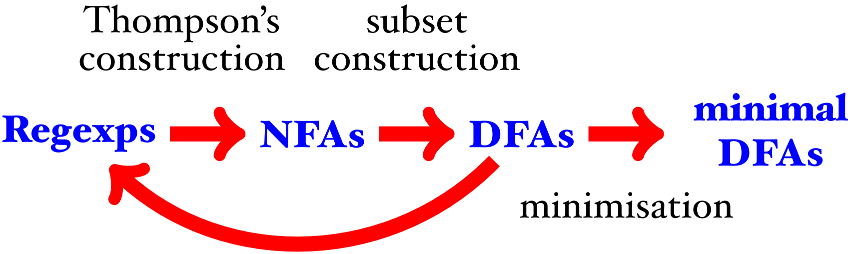
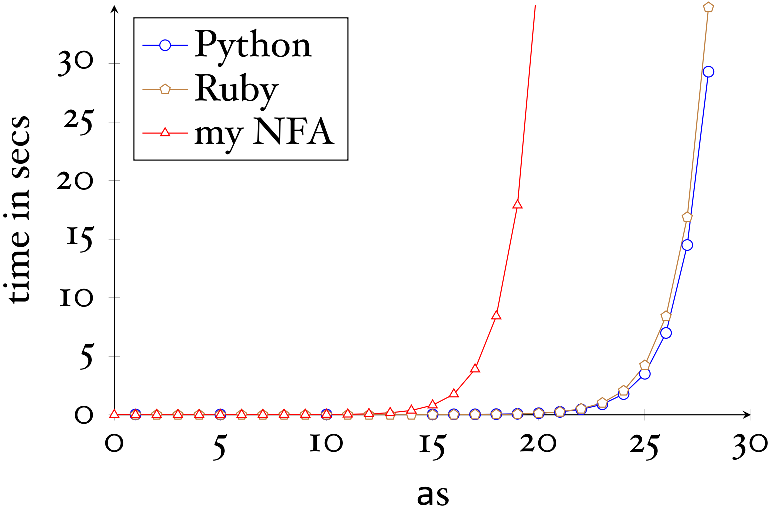
Automata and Formal Languages (4)

Email: christian.urban at kcl.ac.uk
Office: S1.27 (1st floor Strand Building)
Slides: KEATS (also home work is there)

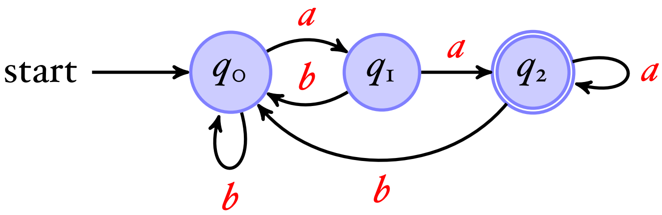
Regexps and Automata



$$(a?\{n\}) \cdot a\{n\}$$

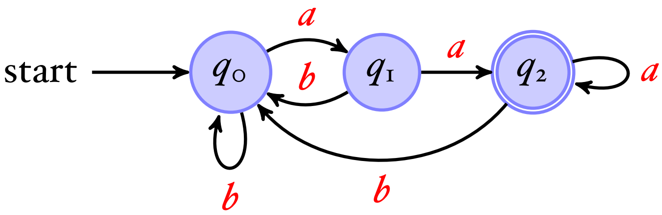

Python
Ruby
my NFA
time in secs
as
0
5
10
15
20
25
30
0
5
10
15
20
25
30

DFA to Rexp



$$
\begin{aligned}
q_0 &= \epsilon + q_0\, b + q_1\, b + q_2\, b \qquad \text{(start state)}\\
q_1 &= q_0\, a\\
q_2 &= q_1\, a + q_2\, a
\end{aligned}
$$

