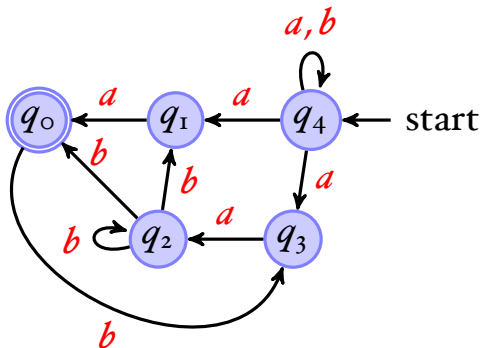
- exchange initial / accepting states
- reverse all edges

Alternatives



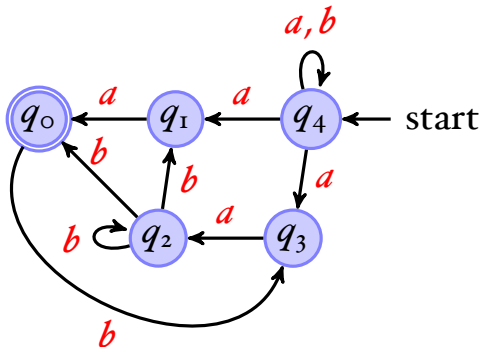
- exchange initial / accepting states
- reverse all edges
- subset construction \Rightarrow DFA

Alternatives



- exchange initial / accepting states
- reverse all edges
- subset construction \Rightarrow DFA
- remove dead states

Alternatives



- exchange initial / accepting states
- reverse all edges
- subset construction \Rightarrow DFA
- remove dead states
- repeat once more \Rightarrow minimal DFA

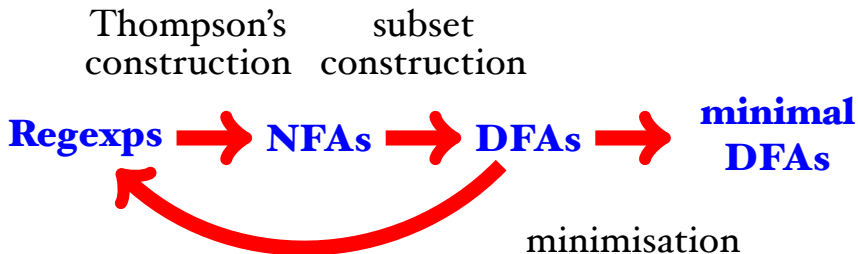
Regexps and Automata

Thompson's construction subset construction

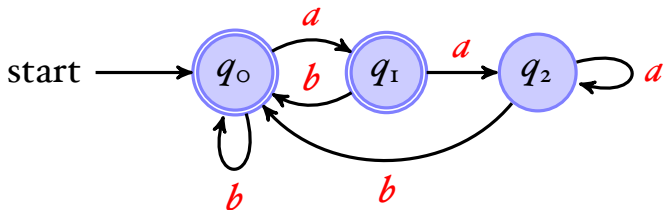
Regexps  **NFAs**  **DFAs**  **minimal DFAs**

