Compilers and Formal Languages (2)

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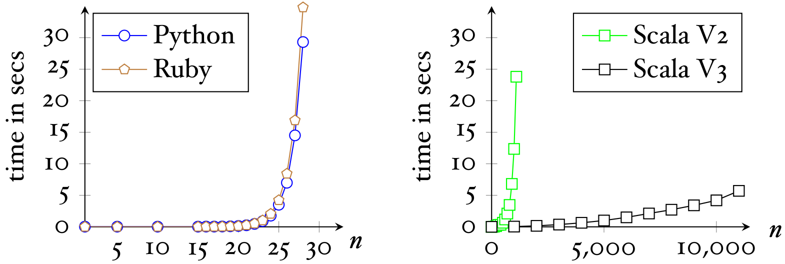
Slides: KEATS (also homework is there)

Lets Implement an Efficient Regular Expression Matcher

Graphs: $a^{?\{n\}} \cdot a^{\{n\}}$ and strings $\underbrace{a\ldots a}_{n}$



In the handouts is a similar graph for $(a^*)^* \cdot b$ and Java 8.

Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
  - $a^{?\{n\}} \cdot a^{\{n\}}$
  - $(a^*)^*$
  - $([a\text{-}z]^+)^*$
  - $(a + a \cdot a)^*$
  - $(a + a^?)^*$
- sometimes also called catastrophic backtracking
- ...I hope you have watched the video by the StackExchange engineer

Languages

- A **Language** is a set of strings, for example

$$\{[], hello, foobar, a, abc\}$$

- **Concatenation** of strings and languages

$$foo @ bar = foobar$$
$$A @ B \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

For example $A = \{foo, bar\}$, $B = \{a, b\}$

$$A @ B = \{fooa, foob, bara, barb\}$$

The Power Operation

- The ***n*th Power** of a language:

$$A^0 \stackrel{\text{def}}{=} \{[]\}$$
$$A^{n+1} \stackrel{\text{def}}{=} A \,@\, A^n$$

For example

$$A^4 = A \,@\, A \,@\, A \,@\, A \qquad (@\,\{[]\})$$
$$A^1 = A \qquad (@\,\{[]\})$$
$$A^0 = \{[]\}$$

Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in A^4?

Homework Question

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How many strings are in A^4?

What if $A = \{[a], [b], [c], []\}$;
how many strings are then in A^4?

The Star Operation

- The **Kleene Star** of a language:

$$A\star \stackrel{\text{def}}{=} \bigcup_{0\leq n} A^n$$

This expands to

$$A^0 \cup A^1 \cup A^2 \cup A^3 \cup A^4 \cup \ldots$$

or

$$\{[]\} \,\cup\, A \,\cup\, A\,@\,A \,\cup\, A\,@\,A\,@\,A \,\cup\, A\,@\,A\,@\,A\,@\,A \cup \ldots$$

The Meaning of a Regular Expression

$$
\begin{array}{rcl}
L(\mathbf{0}) & \stackrel{\text{def}}{=} & \{\} \\
L(\mathbf{1}) & \stackrel{\text{def}}{=} & \{[]\} \\
L(c) & \stackrel{\text{def}}{=} & \{[c]\} \\
L(r_1 + r_2) & \stackrel{\text{def}}{=} & L(r_1) \cup L(r_2) \\
L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \{s_1 @ s_2 \mid s_1 \in L(r_1) \wedge s_2 \in L(r_2)\} \\
L(r^*) & \stackrel{\text{def}}{=} & (L(r))\star \quad \stackrel{\text{def}}{=} \bigcup_{0 \leq n} L(r)^n
\end{array}
$$

L is a function from regular expressions to sets of strings (languages):

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

Questions?

homework (written exam 80%)
coursework (20%; first one today)
submission Fridays @ 18:00 – accepted until Mondays

Semantic Derivative

- The **Semantic Derivative** of a language w.r.t. to a character c:

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \{\} \end{aligned}$$

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$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \{\} \end{aligned}$$

We can extend this definition to strings

$$Ders\,s\,A = \{s' \mid s @ s' \in A\}$$

The Specification for Matching

A regular expression r matches a string s provided

$$s \in L(r)$$

...and the point of the this lecture is to decide this problem as fast as possible (unlike Python, Ruby, Java etc)

Regular Expressions

Their inductive definition:

r	$::=$	$\mathbf{0}$	nothing
	$\mid$	$\mathbf{1}$	empty string / "" / []
	$\mid$	c	single character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

Th

```scala
abstract class Rexp
case object ZERO extends Rexp
case object ONE extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

r	$::=$	$\mathbf{0}$	nothing
	$\mid$	$\mathbf{1}$	empty string / "" / []
	$\mid$	c	single character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

When Are Two Regular Expressions Equivalent?

$$r_1 \equiv r_2 \quad \stackrel{\text{def}}{=} \quad L(r_1) = L(r_2)$$

Concrete Equivalences

$$
\begin{array}{rcl}
(a+b)+c & \equiv & a+(b+c) \\
a+a & \equiv & a \\
a+b & \equiv & b+a \\
(a\cdot b)\cdot c & \equiv & a\cdot(b\cdot c) \\
c\cdot(a+b) & \equiv & (c\cdot a)+(c\cdot b)
\end{array}
$$

Concrete Equivalences

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c\cdot(a+b) & \equiv & (c\cdot a)+(c\cdot b)
\end{array}$$

$$\begin{array}{rcl}
a\cdot a & \not\equiv & a \\
a+(b\cdot c) & \not\equiv & (a+b)\cdot(a+c)
\end{array}$$

Corner Cases

$$
\begin{aligned}
a \cdot \mathbf{0} &\not\equiv a \\
a + \mathbf{1} &\not\equiv a \\
\mathbf{1} &\equiv \mathbf{0}^* \\
\mathbf{1}^* &\equiv \mathbf{1} \\
\mathbf{0}^* &\not\equiv \mathbf{0}
\end{aligned}
$$

Simplification Rules

$$
\begin{aligned}
r + \mathbf{0} &\equiv r \\
\mathbf{0} + r &\equiv r \\
r \cdot \mathbf{1} &\equiv r \\
\mathbf{1} \cdot r &\equiv r \\
r \cdot \mathbf{0} &\equiv \mathbf{0} \\
\mathbf{0} \cdot r &\equiv \mathbf{0} \\
r + r &\equiv r
\end{aligned}
$$

Another Homework Question

- How many basic regular expressions are there to match the string *abcd*?

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- How many if they cannot include **1** and **0**?

