Automata and Formal Languages (2)

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Slides:	KEATS

Languages, Strings

- A language is a set of strings.

  {[], hello, foobar, a, abc}

- The meaning of a regular expression is a set of strings, or language.

Strings

Different ways of writing strings:

$$"hello" \qquad [h,e,l,l,o] \qquad h::e::l::l::o::Nil$$

$$"" \qquad [] \qquad Nil$$

Strings

Different ways of writing strings:

$$"hello" \qquad [h, e, l, l, o] \qquad h :: e :: l :: l :: o :: Nil$$

$$"" \qquad [] \qquad Nil$$

The concatenation operation on strings and sets of strings:

$$"foo" @ "bar" = "foobar"$$

$$A @ B \stackrel{\text{def}}{=} \{s_1 @ s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

Regular Expressions

Their inductive definition:

r	$::=$	$\varnothing$	null
	$\mid$	ϵ	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

Re

Their indu

```
abstract class Rexp

case object NULL extends Rexp
case object EMPTY extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

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	$\mid$	ϵ	empty string / "" / []
	$\mid$	c	character
	$\mid$	$r_1 \cdot r_2$	sequence
	$\mid$	$r_1 + r_2$	alternative / choice
	$\mid$	r^*	star (zero or more)

The Meaning of a Regular Expression

$$
\begin{array}{rcl}
L(\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
L(\epsilon) & \stackrel{\text{def}}{=} & \{''''\} \\
L(c) & \stackrel{\text{def}}{=} & \{''c''\} \\
L(r_1 + r_2) & \stackrel{\text{def}}{=} & L(r_1) \cup L(r_2) \\
L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & L(r_1) \,@\, L(r_2) \\
L(r^*) & \stackrel{\text{def}}{=} & \bigcup_{n \geq 0} L(r)^n
\end{array}
$$

L is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

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L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & L(r_1) @ L(r_2) \\
L(r^*) & \stackrel{\text{def}}{=} & \bigcup_{n \geq 0} L(r)^n
\end{array}
$$

$$
\begin{array}{rcl}
L(r)^0 & \stackrel{\text{def}}{=} & \{""\} \\
L(r)^{n+1} & \stackrel{\text{def}}{=} & L(r) @ L(r)^n
\end{array}
$$

L is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

What is $L(a^*)$?

Reg Exp Equivalences

$$
\begin{array}{lrcl}
 & (a+b)+c & \equiv^? & a+(b+c) \\
 & a+a & \equiv^? & a \\
 & (a\cdot b)\cdot c & \equiv^? & a\cdot(b\cdot c) \\
 & a\cdot a & \equiv^? & a \\
 & \epsilon^* & \equiv^? & \epsilon \\
 & \varnothing^* & \equiv^? & \varnothing \\
\forall\ \text{r.} & r\cdot\epsilon & \equiv^? & r \\
\forall\ \text{r.} & r+\epsilon & \equiv^? & r \\
