# Compilers and Formal Languages

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|---|---|
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| Slides: | KEATS (also home work is there) |

**There are more problems, than there are programs.**

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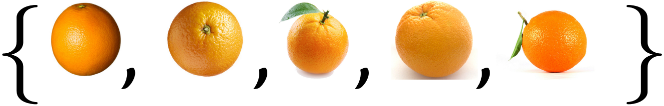
There must be a problem for which there is no program.

# Subsets

If $A \subseteq B$ then $A$ has fewer elements than $B$

$A \subseteq B$ and $B \subseteq A$

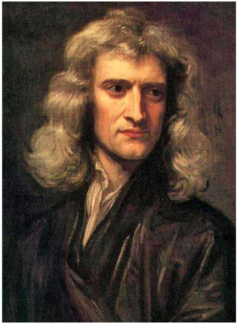
then $A = B$



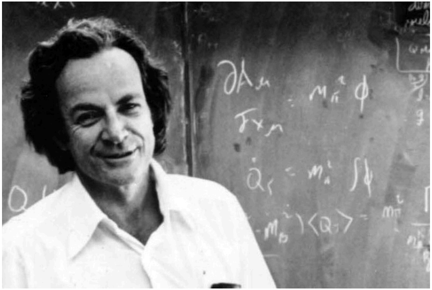
5 elements

3 elements

# Newton vs Feynman



classical physics



quantum physics

# The Goal of the Talk

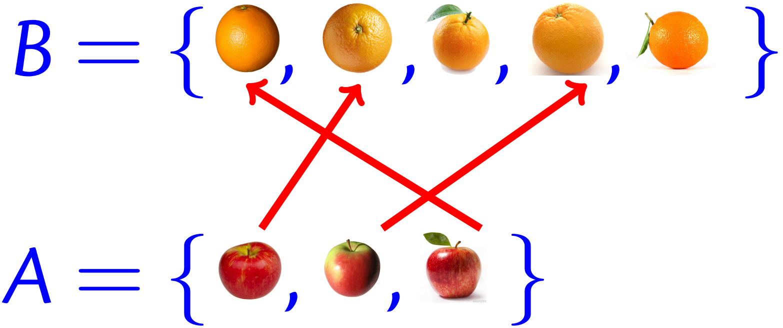
- show you that something very unintuitive happens with very large sets

- convince you that there are more **problems** than **programs**

$B = \{$  $\}$

$A = \{$  $\}$

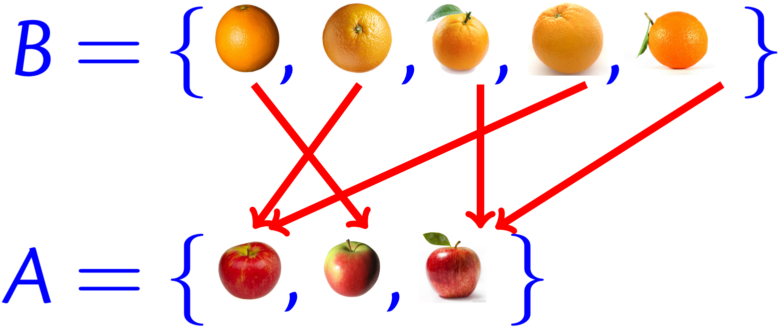
$$|A| = 5, |B| = 3$$



$B = \{\ ,\ ,\ ,\ ,\ \}$

$A = \{\ ,\ ,\ \}$

then $|A| \leq |B|$



for $=$ has to be a **one-to-one** mapping

# Cardinality

$|A| \overset{\text{def}}{=}$ "how many elements"

$$A \subseteq B \Rightarrow |A| \leq |B|$$

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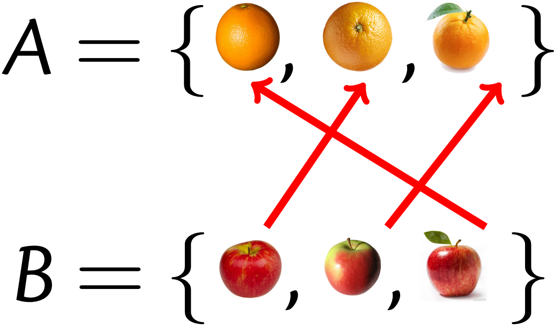
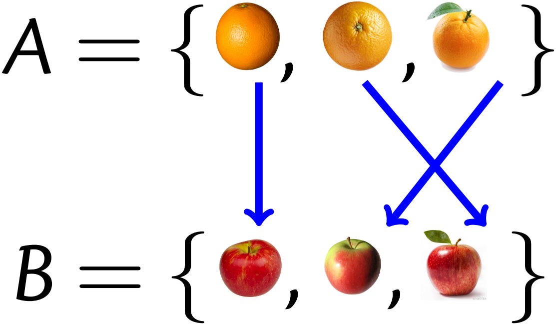
if there is an injective function $f : A \rightarrow B$ then $|A| \leq |B|$