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- How many if they are also not allowed to contain stars?

Another Homework Question

- How many basic regular expressions are there to match the string *abcd*?
- How many if they cannot include $\mathbf{1}$ and $\mathbf{0}$?
- How many if they are also not allowed to contain stars?
- How many if they are also not allowed to contain $\_ + \_$?

Brzozowski's Algorithm (1)

...whether a regular expression can match the empty string:

$$
\begin{aligned}
nullable(\mathbf{0}) &\stackrel{\text{def}}{=} false\\
nullable(\mathbf{1}) &\stackrel{\text{def}}{=} true\\
nullable(c) &\stackrel{\text{def}}{=} false\\
nullable(r_1 + r_2) &\stackrel{\text{def}}{=} nullable(r_1) \vee nullable(r_2)\\
nullable(r_1 \cdot r_2) &\stackrel{\text{def}}{=} nullable(r_1) \wedge nullable(r_2)\\
nullable(r^*) &\stackrel{\text{def}}{=} true
\end{aligned}
$$

The Derivative of a Rexp

If r matches the string $c::s$, what is a regular expression that matches just s?

$der\, c\, r$ gives the answer, Brzozowski 1964

The Derivative of a Rexp

$$\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \mathbf{1} \text{ else } \mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else } (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}$$

The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\mathbf{0}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(\mathbf{1}) & \stackrel{\text{def}}{=} & \mathbf{0}\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if}\ c = d\ \text{then}\ \mathbf{1}\ \text{else}\ \mathbf{0}\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if}\ nullable(r_1)\\
 & & \text{then}\ (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else}\ (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)\\
\\
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}
$$

Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$\begin{aligned} der\,a\,r &= ? \\ der\,b\,r &= ? \\ der\,c\,r &= ? \end{aligned}$$

The Brzozowski Algorithm

$$matches\, r\, s \stackrel{\text{def}}{=} nullable(ders\ s\ r)$$

Brzozowski: An Example

Does r_1 match abc?

Step 1:	build derivative of a and r_1	$(r_2 = der\, a\, r_1)$
Step 2:	build derivative of b and r_2	$(r_3 = der\, b\, r_2)$
Step 3:	build derivative of c and r_3	$(r_4 = der\, c\, r_3)$