\forall\ \text{r.} & r+\varnothing & \equiv^? & r \\
\forall\ \text{r.} & r\cdot\varnothing & \equiv^? & r \\
 & c\cdot(a+b) & \equiv^? & (c\cdot a)+(c\cdot b) \\
 & a^* & \equiv^? & \epsilon+(a\cdot a^*)
\end{array}
$$

Reg Exp Equivalences

$$
\begin{array}{lrcll}
 & (a+b)+c & \equiv^? & a+(b+c) & \text{yes}\\
 & a+a & \equiv^? & a & \\
 & (a\cdot b)\cdot c & \equiv^? & a\cdot(b\cdot c) & \\
 & a\cdot a & \equiv^? & a & \\
 & \epsilon^* & \equiv^? & \epsilon & \\
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\forall\ \text{r.} & r\cdot\epsilon & \equiv^? & r & \\
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$$

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\end{array}
$$

Reg Exp Equivalences

	$(a + b) + c$	$\equiv^?$	$a + (b + c)$	yes
	$a + a$	$\equiv^?$	a	yes
	$(a \cdot b) \cdot c$	$\equiv^?$	$a \cdot (b \cdot c)$	yes
	$a \cdot a$	$\equiv^?$	a	
	ϵ^*	$\equiv^?$	ϵ	
	$\varnothing^*$	$\equiv^?$	$\varnothing$	
∀ r.	$r \cdot \epsilon$	$\equiv^?$	r	
∀ r.	$r + \epsilon$	$\equiv^?$	r	
∀ r.	$r + \varnothing$	$\equiv^?$	r	
∀ r.	$r \cdot \varnothing$	$\equiv^?$	r	
	$c \cdot (a + b)$	$\equiv^?$	$(c \cdot a) + (c \cdot b)$	
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$\forall$ r.	$r\cdot\epsilon$	$\equiv^?$	r	
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	a^*	$\equiv^?$	$\epsilon + (a \cdot a^*)$	yes

The Specification for Matching

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$$s \in L(r)$$

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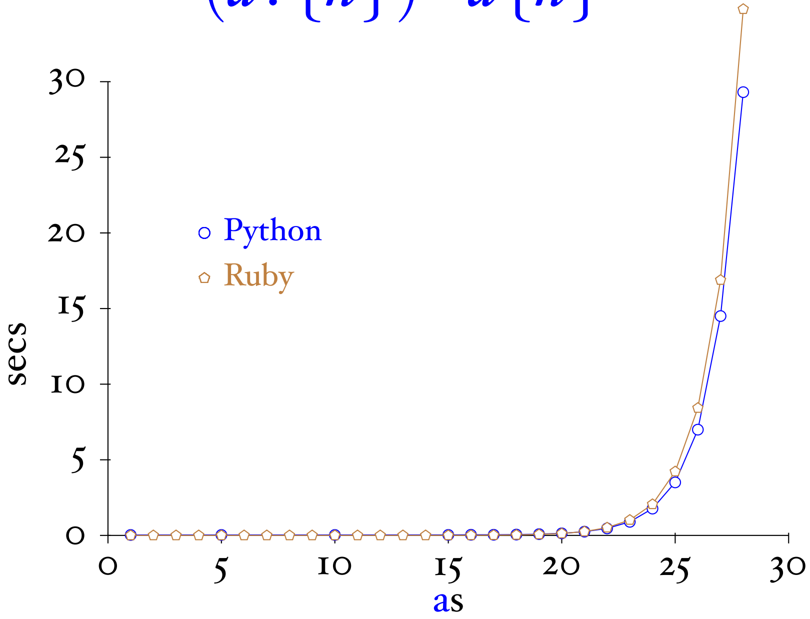
$(a?\{n\}) \cdot a\{n\}$



Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
  - $(a?\{n\}) \cdot a\{n\}$
  - $(a^+)^+$
  - $([a\text{-}z]^+)^*$
  - $(a + a \cdot a)^+$
  - $(a + a?)^+$

A Matching Algorithm

...whether a regular expression can match the empty string:

$$
\begin{array}{lcl}
nullable(\varnothing) & \stackrel{\text{def}}{=} & false \\
nullable(\epsilon) & \stackrel{\text{def}}{=} & true \\
nullable(c) & \stackrel{\text{def}}{=} & false \\
nullable(r_1 + r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \vee nullable(r_2) \\
nullable(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \wedge nullable(r_2) \\
nullable(r^*) & \stackrel{\text{def}}{=} & true
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$$

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nullable(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & nullable(r_1) \wedge nullable(r_2) \\
nu\ldots & \stackrel{\text{def}}{=} &
\end{array}
$$