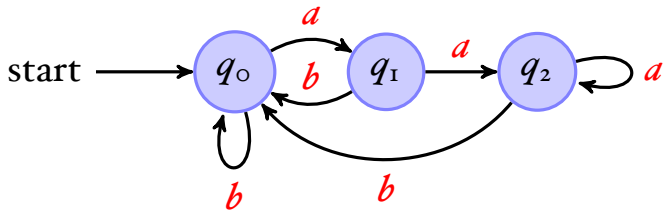
minimisation

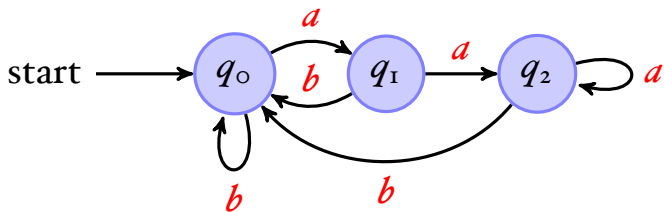
Regexps and Automata



DFA to Rexp





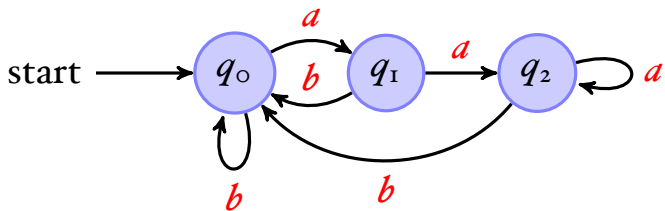


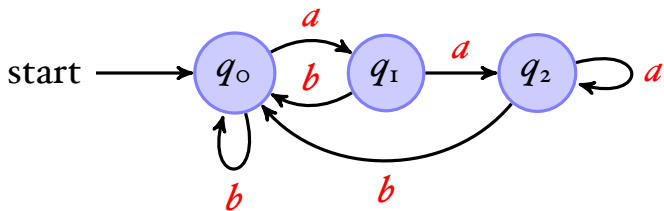
You know how to solve since school days, no?

$$q_0 = 2q_0 + 3q_1 + 4q_2$$

$$q_1 = 2q_0 + 3q_1 + 1q_2$$

$$q_2 = 1q_0 + 5q_1 + 2q_2$$

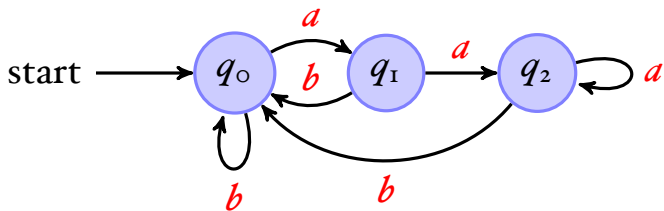




$$q_0 = \mathbf{I} + q_0 b + q_1 b + q_2 b$$

$$q_1 = q_0 a$$

$$q_2 = q_1 a + q_2 a$$



$$q_0 = \mathbf{I} + q_0 b + q_1 b + q_2 b$$

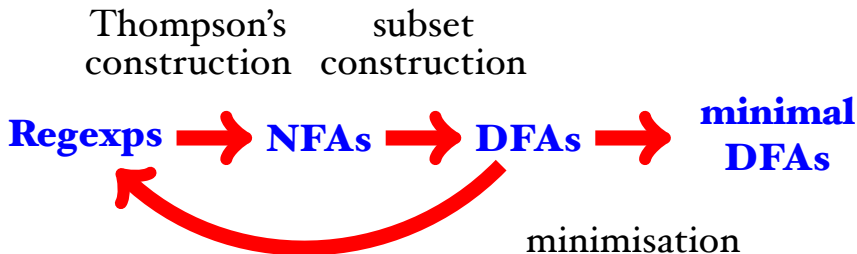
$$q_1 = q_0 a$$

$$q_2 = q_1 a + q_2 a$$

Arden's Lemma:

$$\text{If } q = qr + s \text{ then } q = sr^*$$

Regexps and Automata



Regular Languages (3)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

Regular Languages (3)

A language is **regular** iff there exists a regular expression that recognises all its strings.

or **equivalently**

A language is **regular** iff there exists a deterministic finite automaton that recognises all its strings.

Why is every finite set of strings a regular language?

Given the function

$$\mathit{rev}(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0}$$

$$\mathit{rev}(\mathbf{I}) \stackrel{\text{def}}{=} \mathbf{I}$$

$$\mathit{rev}(c) \stackrel{\text{def}}{=} c$$

$$\mathit{rev}(r_1 + r_2) \stackrel{\text{def}}{=} \mathit{rev}(r_1) + \mathit{rev}(r_2)$$

$$\mathit{rev}(r_1 \cdot r_2) \stackrel{\text{def}}{=} \mathit{rev}(r_2) \cdot \mathit{rev}(r_1)$$

$$\mathit{rev}(r^*) \stackrel{\text{def}}{=} \mathit{rev}(r)^*$$

and the set

$$\mathit{Rev} A \stackrel{\text{def}}{=} \{s^{-1} \mid s \in A\}$$

prove whether

$$L(\mathit{rev}(r)) = \mathit{Rev}(L(r))$$