Step 4:	the string is exhausted: test whether r_4 can recognise the empty string	$(nullable(r_4))$
Output:	result of the test $\Rightarrow$ $true$ or $false$	

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\, a\, (L(r_1))$

The Idea of the Algorithm

If we want to recognise the string abc with regular expression r_1 then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$

The Idea of the Algorithm

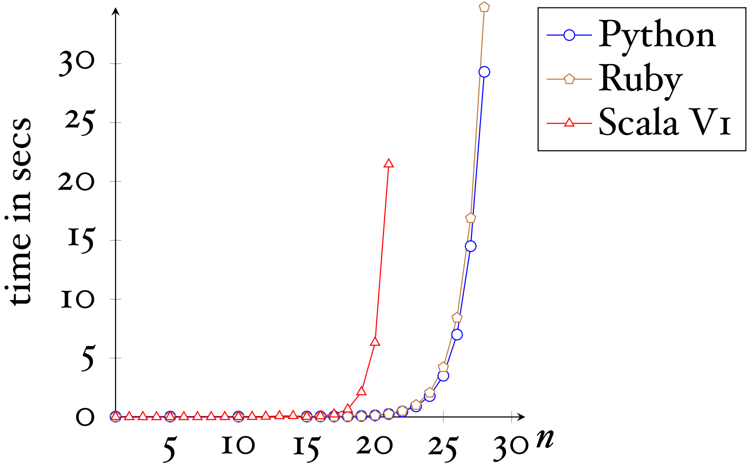
If we want to recognise the string abc with regular expression r_1 then

1. $Der\, a\, (L(r_1))$
2. $Der\, b\, (Der\, a\, (L(r_1)))$
3. $Der\, c\, (Der\, b\, (Der\, a\, (L(r_1))))$

4. finally we test whether the empty string is in this set; same for $Ders\, abc\, (L(r_1))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

Oops... $a^{?\{n\}} \cdot a^{\{n\}}$



A Problem

We represented the "n-times" $a^{\{n\}}$ as a sequence regular expression:

$$
\begin{array}{rl}
1: & a\\
2: & a \cdot a\\
3: & a \cdot a \cdot a\\
 & \ldots\\
13: & a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a\\
 & \ldots\\
20: &
\end{array}
$$

This problem is aggravated with $a^?$ being represented as $a + \mathbf{1}$.

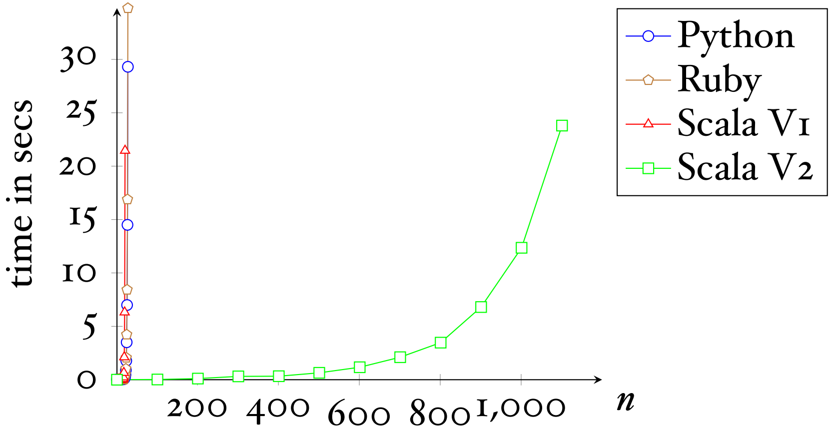
Solving the Problem

What happens if we extend our regular expressions with explicit constructors

$$
\begin{array}{rcl}
r & ::= & \ldots \\
  & | & r^{\{n\}} \\
  & | & r^{?}
\end{array}
$$

What is their meaning?
What are the cases for *nullable* and *der*?

Brzozowski: $a^{?\{n\}} \cdot a^{\{n\}}$



Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$
\begin{aligned}
der\, a\, r &= ((\mathbf{1} \cdot b) + \mathbf{0}) \cdot r \\
der\, b\, r &= ((\mathbf{0} \cdot b) + \mathbf{1}) \cdot r \\
der\, c\, r &= ((\mathbf{0} \cdot b) + \mathbf{0}) \cdot r
\end{aligned}
$$

What are these regular expressions equivalent to?

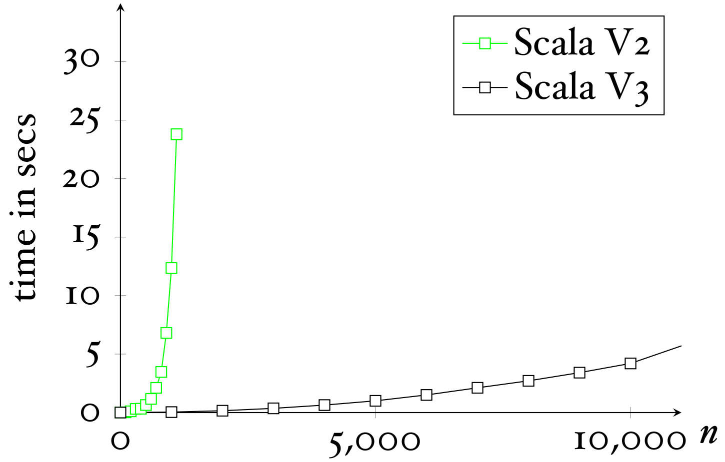
Simplification Rules

$$
\begin{array}{rcl}
r + \mathbf{0} & \Rightarrow & r \\
\mathbf{0} + r & \Rightarrow & r \\
r \cdot \mathbf{1} & \Rightarrow & r \\
\mathbf{1} \cdot r & \Rightarrow & r \\
r \cdot \mathbf{0} & \Rightarrow & \mathbf{0} \\
\mathbf{0} \cdot r & \Rightarrow & \mathbf{0} \\
r + r & \Rightarrow & r
\end{array}
$$

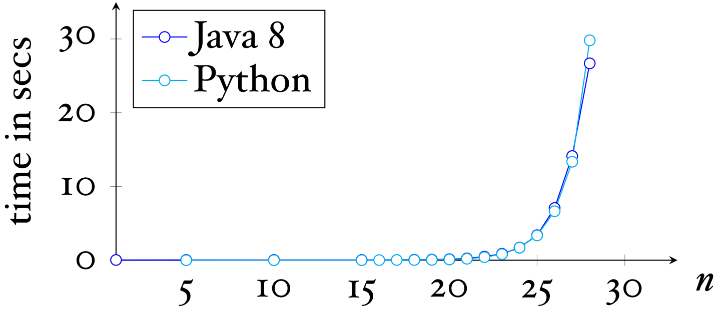
```
def ders(s: List[Char], r: Rexp) : Rexp = s match {
  case Nil => r
  case c::s => ders(s, simp(der(c, r)))
}
```