```
def nullable (r: Rexp) : Boolean = r match {
  case NULL => false
  case EMPTY => true
  case CHAR(_) => false
  case ALT(r1, r2) => nullable(r1) || nullable(r2)
  case SEQ(r1, r2) => nullable(r1) && nullable(r2)
  case STAR(_) => true
}
```

The Derivative of a Rexp

If r matches the string $c :: s$, what is a regular expression that matches s?

$der\, c\, r$ gives the answer

The Derivative of a Rexp (2)

$$
\begin{array}{lcl}
der\, c\, (\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
der\, c\, (\epsilon) & \stackrel{\text{def}}{=} & \varnothing \\
der\, c\, (d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \epsilon \text{ else } \varnothing \\
der\, c\, (r_1 + r_2) & \stackrel{\text{def}}{=} & der\, c\, r_1 + der\, c\, r_2 \\
der\, c\, (r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } (der\, c\, r_1) \cdot r_2 + der\, c\, r_2 \\
 & & \text{else } (der\, c\, r_1) \cdot r_2 \\
der\, c\, (r^*) & \stackrel{\text{def}}{=} & (der\, c\, r) \cdot (r^*)
\end{array}
$$

The Derivative of a Rexp (2)

$$
\begin{array}{lcl}
der\,c\,(\varnothing) & \stackrel{\text{def}}{=} & \varnothing\\
der\,c\,(\epsilon) & \stackrel{\text{def}}{=} & \varnothing\\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \epsilon \text{ else } \varnothing\\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1)\\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2\\
 & & \text{else } (der\,c\,r_1) \cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)\\
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}
$$

The Derivative of a Rexp (2)

$$der\, c\, (\varnothing) \stackrel{\text{def}}{=} \varnothing$$

$$der\, c\, (\epsilon) \stackrel{\text{def}}{=} \varnothing$$

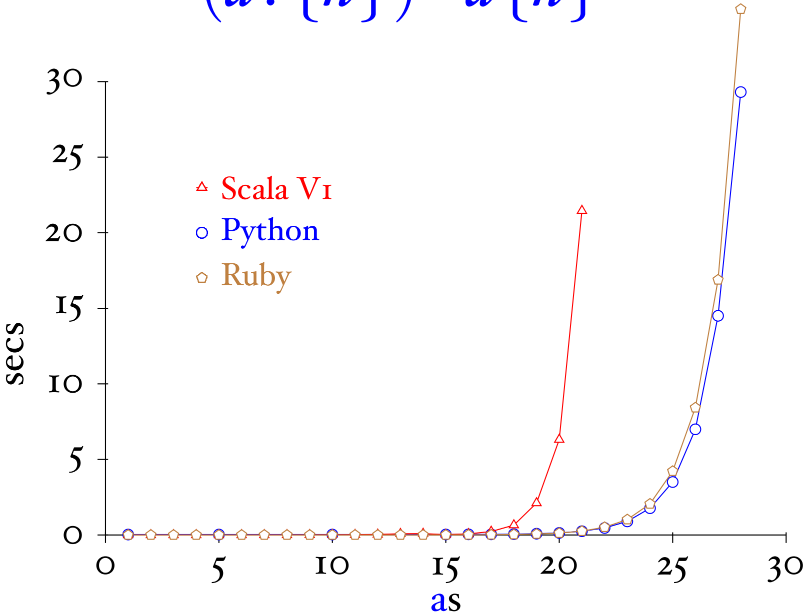
```
def der (r: Rexp, c: Char) : Rexp = r match {
  case NULL => NULL
  case EMPTY => NULL
  case CHAR(d) => if (c == d) EMPTY else NULL
  case ALT(r1, r2) => ALT(der(r1, c), der(r2, c))
  case SEQ(r1, r2) =>
    if (nullable(r1)) ALT(SEQ(der(r1, c), r2), der(r2, c))
    else SEQ(der(r1, c), r2)
  case STAR(r) => SEQ(der(r, c), STAR(r))
}