DFA to Rexp



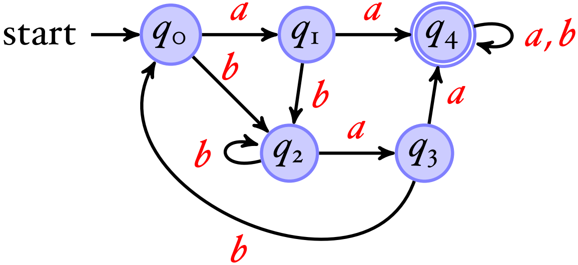
$$
\begin{aligned}
q_0 &= \epsilon + q_0\,b + q_1\,b + q_2\,b \qquad \text{(start state)}\\
q_1 &= q_0\,a\\
q_2 &= q_1\,a + q_2\,a
\end{aligned}
$$

Arden's Lemma:

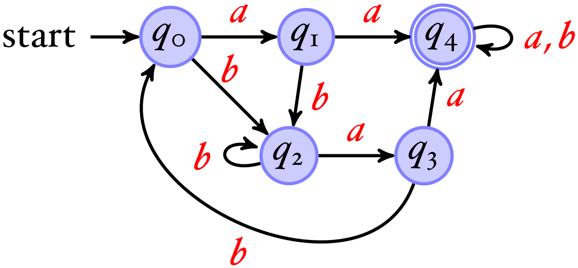
If $q = q\,r + s$ then $q = s\,r^*$

DFA Minimisation

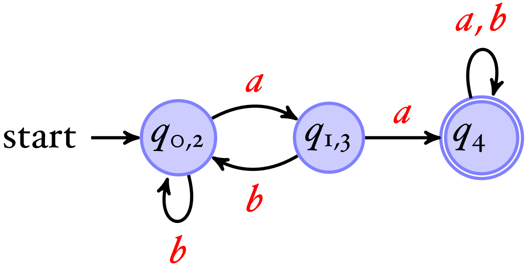
1. Take all pairs (q, p) with $q \neq p$
2. Mark all pairs that accepting and non-accepting states
3. For all unmarked pairs (q, p) and all characters c test whether
$$(\delta(q,c), \delta(p,c))$$
are marked. If yes, then also mark (q, p).
4. Repeat last step until no change.
5. All unmarked pairs can be merged.