```scala
def simp(r: Rexp) : Rexp = r match {
  case ALT(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, r2s) => r2s
      case (r1s, ZERO) => r1s
      case (r1s, r2s) =>
        if (r1s == r2s) r1s else ALT(r1s, r2s)
    }
  }
  case SEQ(r1, r2) => {
    (simp(r1), simp(r2)) match {
      case (ZERO, _) => ZERO
      case (_, ZERO) => ZERO
      case (ONE, r2s) => r2s
      case (r1s, ONE) => r1s
      case (r1s, r2s) => SEQ(r1s, r2s)
    }
  }
  case r => r
}
```

Brzozowski: $a^{?\{n\}} \cdot a^{\{n\}}$



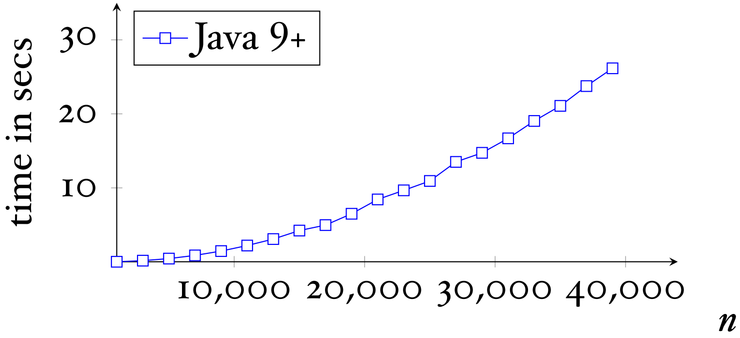
Another Example in Java 8 and Python



Regex: $(a^*)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$

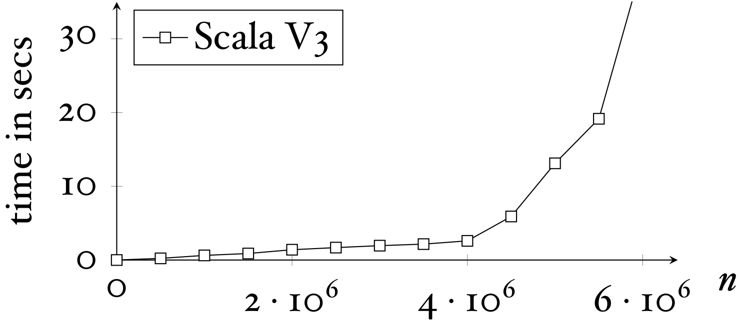
Same Example in Java 9+



Regex: $(a^*)^* \cdot b$

Strings of the form $\underbrace{a \ldots a}_{n}$

and with Brzozowski



Regex: $(a^*)^* \cdot b$
Strings of the form $\underbrace{a \ldots a}_{n}$

What is good about this Alg.

- extends to most regular expressions, for example $\sim r$ (next slide)
- is easy to implement in a functional language (slide after)
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is relatively new work)
- we can prove its correctness...

Negation of Regular Expr's

- $\sim r$ (everything that r cannot recognise)
- $L(\sim r) \stackrel{\text{def}}{=} UNIV - L(r)$
- $nullable(\sim r) \stackrel{\text{def}}{=} not\,(nullable(r))$
- $der\,c\,(\sim r) \stackrel{\text{def}}{=} \sim (der\,c\,r)$

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Used often for recognising comments:

$$/ \cdot * \cdot (\sim ([a\text{-}z]^* \cdot * \cdot / \cdot [a\text{-}z]^*)) \cdot * \cdot /$$

Coursework

Strand 1:

- Submission on Friday 12 October
  accepted until Monday 15 @ 18:00
- source code needs to be submitted as well
- you can re-use my Scala code from KEATS
  or use any programming language you like
- https://nms.kcl.ac.uk/christian.urban/ProgInScala2ed.pdf

Proofs about Rexps

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \mathbf{0} \\
  & \mid & \mathbf{1} \\
  & \mid & c \\
  & \mid & r_1 \cdot r_2 \\
  & \mid & r_1 + r_2 \\
  & \mid & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{c}$