def ders (s: List[Char], r: Rexp) : Rexp = s match {
  case Nil => r
  case c::s => ders(s, der(c, r))
}
```

Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$der\, a\, r$$
$$der\, b\, r$$

$(a?\{n\}) \cdot a\{n\}$



Proofs about Rexps

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \varnothing \\
 & | & \epsilon \\
 & | & c \\
 & | & r_1 \cdot r_2 \\
 & | & r_1 + r_2 \\
 & | & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions r then ...

Proofs about Rexp (2)

- P holds for $\varnothing$, ϵ and $\mathbf{c}$
- P holds for $r_1 + r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for $r_1 \cdot r_2$ under the assumption that P already holds for r_1 and r_2.
- P holds for r^* under the assumption that P already holds for r.

Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$\textit{nullable}(r) \text{ if and only if } "" \in L(r)$$

Proofs about Rexp (4)

Let $Der\,c\,A$ be the set defined as

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

We can prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

by induction on r.

Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings s then ...

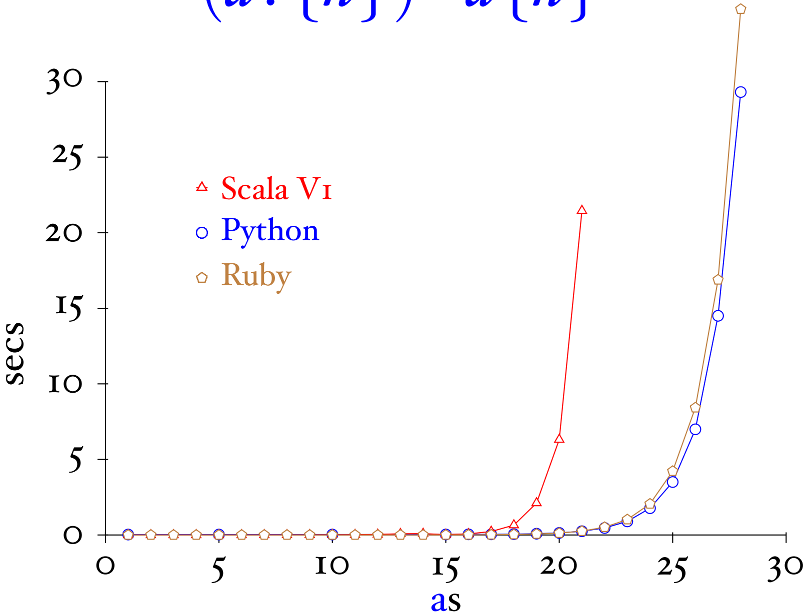
- P holds for the empty string, and
- P holds for the string $c::s$ under the assumption that P already holds for s

Proofs about Strings (2)

We can finally prove

$$\mathit{matcher}(r, s) \text{ if and only if } s \in L(r)$$