$$\forall xy.\, f(x) = f(y) \Rightarrow x = y$$



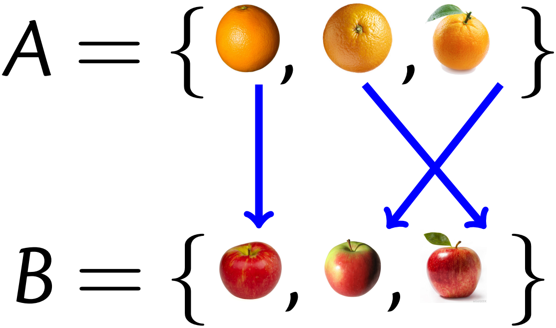
$A = \{\ ,\ ,\ \}$

$B = \{\ ,\ ,\ \}$



$A = \{\;,\;,\;\}$

$B = \{\;,\;,\;\}$



then $|A| = |B|$

# Natural Numbers

$$\mathbb{N} \overset{\text{def}}{=} \{0, 1, 2, 3, \ldots\ldots\}$$

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$A$ is countable iff

$$|A| \leq |\mathbb{N}|$$

# First Question

$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$ ?

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$$|\mathbb{N} - \{0\}| \quad ? \quad |\mathbb{N}|$$

$\geq$ or $\leq$ or $=$ ?

$x \mapsto x + 1$,
$|\mathbb{N} - \{0\}| = |\mathbb{N}|$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \overset{\text{def}}{=}$ odd numbers $\{1, 3, 5......\}$

$$|\mathbb{N} - \{0, 1\}| \quad ? \quad |\mathbb{N}|$$

$$|\mathbb{N} - \mathbb{O}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{O} \overset{\text{def}}{=}$ odd numbers $\{1, 3, 5.....\}$
$\mathbb{E} \overset{\text{def}}{=}$ even numbers $\{0, 2, 4.....\}$

$$|\mathbb{N} \cup -\mathbb{N}| \quad ? \quad |\mathbb{N}|$$

$\mathbb{N} \overset{\text{def}}{=}$ positive numbers
$\{0, 1, 2, 3, ......\}$
$-\mathbb{N} \overset{\text{def}}{=}$ negative numbers
$\{0, -1, -2, -3, ......\}$

$A$ is countable if there exists an injective $f : A \rightarrow \mathbb{N}$

$A$ is uncountable if there does not exist an injective $f : A \rightarrow \mathbb{N}$

countable: $|A| \leq |\mathbb{N}|$
uncountable: $|A| > |\mathbb{N}|$

$A$ is countable if there exists an injective $f : A \to \mathbb{N}$

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countable: $|A| \leq |\mathbb{N}|$
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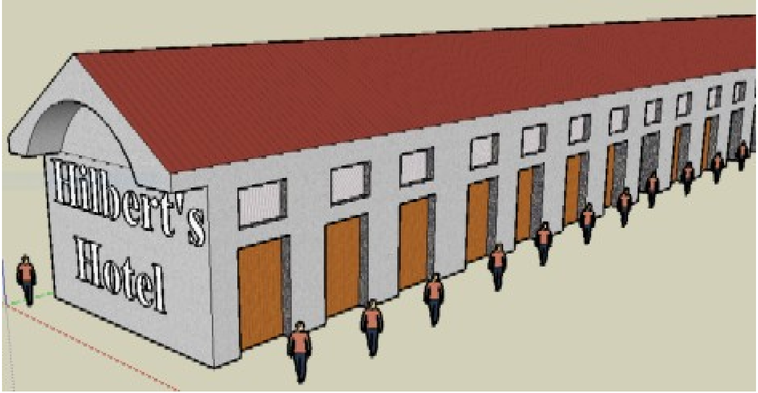
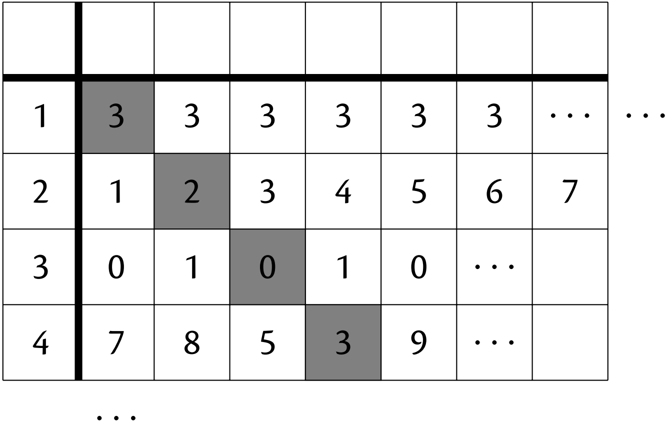
Does there exist such an