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$nullable(r) \text{ if and only if } [] \in L(r)$$

Proofs about Rexp (4)

$$
\begin{aligned}
rev(\mathbf{0}) &\stackrel{\text{def}}{=} \mathbf{0}\\
rev(\mathbf{1}) &\stackrel{\text{def}}{=} \mathbf{1}\\
rev(c) &\stackrel{\text{def}}{=} c\\
rev(r_1 + r_2) &\stackrel{\text{def}}{=} rev(r_1) + rev(r_2)\\
rev(r_1 \cdot r_2) &\stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1)\\
rev(r^*) &\stackrel{\text{def}}{=} rev(r)^*
\end{aligned}
$$

We can prove

$$L(rev(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on r.

Correctness Proof for our Matcher

- We started from

$$\begin{array}{rl} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\, s\, (L(r)) \end{array}$$

Correctness Proof for our Matcher

- We started from

$$\begin{array}{rl} & s \in L(r) \\ \Leftrightarrow & [] \in Ders\,s\,(L(r)) \end{array}$$

- if we can show $Ders\,s\,(L(r)) = L(ders\,s\,r)$ we have

$$\begin{array}{rl} \Leftrightarrow & [] \in L(ders\,s\,r) \\ \Leftrightarrow & nullable(ders\,s\,r) \\ \stackrel{\text{def}}{=} & matches\,s\,r \end{array}$$

Proofs about Rexp (5)

Let $Der\,c\,A$ be the set defined as

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

We can prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

by induction on r.

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

We can then prove

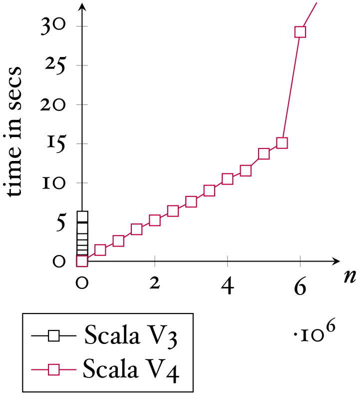
$$Ders\,s\,(L(r)) = L(ders\,s\,r)$$

We can finally prove

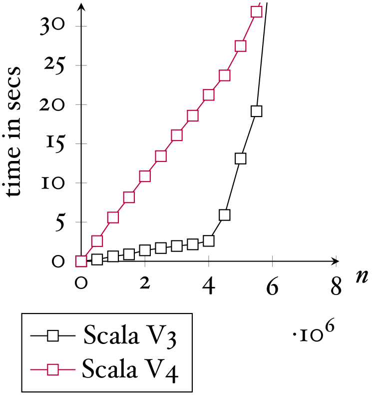
$$matches\,s\,r \text{ if and only if } s \in L(r)$$

Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$



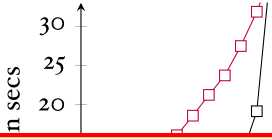
Graph: $(a^*)^* \cdot b$



Epilogue

Graph: $a^{?\{n\}} \cdot a^{\{n\}}$

Graph: $(a^*)^* \cdot b$




```
def ders2(s: List[Char], r: Rexp) : Rexp = (s, r) match {
  case (Nil, r) => r
  case (s, ZERO) => ZERO
  case (s, ONE) => if (s == Nil) ONE else ZERO
  case (s, CHAR(c)) => if (s == List(c)) ONE else
                       if (s == Nil) CHAR(c) else ZERO
  case (s, ALT(r1, r2)) => ALT(ders2(s, r2), ders2(s, r2))
  case (c::s, r) => ders2(s, simp(der(c, r)))
}
```