$(a?\{n\}) \cdot a\{n\}$



A Problem

We represented the "n-times" $a\{n\}$ as a sequence regular expression:

$$
\begin{array}{rl}
1: & a \\
2: & a \cdot a \\
3: & a \cdot a \cdot a \\
 & \ldots \\
13: & a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \\
 & \ldots \\
20: &
\end{array}
$$

This problem is aggravated with $a?$ being represented as $\epsilon + a$.

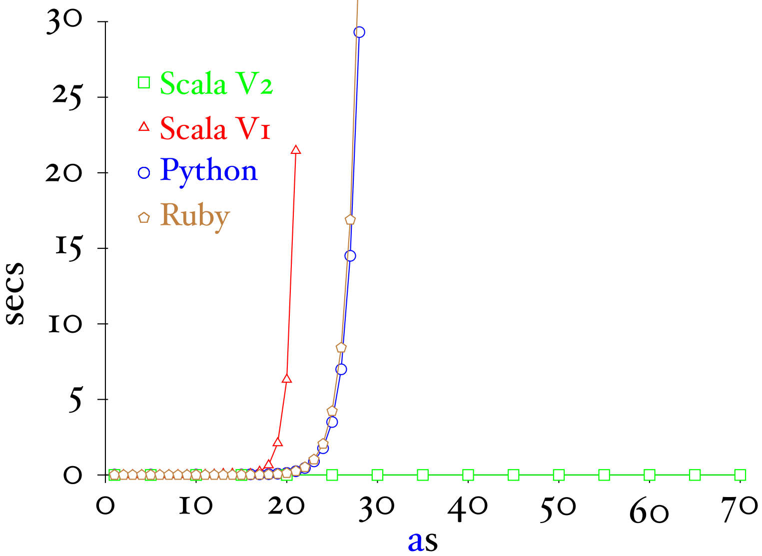
Solving the Problem

What happens if we extend our regular expressions

$$\begin{array}{rcl} r & ::= & \ldots \\ & | & r\{n\} \\ & | & r? \end{array}$$

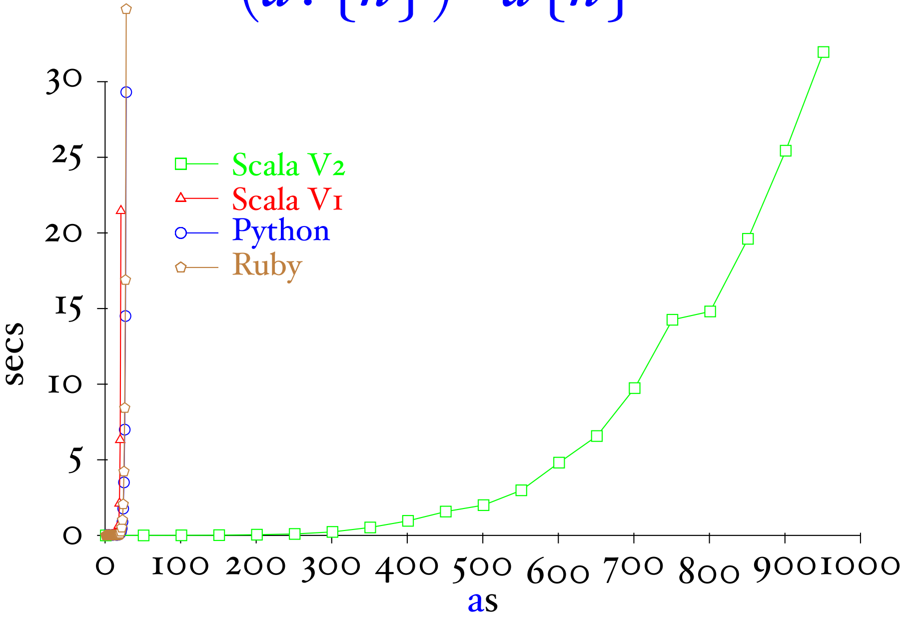
What is their meaning? What are the cases for *nullable* and *der*?

$$(a?\{n\}) \cdot a\{n\}$$



Scala V2
Scala V1
Python
Ruby
secs
as
0
5
10
15
20
25
30
0
10
20
30
40
50
60
70

$(a?\{n\}) \cdot a\{n\}$



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400
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600
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800
900
1000
as

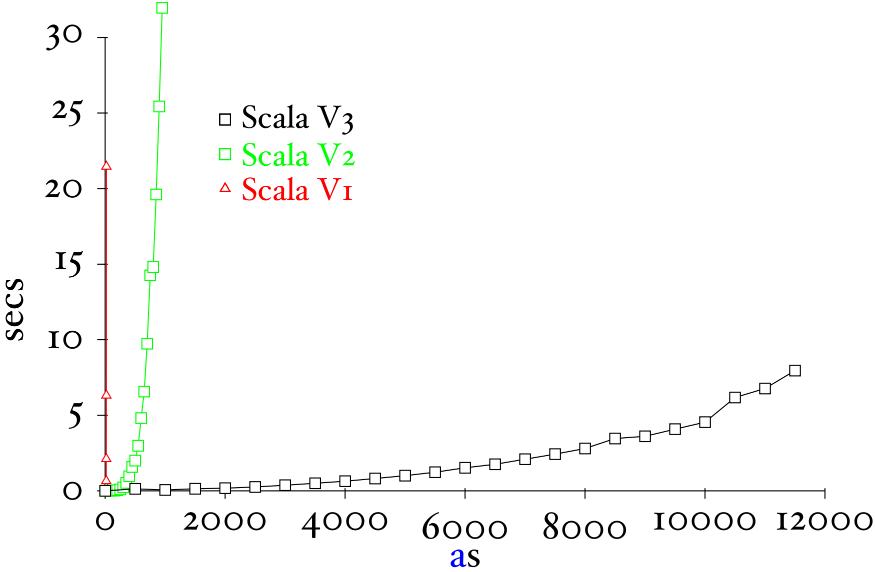
Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$
\begin{aligned}
der\,a\,r &= ((\epsilon \cdot b) + \varnothing) \cdot r \\
der\,b\,r &= ((\varnothing \cdot b) + \epsilon) \cdot r
\end{aligned}
$$

What are these regular expressions equal to?

$(a?\{n\}) \cdot a\{n\}$



30
25
20
15
10
5
0
secs
Scala V3
Scala V2
Scala V1
0
2000
4000
6000
8000
10000
12000
as