q_1				
q_2				
q_3				
q_4	$\star$	$\star$	$\star$	$\star$
	q_0	q_1	q_2	q_3

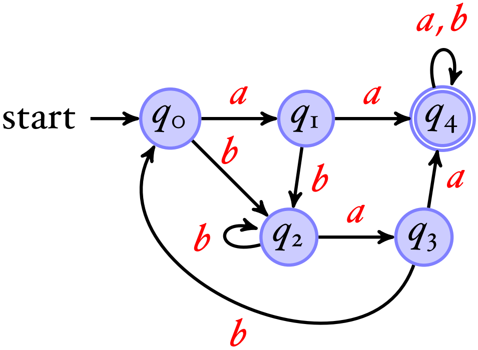


	q_0	q_1	q_2	q_3
q_1	★			
q_2		★		
q_3	★		★	
q_4	★	★	★	★

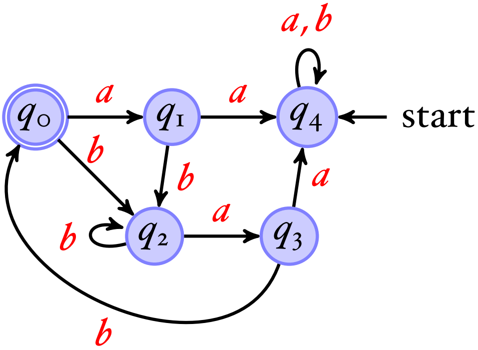


minimal automaton

Alternatives

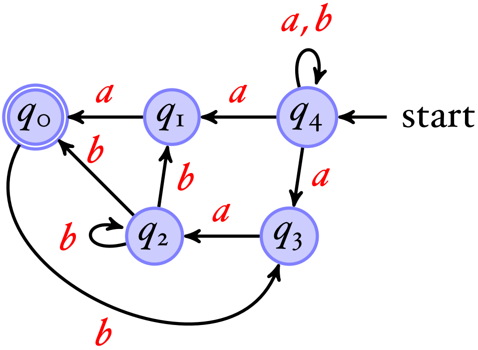


Alternatives



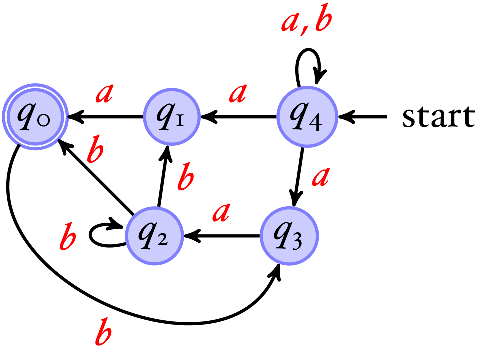
- exchange initial / accepting states

Alternatives



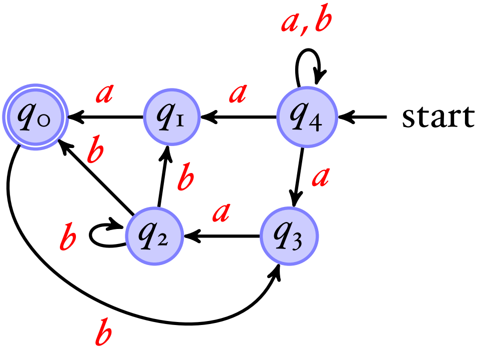
- exchange initial / accepting states
- reverse all edges

Alternatives



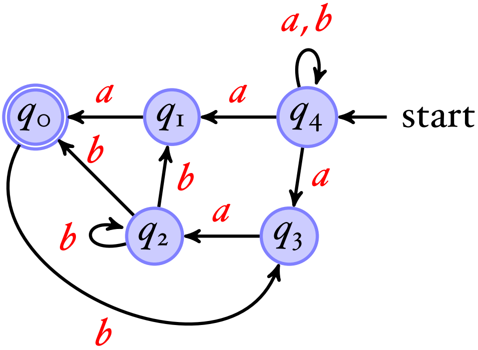
- exchange initial / accepting states
- reverse all edges
- subset construction $\Rightarrow$ DFA

Alternatives



- exchange initial / accepting states
- reverse all edges
- subset construction $\Rightarrow$ DFA
- repeat once more

Alternatives



- exchange initial / accepting states
- reverse all edges
- subset construction ⇒ DFA
- repeat once more ⇒ minimal DFA

Regular Languages

Two equivalent definitions:

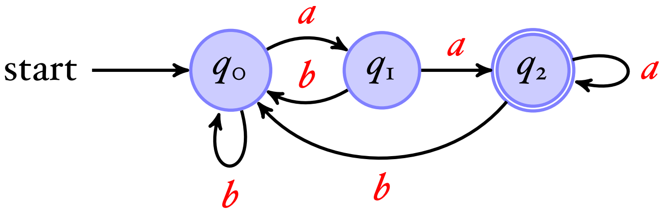
A language is regular iff there exists a regular expression that recognises all its strings.

A language is regular iff there exists an automaton that recognises all its strings.

for example $a^n b^n$ is not regular

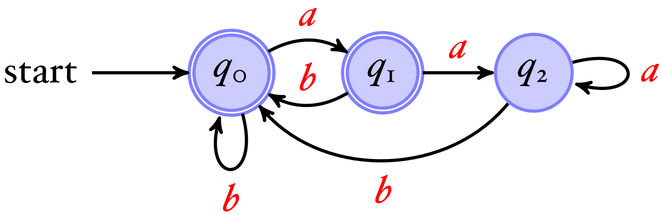
Negation

Regular languages are closed under negation:



Negation

Regular languages are closed under negation:



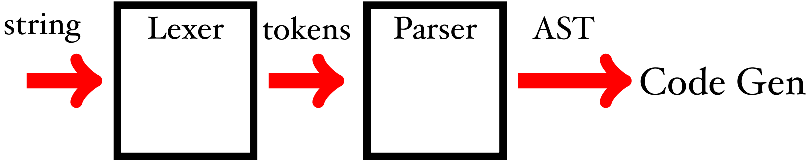
But requires that the automaton is completed!

```
write "Fib";
read n;
minus1 := 0;
minus2 := 1;
while n > 0 do {
       temp := minus2;
       minus2 := minus1 + minus2;
       minus1 := temp;
       n := n - 1
};
write "Result";
write minus2
```