# Hilbert's Hotel



- ...has as many rooms as there are natural numbers

# Real Numbers between 0 and 1



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 3 | 3 | 3 | 3 | . . . |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 0 | 1 | 0 | 1 | 0 | . . . | |
| 4 | 7 | 8 | 5 | 3 | 9 | . . . | |

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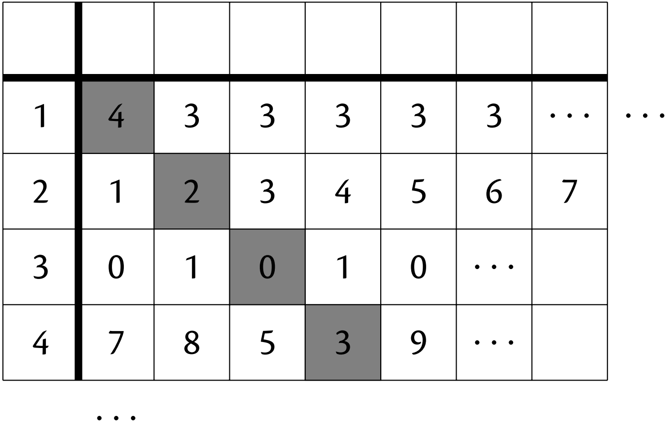
# Real Numbers between 0 and 1



| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 3 | 3 | 3 | 3 | . . . | . . . |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
| 3 | 0 | 1 | 0 | 1 | 0 | . . . | | |
| 4 | 7 | 8 | 5 | 3 | 9 | . . . | | |

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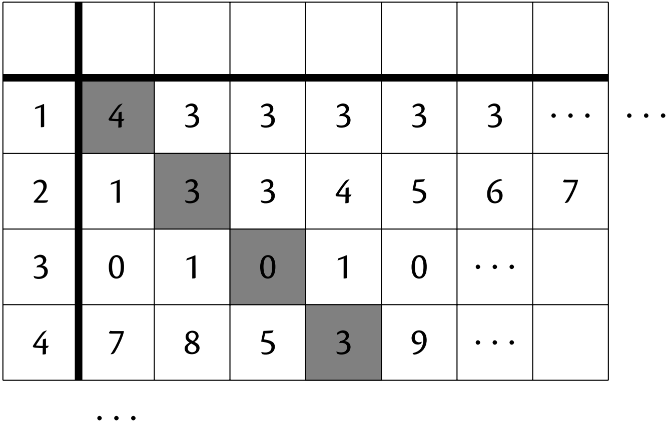

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 3 | 3 | 3 | 3 | . . . |
| 2 | 1 | 3 | 3 | 4 | 5 | 6 | 7 |
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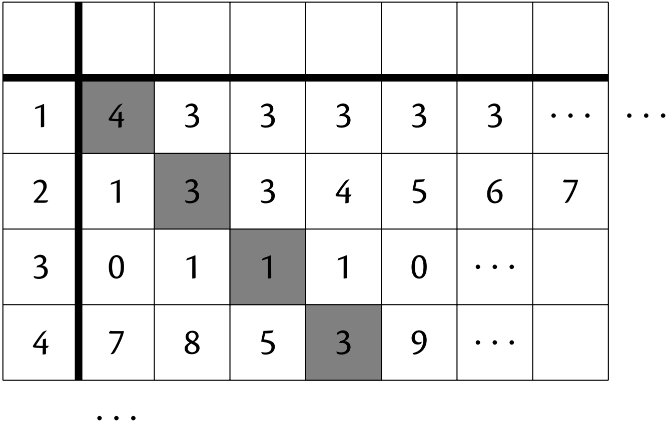
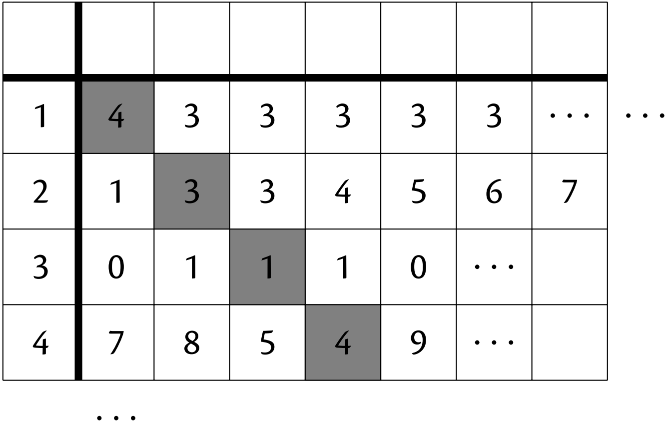


| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 3 | 3 | 3 | 3 | … |
| 2 | 1 | 3 | 3 | 4 | 5 | 6 | 7 |
| 3 | 0 | 1 | 1 | 1 | 0 | … | |
| 4 | 7 | 8 | 5 | 3 | 9 | … | |

…

…

# Real Numbers between 0 and 1



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 3 | 3 | 3 | 3 | . . . |
| 2 | 1 | 3 | 3 | 4 | 5 | 6 | 7 |
| 3 | 0 | 1 | 1 | 1 | 0 | . . . | |
| 4 | 7 | 8 | 5 | 4 | 9 | . . . | |

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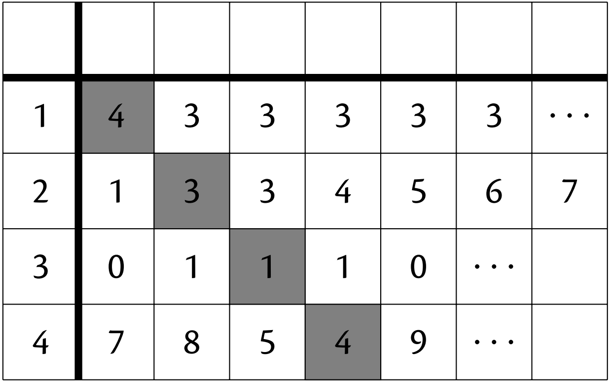
# Real Numbers between 0 and 1



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 3 | 3 | 3 | 3 | 3 | … |
| 2 | 1 | 3 | 3 | 4 | 5 | 6 | 7 |
| 3 | 0 | 1 | 1 | 1 | 0 | … | |
| 4 | 7 | 8 | 5 | 4 | 9 | … | |

…

…

$$|\mathbb{N}| < |R|$$

# The Set of Problems

$\aleph_0$

| | 0 | 1 | 2 | 3 | 4 | 5 | … | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | … | … |
| 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | |
| 3 | 0 | 0 | 0 | 0 | 0 | … | | |
| 4 | 1 | 1 | 0 | 1 | 1 | … | | |

…

# The Set of Problems

$\aleph_0$

|   | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | $\cdots$ |
| 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | $\cdots$ |   |
| 4 | 1 | 1 | 0 | 1 | 1 | $\cdots$ |   |

$\cdots$

$\cdots$

$$|\text{Progs}| = |\mathbb{N}| < |\text{Probs}|$$

# Halting Problem

Assume a program $H$ that decides for all programs $A$ and all input data $D$ whether

- $H(A, D) \overset{\text{def}}{=} 1$ iff $A(D)$ terminates
- $H(A, D) \overset{\text{def}}{=} 0$ otherwise

# Halting Problem (2)

Given such a program $H$
define the following program
C: for all programs $A$

- $C(A) \overset{\text{def}}{=} 0$ iff $H(A, A) = 0$
- $C(A) \overset{\text{def}}{=}$ loops otherwise

# Contradiction

$H(C, C)$ is either $0$ or $1$.

- $H(C, C) = 1 \overset{\text{def } H}{\Rightarrow} C(C)\downarrow \overset{\text{def } C}{\Rightarrow}$ $H(C, C) = 0$
- $H(C, C) = 0 \overset{\text{def } H}{\Rightarrow} C(C)$ loops $\overset{\text{def } C}{\Rightarrow}$

$H(C, C) = 1$

Contradiction in both cases. So $H$ cannot exist.

# Take Home Points

- there are sets that are more infinite than others

- even with the most powerful computer we can imagine, there are problems that cannot be solved by any program

- in CS we actually hit quite often such problems (halting problem)