??

```
write "Input a number ";
read n;
while n > 1 do {
  if n % 2 == 0
  then n := n/2
  else n := 3*n+1;
};
write "Yes";
```

A Compiler



"if true then then 42 else +"

KEYWORD:

if, then, else,

WHITESPACE:

" ", \n,

IDENT:

$LETTER \cdot (LETTER + DIGIT + \_)^*$

NUM:

$(NONZERODIGIT \cdot DIGIT^*) + 0$

OP:

+

COMMENT:

$/* \cdot \sim(ALL^* \cdot (*/) \cdot ALL^*) \cdot */$

"if true then then 42 else +"

```
KEYWORD(if),
WHITESPACE,
IDENT(true),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
KEYWORD(then),
WHITESPACE,
NUM(42),
WHITESPACE,
KEYWORD(else),
WHITESPACE,
OP(+)
```

”if true then then 42 else +”

```
KEYWORD(if),
IDENT(true),
KEYWORD(then),
KEYWORD(then),
NUM(42),
KEYWORD(else),
OP(+)
```

There is one small problem with the tokenizer. How should we tokenize:

"x-3"

```
ID: ...
OP:
  "+", "-"
NUM:
  (NONZERODIGIT · DIGIT*) + "0"
NUMBER:
  NUM + ("-" · NUM)
```

POSIX: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as the next token.
- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

POSIX: Two Rules

- Longest match rule ("maximal munch rule"): The longest initial substring matched by any regular expression is taken as the next token.

- Rule priority: For a particular longest initial substring, the first regular expression that can match determines the token.

most posix matchers are buggy
http://www.haskell.org/haskellwiki/Regex_Posix

Sulzmann Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\ a} r_2$$

Sulzmann Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3$$

Sulzmann Matcher

We want to match the string abc using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4$$

Sulzmann Matcher

We want to match the string *abc* using r_1:

$$r_1 \xrightarrow{der\,a} r_2 \xrightarrow{der\,b} r_3 \xrightarrow{der\,c} r_4\ nullable$$

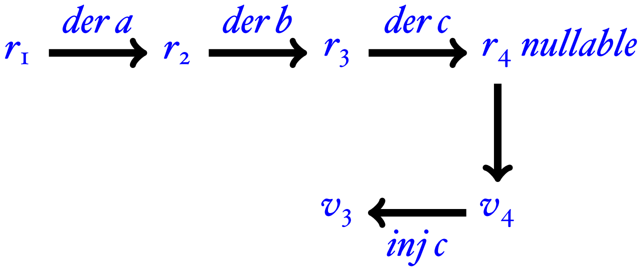
Sulzmann Matcher

We want to match the string *abc* using r_1:

$$
\begin{array}{ccccccc}
r_1 & \xrightarrow{der\ a} & r_2 & \xrightarrow{der\ b} & r_3 & \xrightarrow{der\ c} & r_4 \ nullable \\
 & & & & & & \downarrow \\
 & & & & & & v_4
\end{array}
$$

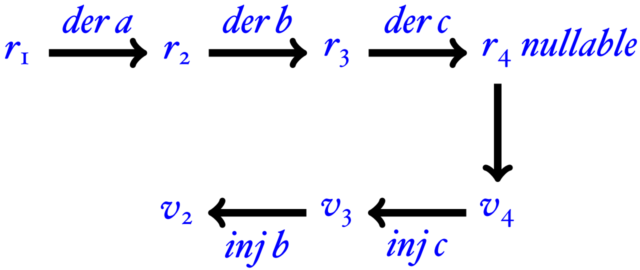
Sulzmann Matcher

We want to match the string *abc* using r_1:



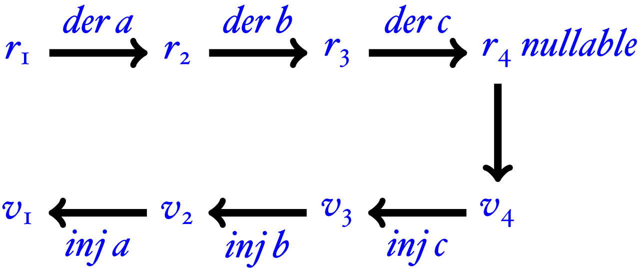
Sulzmann Matcher

We want to match the string *abc* using r_1:



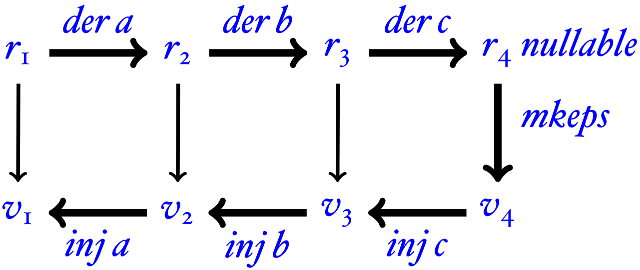
Sulzmann Matcher

We want to match the string *abc* using r_1:



Sulzmann Matcher

We want to match the string *abc* using r_1:



Regexes and Values

Regular expressions and their corresponding values:

$$
\begin{array}{rcl}
r & ::= & \varnothing \\
 & | & \epsilon \\
 & | & c \\
 & | & r_1 \cdot r_2 \\
 & | & r_1 + r_2 \\
 & & \\
 & | & r^*
\end{array}
\qquad
\begin{array}{rcl}
v & ::= & \\
 & & \mathit{Empty} \\
 & | & \mathit{Char}(c) \\
 & | & \mathit{Seq}(v_1, v_2) \\
 & | & \mathit{Left}(v) \\
 & | & \mathit{Right}(v) \\
 & | & [] \\
 & | & [v_1, \ldots v_n]
\end{array}
$$

Mkeps

Finding a (posix) value for recognising the empty string:

$$
\begin{array}{lcl}
mkeps\,\epsilon & \stackrel{\text{def}}{=} & Empty\\
mkeps\,r_1 + r_2 & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } Left(mkeps(r_1))\\
 & & \text{else } Right(mkeps(r_2))\\
mkeps\,r_1 \cdot r_2 & \stackrel{\text{def}}{=} & Seq(mkeps(r_1), mkeps(r_2))\\
mkeps\,r^* & \stackrel{\text{def}}{=} & []
\end{array}
$$

Inject

Injecting ("Adding") a character to a value

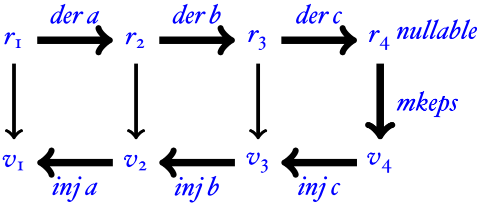
$$
\begin{array}{ll}
inj\,(c)\;c\;Empty & \stackrel{\text{def}}{=} Char\;c\\
inj\,(r_1 + r_2)\;c\;Left(v) & \stackrel{\text{def}}{=} Left(inj\;r_1\;c\;v)\\
inj\,(r_1 + r_2)\;c\;Right(v) & \stackrel{\text{def}}{=} Right(inj\;r_2\;c\;v)\\
inj\,(r_1 \cdot r_2)\;c\;Seq(v_1, v_2) & \stackrel{\text{def}}{=} Seq(inj\;r_1\;c\;v_1, v_2)\\
inj\,(r_1 \cdot r_2)\;c\;Left(Seq(v_1, v_2)) & \stackrel{\text{def}}{=} Seq(inj\;r_1\;c\;v_1, v_2)\\
inj\,(r_1 \cdot r_2)\;c\;Right(v) & \stackrel{\text{def}}{=} Seq(mkeps(r_1), inj\;r_2\;c\;v)\\
inj\,(r^*)\;c\;Seq(v, vs) & \stackrel{\text{def}}{=} inj\;r\;c\;v\;::\;vs
\end{array}
$$

inj: 1st arg $\mapsto$ a rexp; 2nd arg $\mapsto$ a character; 3rd arg $\mapsto$ a value

Lexing

$$\begin{aligned} lex\; r\; [] &\stackrel{\text{def}}{=} \text{if } nullable(r) \text{ then } mkeps(r) \text{ else } error \\ lex\; r\; c :: s &\stackrel{\text{def}}{=} inj\; r\; c\; lex(der(c, r), s) \end{aligned}$$

lex: returns a value



Records

- new regex: $(x : r)$ new value: $Rec(x, v)$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$
- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\, c\, (x : r) \stackrel{\text{def}}{=} (x : der\, c\, r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\, (x : r)\, c\, v \stackrel{\text{def}}{=} Rec(x, inj\, r\, c\, v)$

Records

- new regex: $(x : r)$ new value: $Rec(x, v)$
- $nullable(x : r) \stackrel{\text{def}}{=} nullable(r)$
- $der\, c\, (x : r) \stackrel{\text{def}}{=} (x : der\, c\, r)$
- $mkeps(x : r) \stackrel{\text{def}}{=} Rec(x, mkeps(r))$
- $inj\, (x : r)\, c\, v \stackrel{\text{def}}{=} Rec(x, inj\, r\, c\, v)$

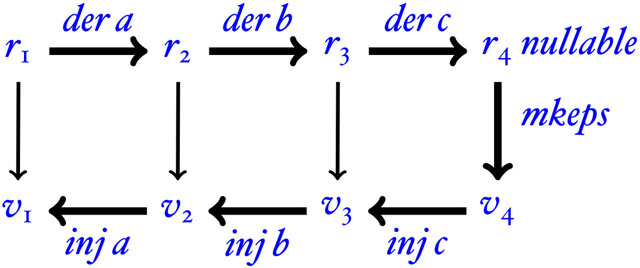
for extracting subpatterns $(z : ((x : ab) + (y : ba))$

While Tokens

```
WHILE_REGS  def=  (("k" : KEYWORD) +
                   ("i" : ID) +
                   ("o" : OP) +
                   ("n" : NUM) +
                   ("s" : SEMI) +
                   ("p" : (LPAREN + RPAREN)) +
                   ("b" : (BEGIN + END)) +
                   ("w" : WHITESPACE))*
```

Simplification

- If we simplify after the derivative, then we are builing the value for the simplified regular expression, but *not* for the original regular expression.



Rectification

			rectification functions:
$r \cdot \varnothing$	$\mapsto$	$\varnothing$	
$\varnothing \cdot r$	$\mapsto$	$\varnothing$	
$r \cdot \epsilon$	$\mapsto$	r	$\lambda f_1 f_2 v.\, Seq(f_1\, v, f_2\, Empty)$
$\epsilon \cdot r$	$\mapsto$	r	$\lambda f_1 f_2 v.\, Seq(f_1\, Empty, f_2\, v)$
$r + \varnothing$	$\mapsto$	r	$\lambda f_1 f_2 v.\, Left(f_1\, v)$
$\varnothing + r$	$\mapsto$	r	$\lambda f_1 f_2 v.\, Right(f_2\, v)$
$r + r$	$\mapsto$	r	$\lambda f_1 f_2 v.\, Left(f_1\, v)$

Rectification

rectification functions:

$r \cdot \varnothing$	$\mapsto$	$\varnothing$	
$\varnothing \cdot r$	$\mapsto$	$\varnothing$	
$r \cdot \epsilon$	$\mapsto$	r	$\lambda f_1 f_2 v. Seq(f_1\, v, f_2\, Empty)$
$\epsilon \cdot r$	$\mapsto$	r	$\lambda f_1 f_2 v. Seq(f_1\, Empty, f_2\, v)$
$r + \varnothing$	$\mapsto$	r	$\lambda f_1 f_2 v. Left(f_1\, v)$
$\varnothing + r$	$\mapsto$	r	$\lambda f_1 f_2 v. Right(f_2\, v)$
$r + r$	$\mapsto$	r	$\lambda f_1 f_2 v. Left(f_1\, v)$

old *simp* returns a rexp;
new *simp* returns a rexp and a rectification fun.

Lexing with Simplification

$$
\begin{array}{lcl}
lex\; r\; [] & \stackrel{\text{def}}{=} & \text{if } nullable(r) \text{ then } mkeps(r) \text{ else } error \\
lex\; r\; c :: s & \stackrel{\text{def}}{=} & \text{let } (r', frect) = simp(der(c, r)) \\
 & & inj\; r\; c\; (frect(lex(r', s)))
\end{array}
$$